Fibers Buckling in Liquid Drops

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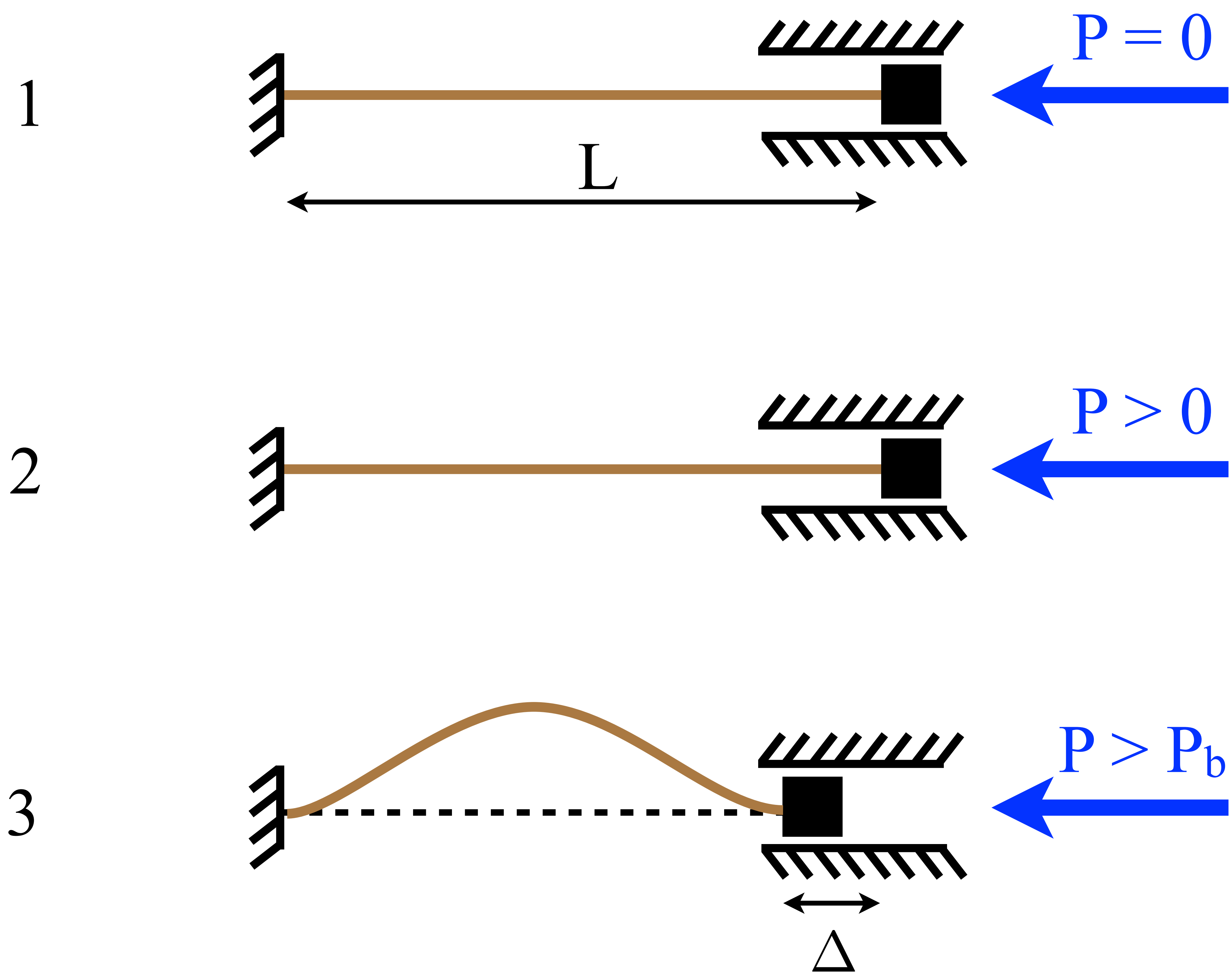
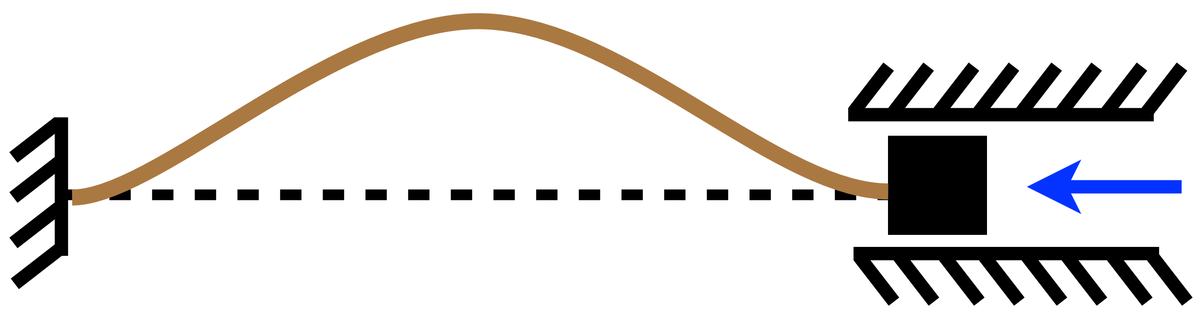
Classical Buckling



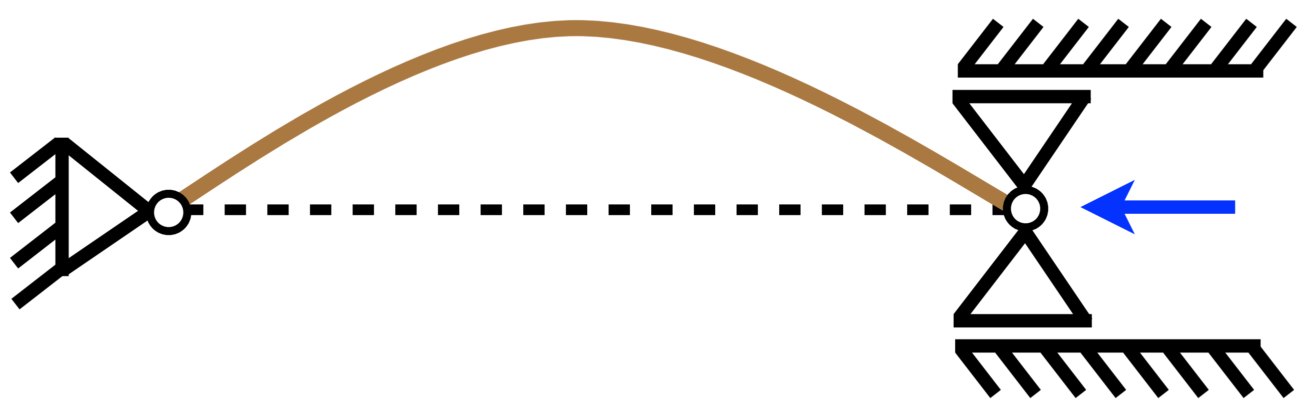
Classical Buckling

clamped



$$P_b = 4\pi^2 \frac{EI}{L^2}$$

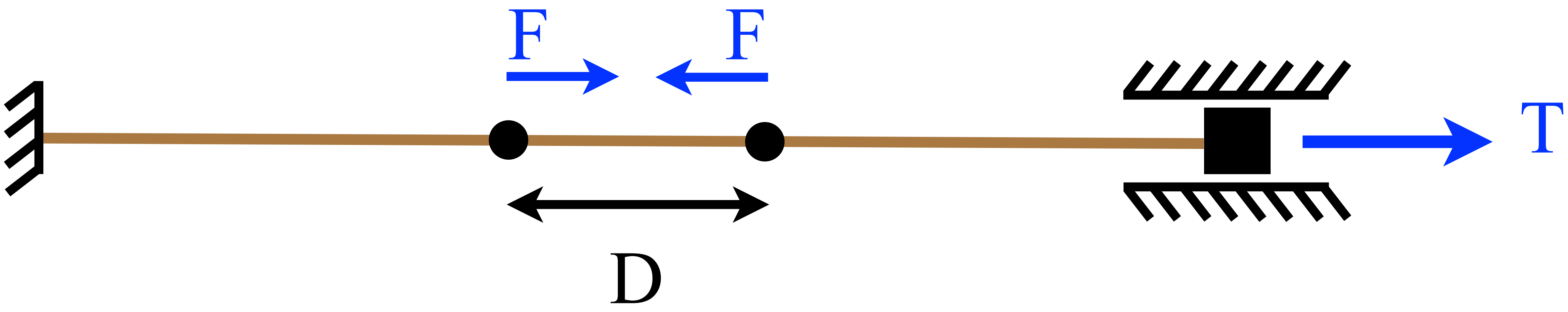
pinned



$$P_b = \pi^2 \frac{EI}{L^2}$$

Remark: L is the total arc-length of the beam

Buckling due to internal compression

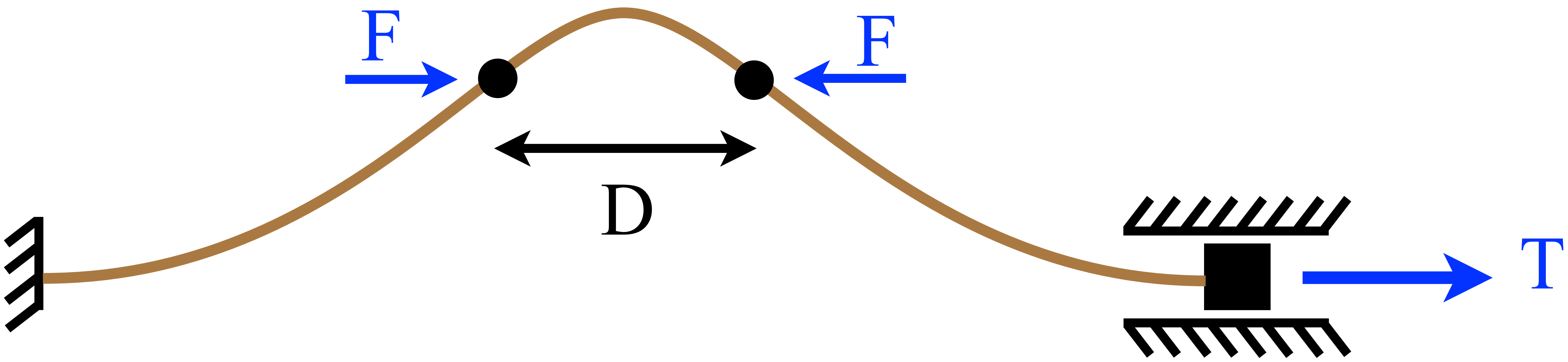


inner part of beam is in compression $P = F - T$

does everything only depend on P ?

Low tension T

$$\frac{TD^2}{EI} \ll 1$$

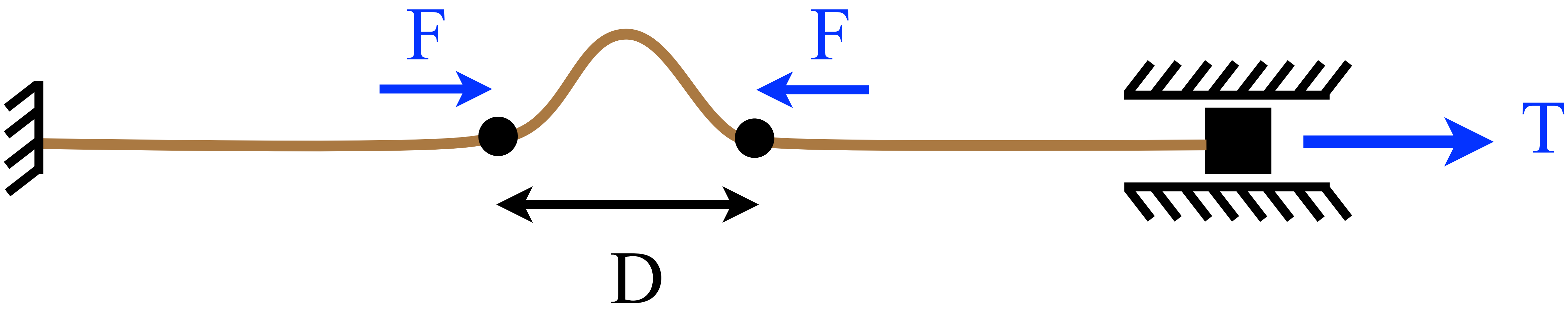


pinned $\quad P_b = F - T = \pi^2 \frac{EI}{D^2}$

$$F \simeq \pi^2 \frac{EI}{D^2}$$

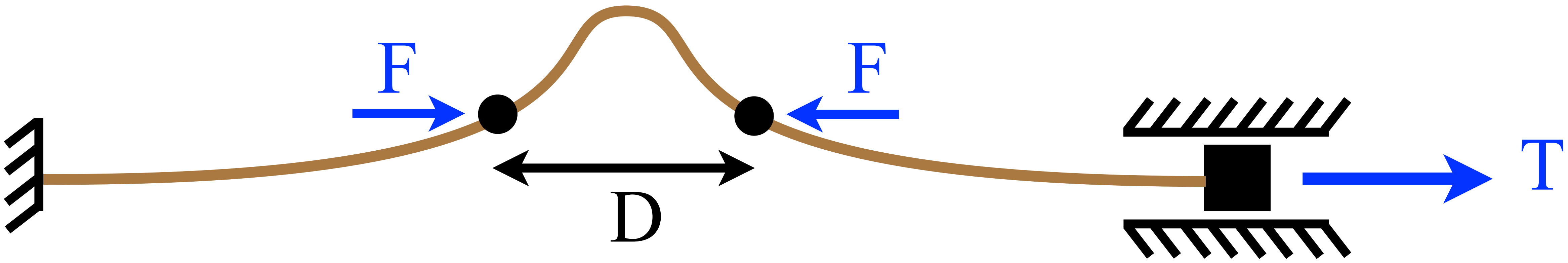
Large tension T

$$\frac{TD^2}{EI} \gg 1$$



clamped $P_b = F - T = 4\pi^2 \frac{EI}{D^2}$

Exact computation

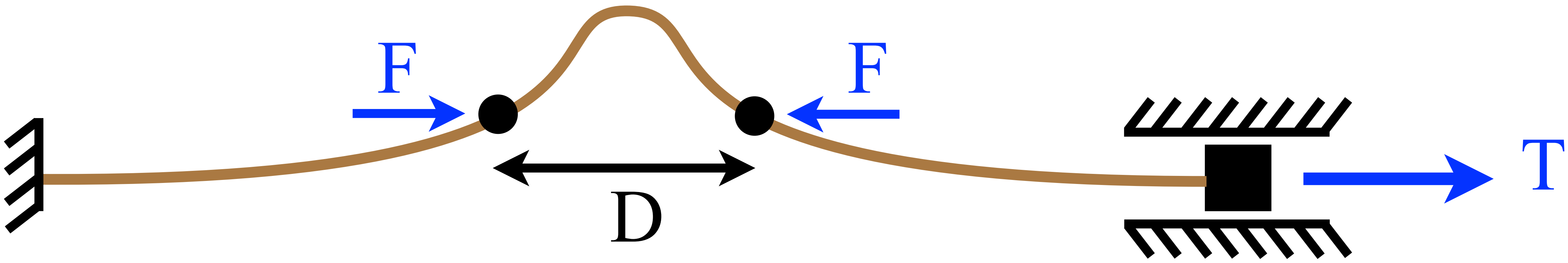


$$\sqrt{f-t}\,\tanh\frac{(\ell-1)\sqrt{t}}{2}+\sqrt{t}\,\tan\frac{\sqrt{f}}{2}=0$$

with: $f=\dfrac{FD^2}{EI}$ $\quad t=\dfrac{TD^2}{EI}$ $\quad \ell=\dfrac{L}{D}$

Large L limit

$L \gg D$

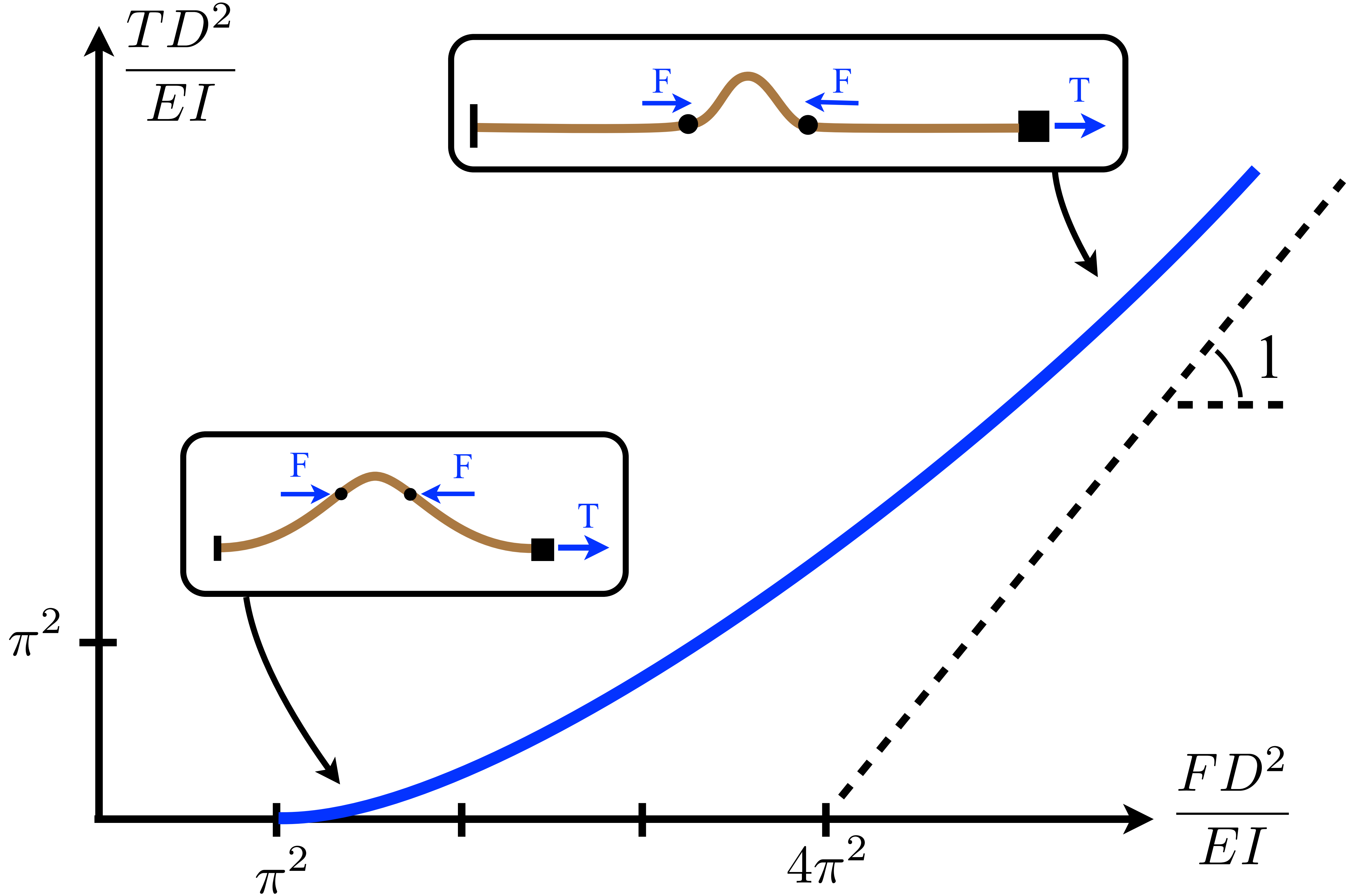


$$\sqrt{f-t} + \sqrt{t}\,\tan\frac{\sqrt{f}}{2} = 0$$

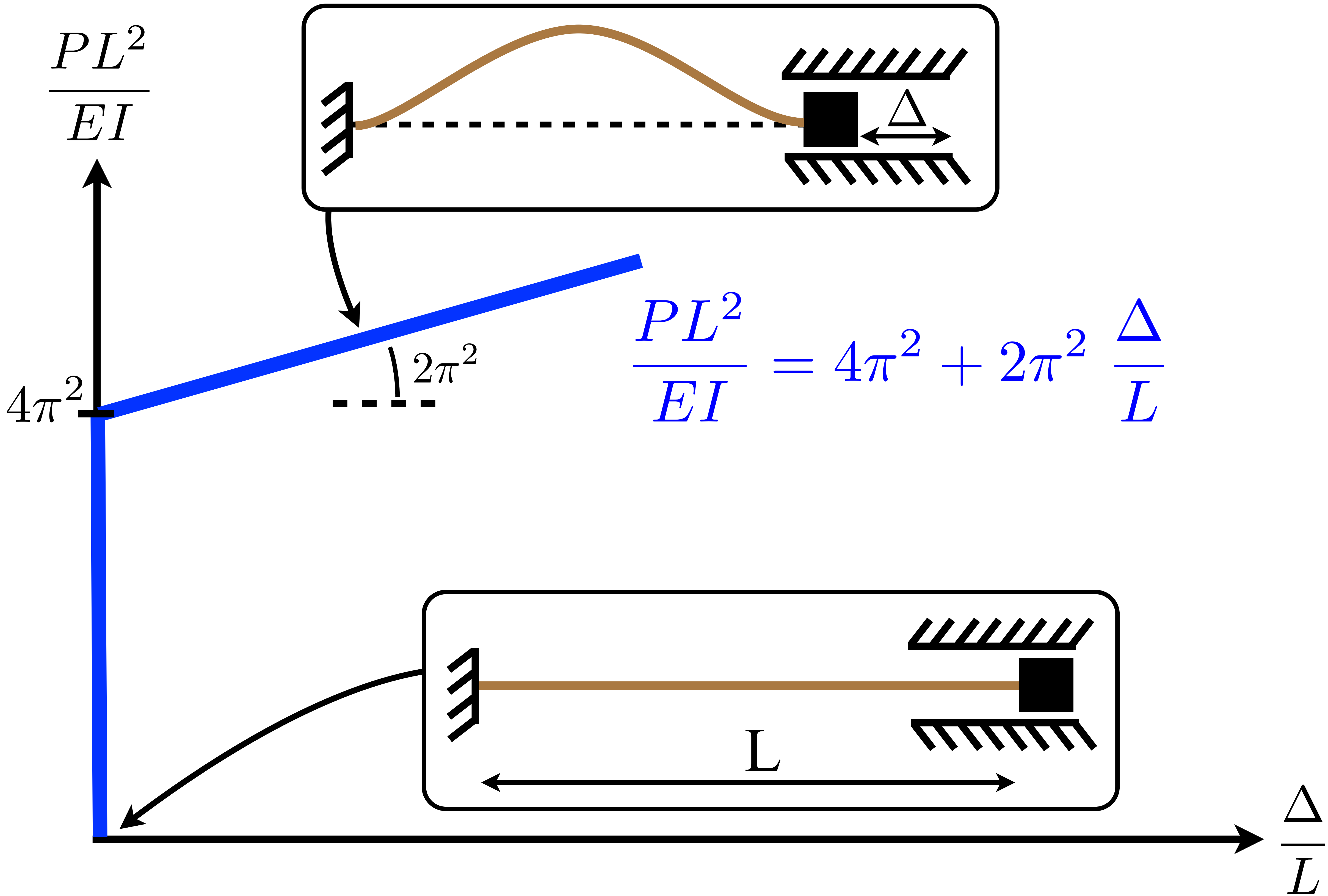
with: $f = \dfrac{FD^2}{EI}$ $\qquad t = \dfrac{TD^2}{EI}$

Buckling due to internal compression

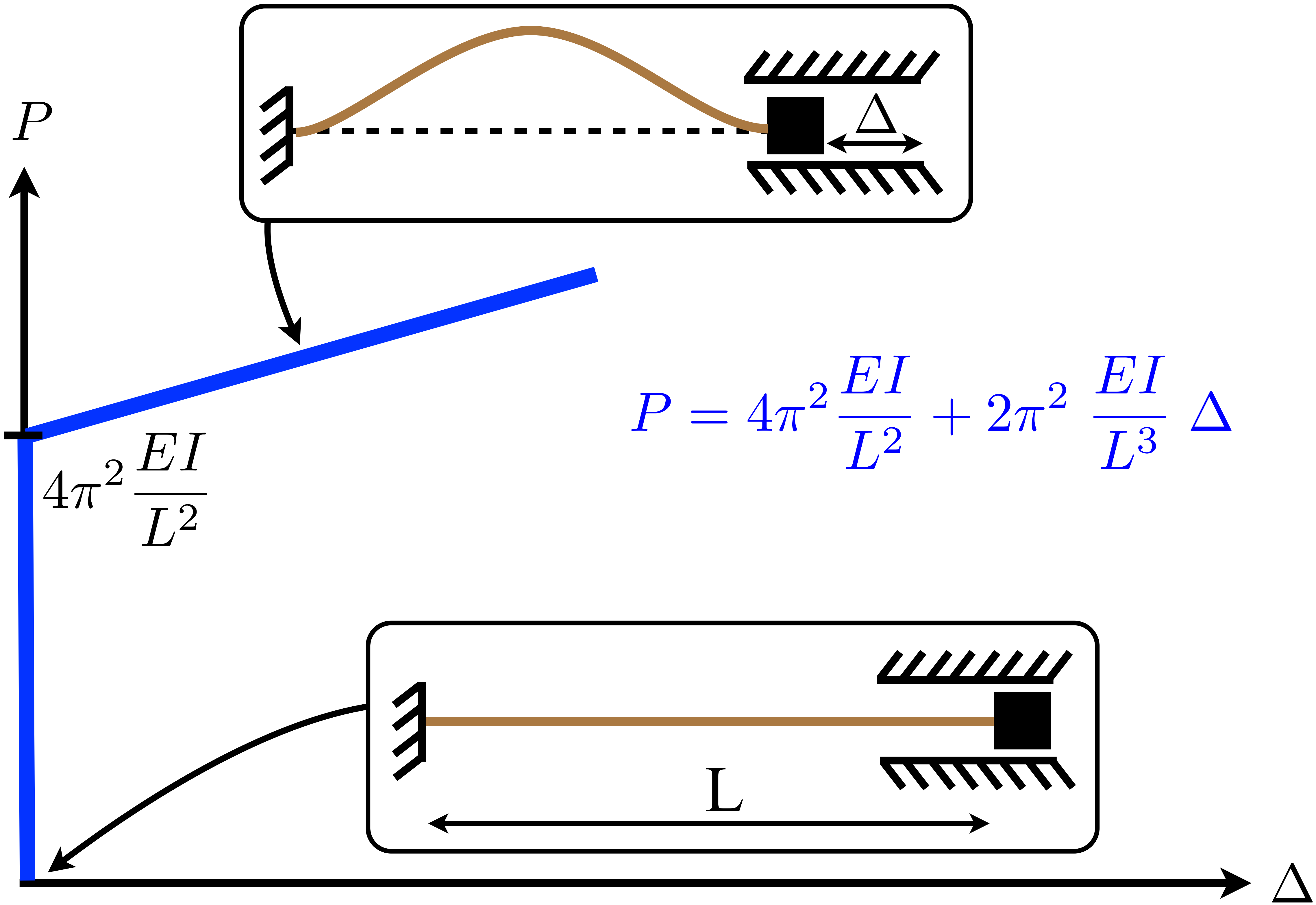
large L limit
$L \gg D$



Initial post-buckling regime



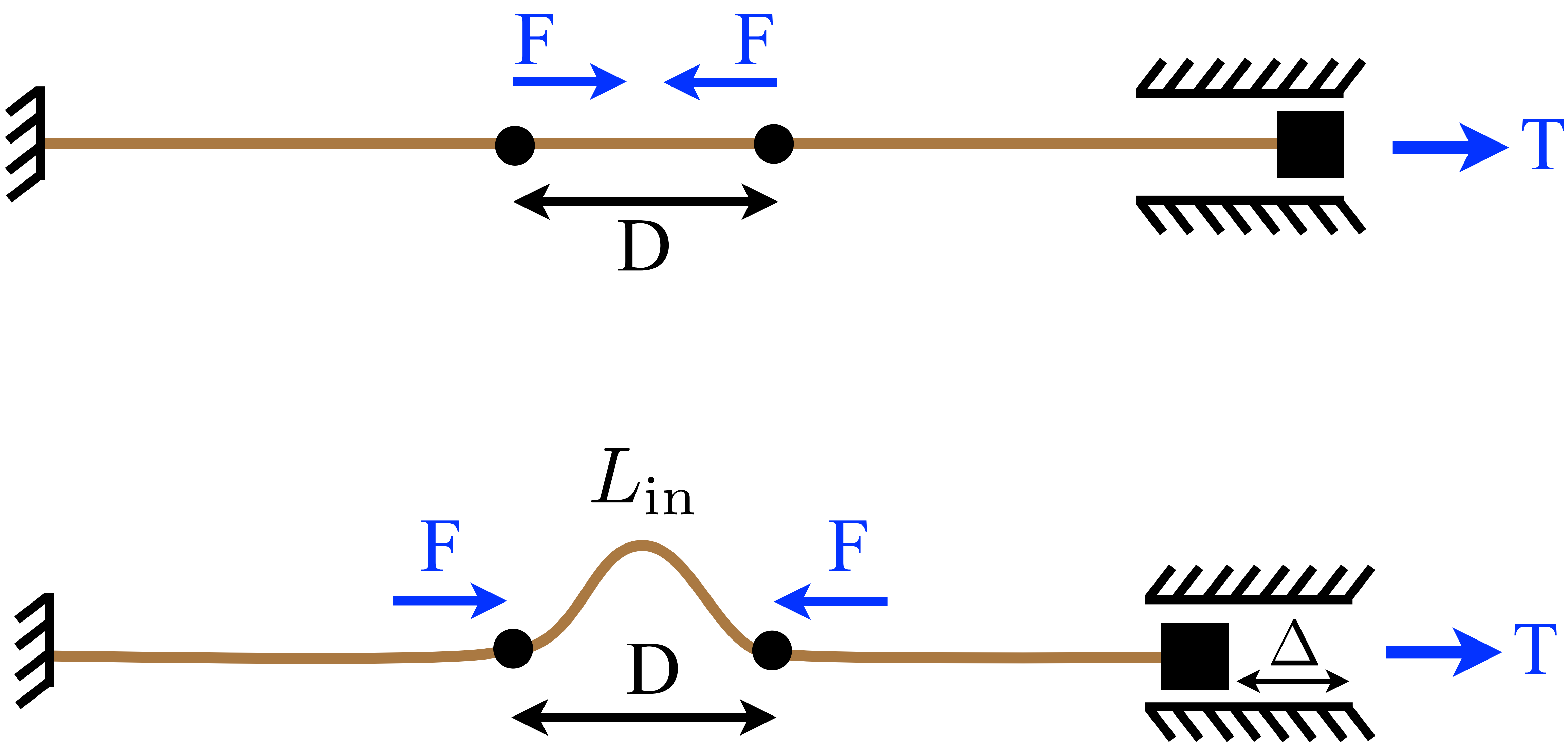
Initial post-buckling regime



Initial post-buckling regime

large T limit

$$T \gg \frac{EI}{D^2}$$



$$\frac{PL_{\text{in}}^2}{EI} = 4\pi^2 + 2\pi^2 \frac{\Delta}{L_{\text{in}}}$$

with

$$L_{\text{in}} = D + \Delta$$

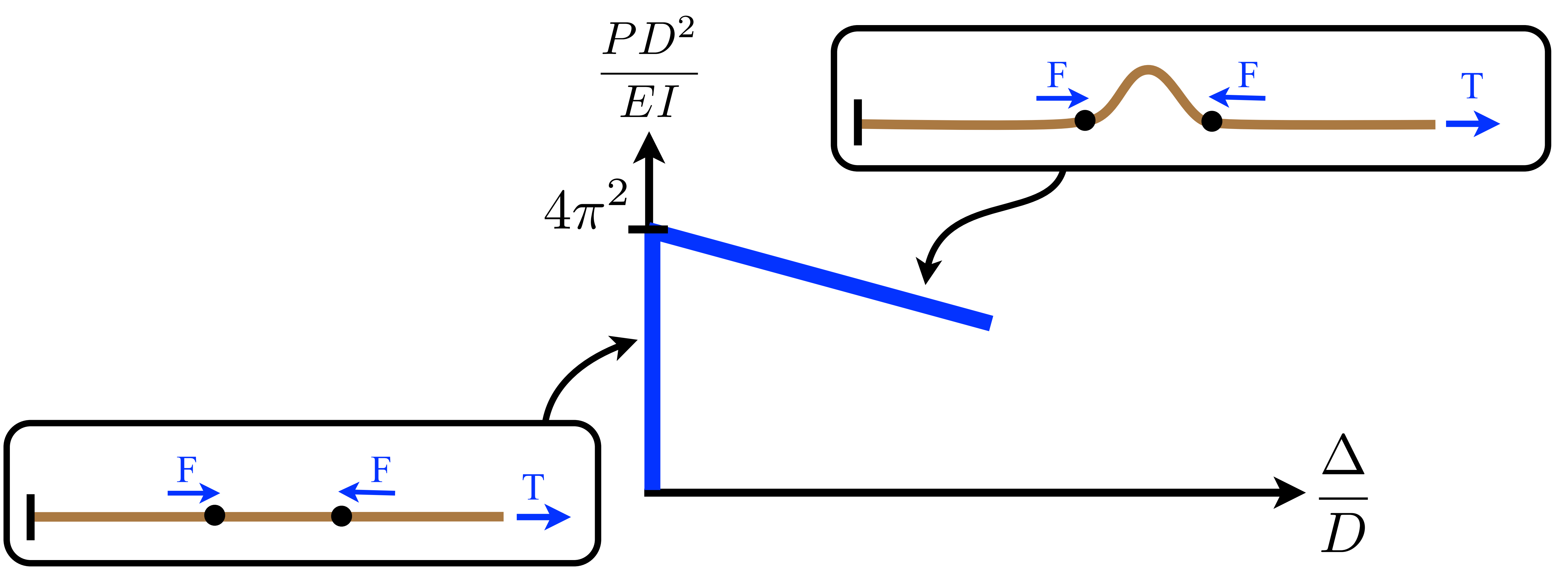
Initial post-buckling regime

large T limit

$$T \gg \frac{EI}{D^2}$$

$$\frac{PL_{\text{in}}^2}{EI} = 4\pi^2 + 2\pi^2 \frac{\Delta}{L_{\text{in}}} \quad \text{with} \quad L_{\text{in}} = D + \Delta$$

$$\frac{PD^2}{EI} \simeq 4\pi^2 - 6\pi^2 \frac{\Delta}{D} \quad \text{for} \quad \frac{\Delta}{D} \ll 1$$

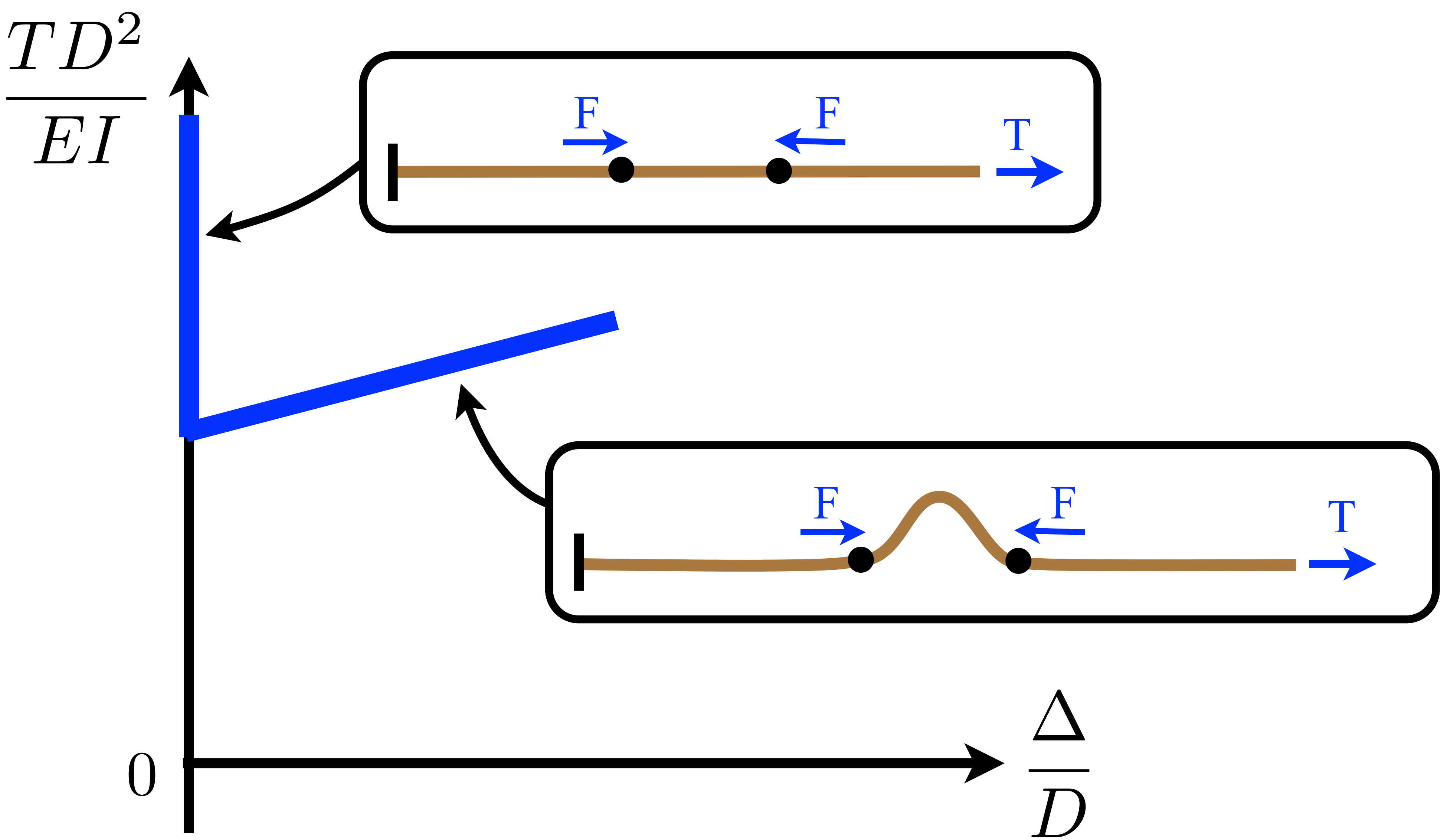


Initial post-buckling regime

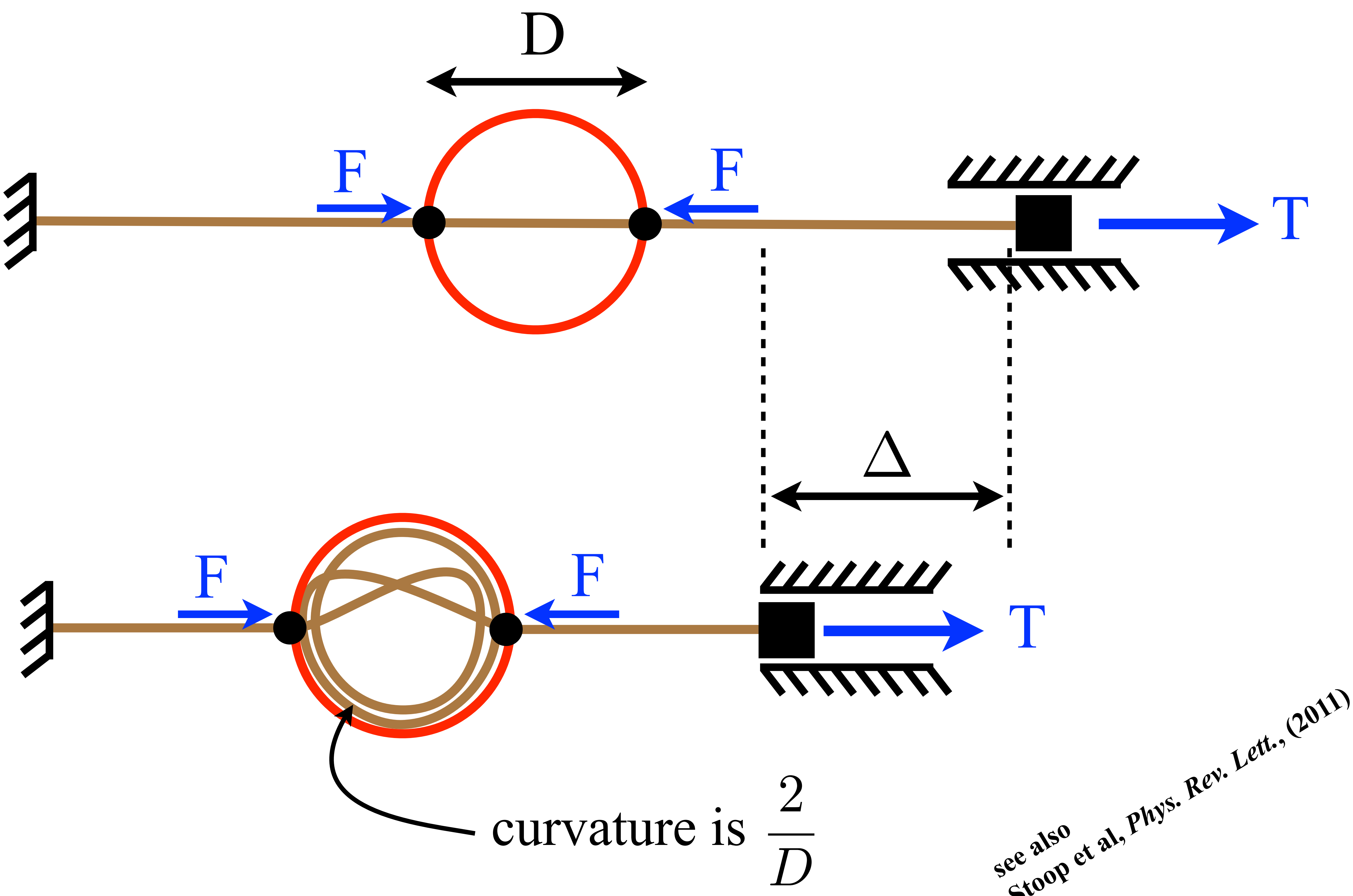
large T limit

$$T \gg \frac{EI}{D^2}$$

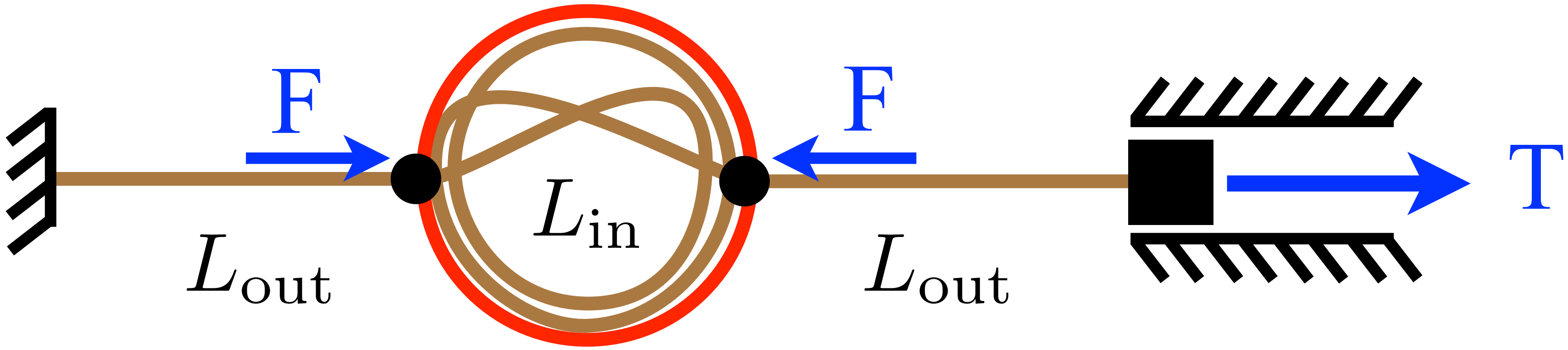
We plot $T = F - P$ for a fixed, given value of F



Buckling inside a disk

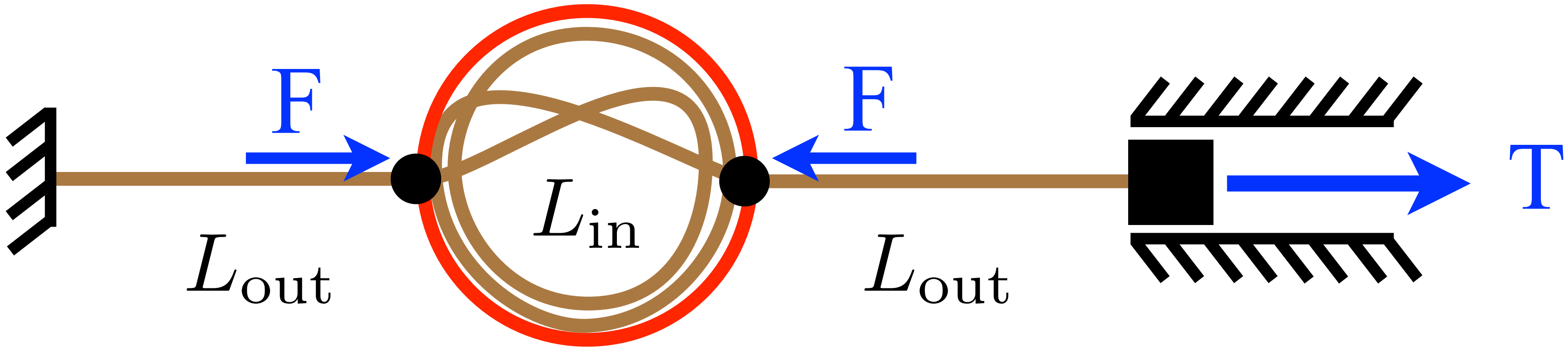


Far post-buckling regime



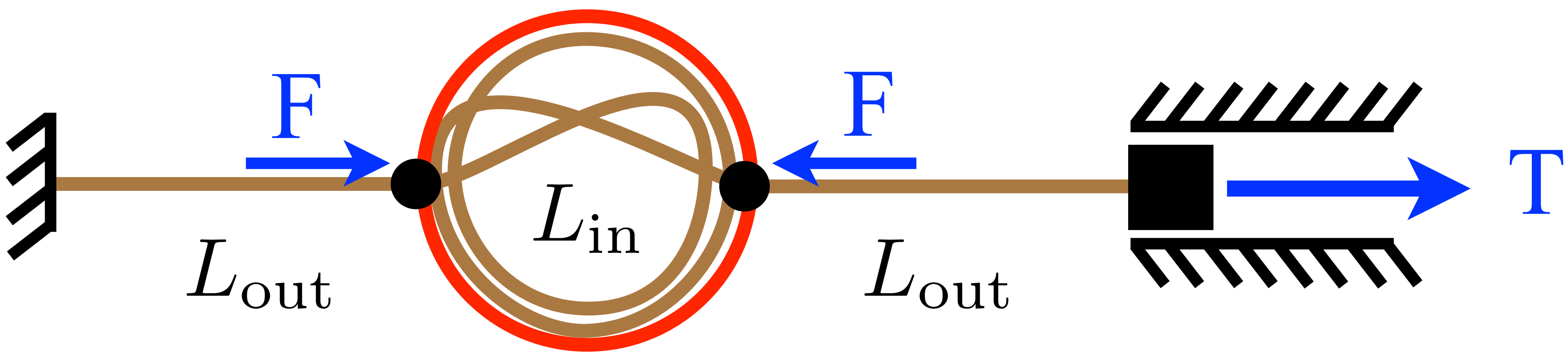
$$V(L_{\mathrm{in}}) = \frac{1}{2}\, EI \left(\frac{2}{D}\right)^2 L_{\mathrm{in}}$$

Far post-buckling regime



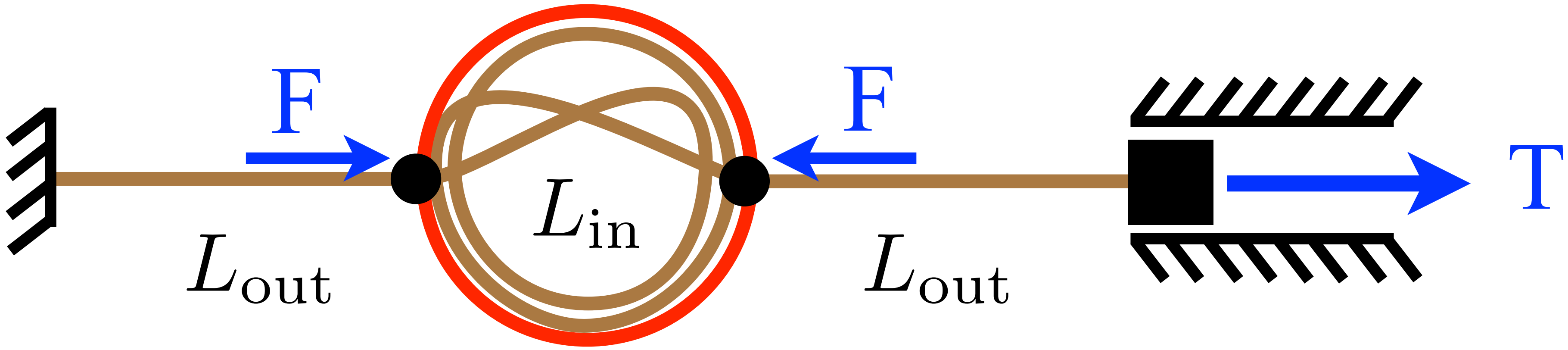
$$V(L_{\rm in}) = \frac{1}{2}\, EI\, \left(\frac{2}{D}\right)^2 L_{\rm in} + T\, L_{\rm in}$$

Far post-buckling regime



$$V(L_{\rm in}) = \frac{1}{2}\, EI\, \left(\frac{2}{D}\right)^2 L_{\rm in} + T\, L_{\rm in} - F\, L_{\rm in}$$

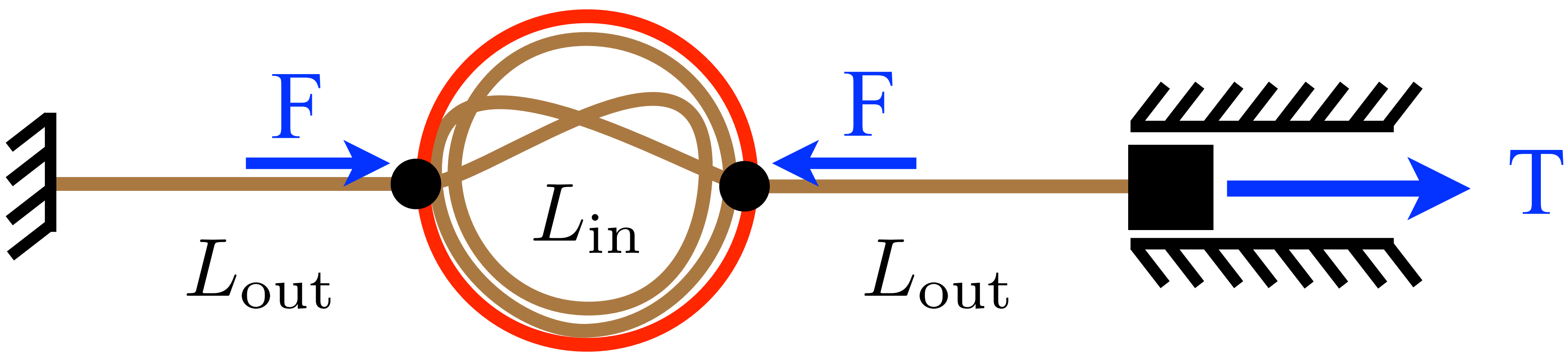
Far post-buckling regime



$$V(L_{\mathrm{in}}) = \frac{1}{2}\,EI\,\left(\frac{2}{D}\right)^2 L_{\mathrm{in}} + T\,L_{\mathrm{in}} - F\,L_{\mathrm{in}}$$

$$\frac{\partial V}{\partial L_{\mathrm{in}}} = \frac{1}{2}\,EI\,\left(\frac{2}{D}\right)^2 + T - F = 0$$

Far post-buckling regime

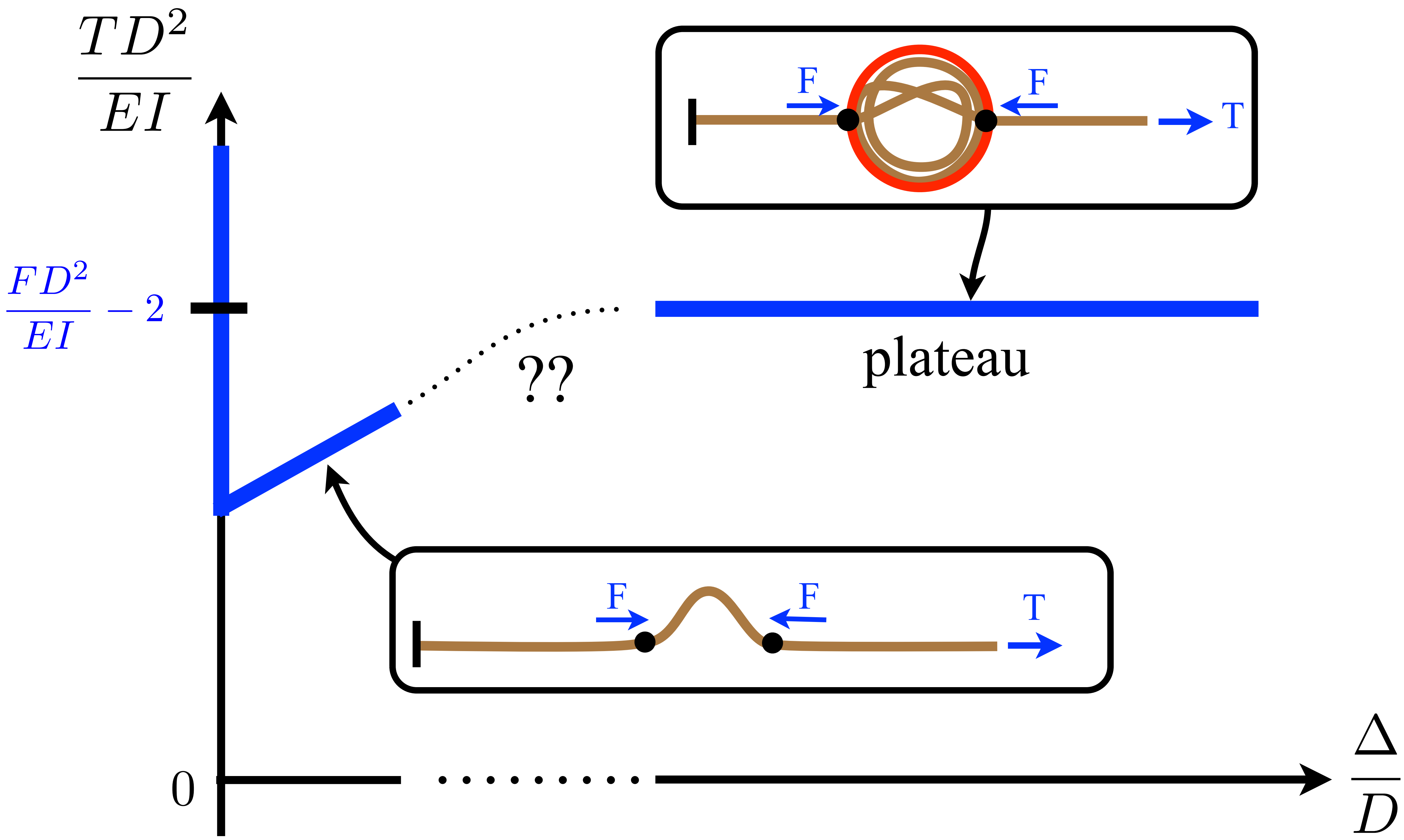


$$V(L_{\text{in}}) = \frac{1}{2}\,EI\,\left(\frac{2}{D}\right)^2 L_{\text{in}} + T\,L_{\text{in}} - F\,L_{\text{in}}$$

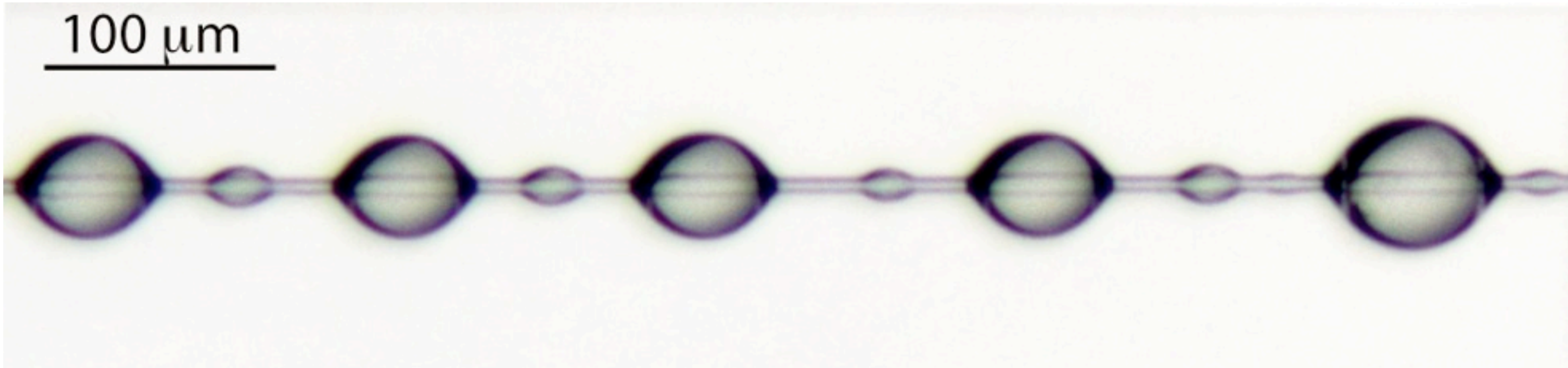
$$\frac{\partial V}{\partial L_{\text{in}}} = \frac{1}{2}\,EI\,\left(\frac{2}{D}\right)^2 + T - F = 0$$

$$\Rightarrow\ T_p = F - 2\,\frac{EI}{D^2} \qquad \text{(plateau force)}$$

Bifurcation diagram



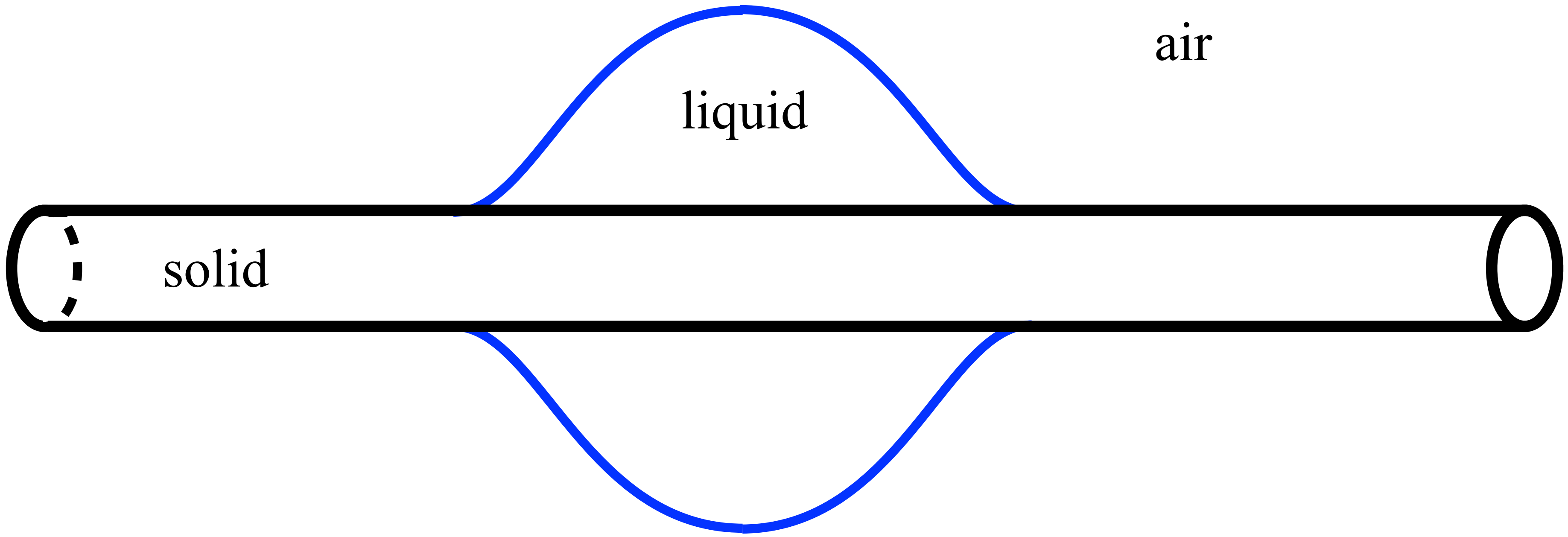
Liquid drops on elastic fiber



surface tension <=> interface energy

P.-G. de Gennes et al, *Capillarity and wetting phenomena*, 2004
E. Lorenceau et al, *Wetting of fibers,* 2006

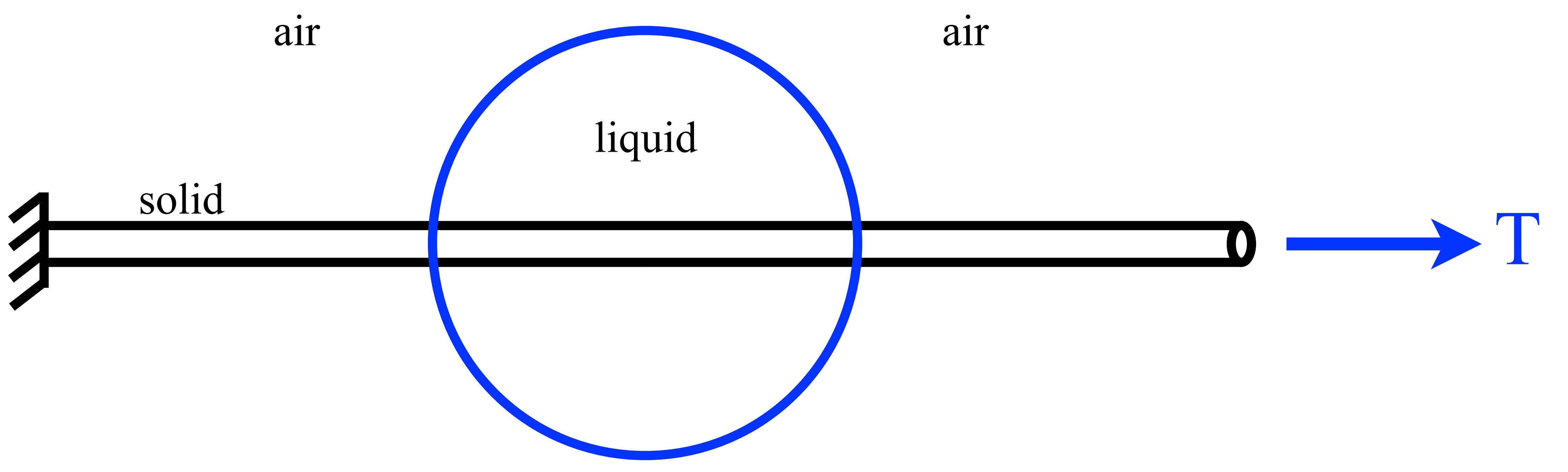
Liquid drops on elastic fiber



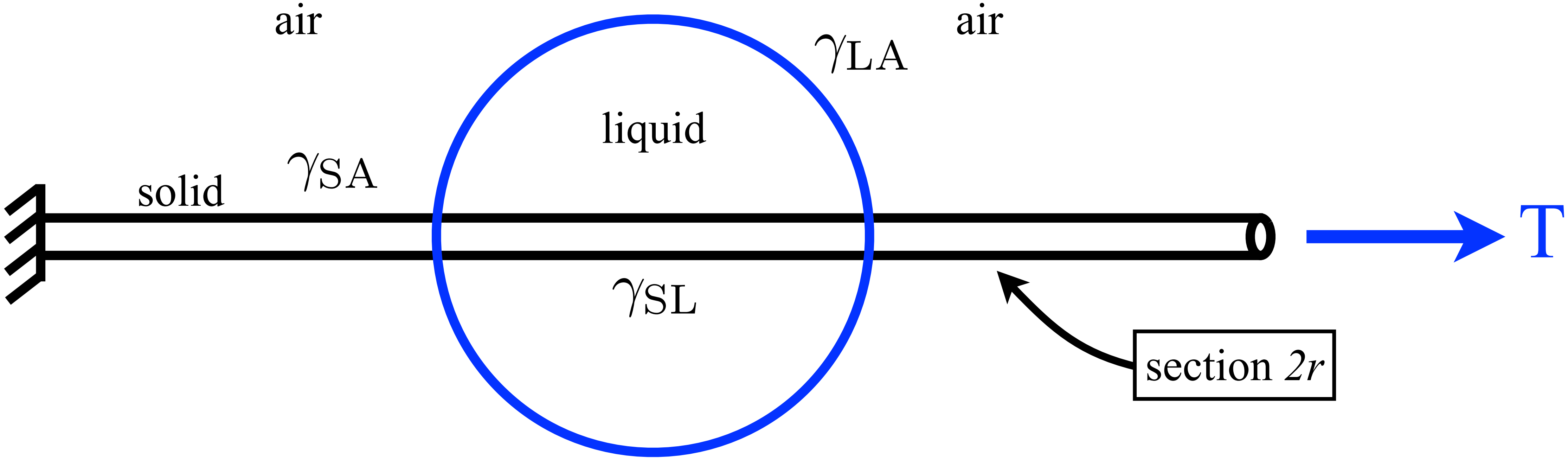
unduloid (constant mean curvature)

J. Plateau, *Statique expérimentale des liquides*, 1873
H. Poincaré, *Capillarité*, 1895
B. Carroll, *Liquid drops on thin cylinders*, Langmuir, 1986

Liquid drops on elastic fiber



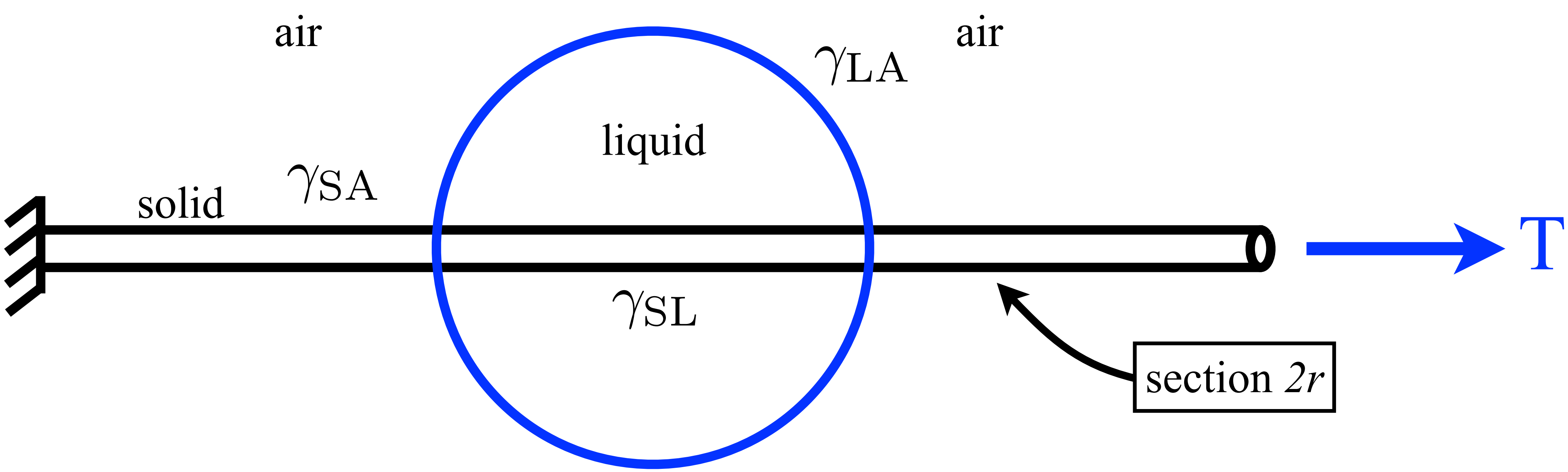
Liquid drops on elastic fiber



γ_{SA} energy per unit area of the interface *solid-air*

$$V(\kappa(s), L_{\mathrm{in}}) = \frac{1}{2} EI \int_0^{L_{\mathrm{in}}} \kappa^2(s, L_{\mathrm{in}}) ds + T\, L_{\mathrm{in}}$$

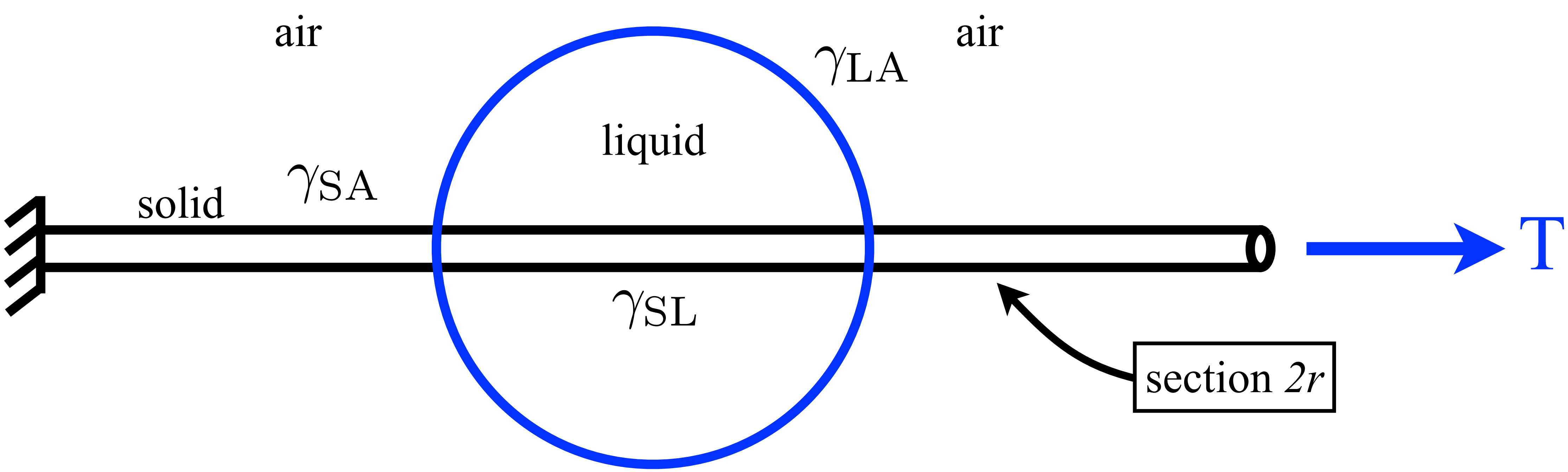
Liquid drops on elastic fiber



γ_{SA} energy per unit area of the interface *solid-air*

$$V(\kappa(s), L_{\mathrm{in}}) = \frac{1}{2} EI \int_0^{L_{\mathrm{in}}} \kappa^2(s, L_{\mathrm{in}}) ds + T\, L_{\mathrm{in}} + 2\pi r\, \gamma_{\mathrm{SL}}\, L_{\mathrm{in}} + 2\pi r\, \gamma_{\mathrm{SA}}\, (L - L_{\mathrm{in}}) + \pi D^2\, \gamma_{\mathrm{LA}}$$

Liquid drops on elastic fiber

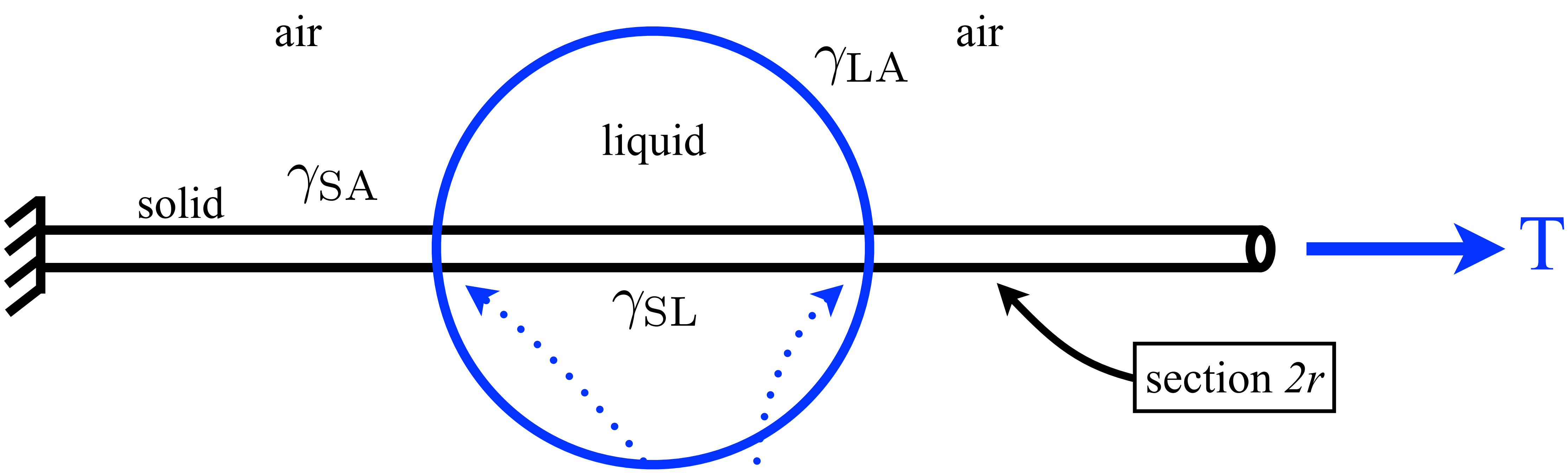


γ_{SA} energy per unit area of the interface *solid-air*

$$V(\kappa(s), L_{\mathrm{in}}) = \frac{1}{2}\, EI \int_0^{L_{\mathrm{in}}} \kappa^2(s, L_{\mathrm{in}}) ds + T\, L_{\mathrm{in}}$$

$$- 2\pi r\, (\gamma_{\mathrm{SA}} - \gamma_{\mathrm{SL}})\, L_{\mathrm{in}} + \mathrm{constants}$$

Liquid drops on elastic fiber



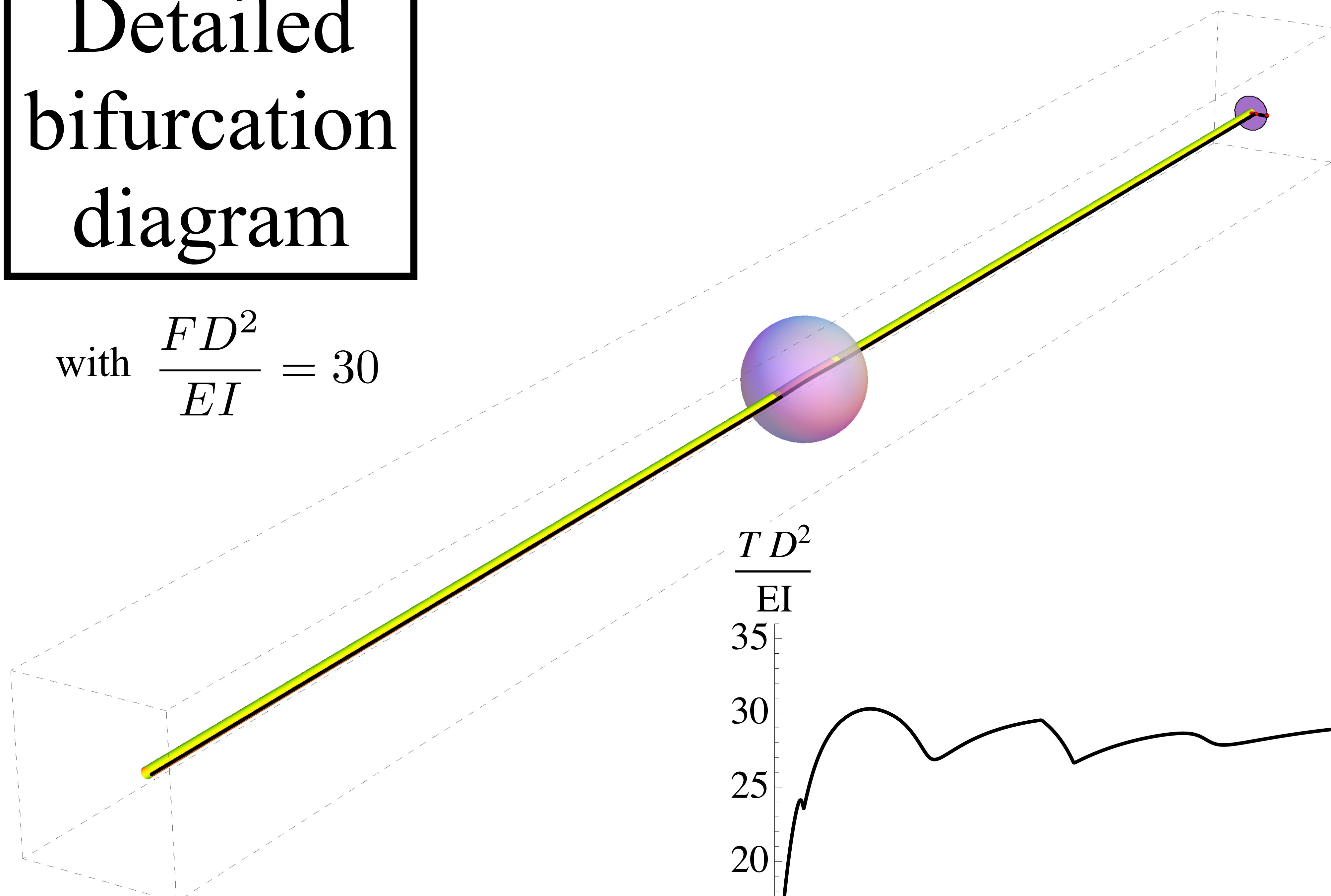
$\gamma_{\rm SA}$ energy per unit area of the interface *solid-air*

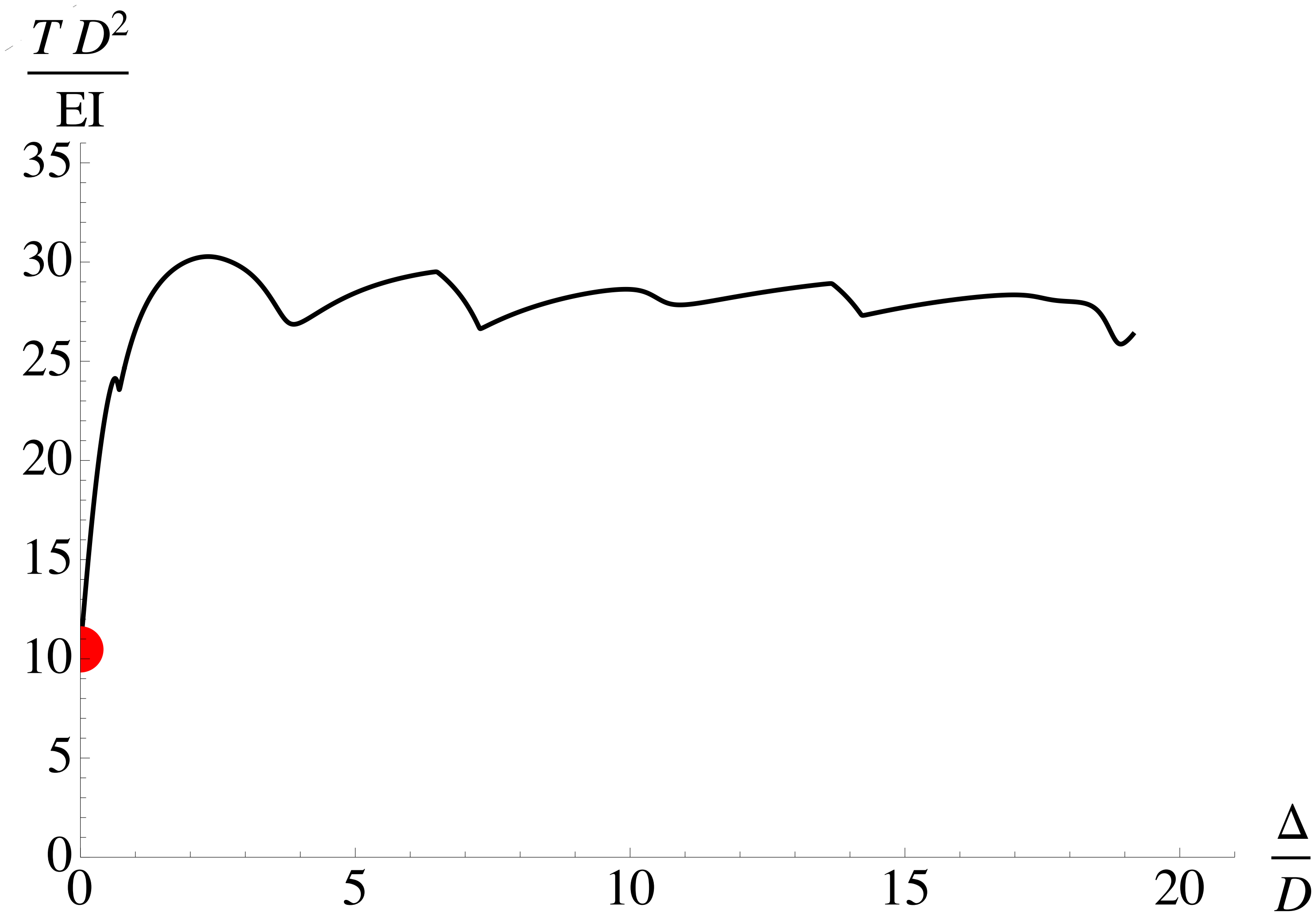
$$V(\kappa(s), L_{\rm in}) = \frac{1}{2}\, EI \int_0^{L_{\rm in}} \kappa^2(s, L_{\rm in}) ds + T\, L_{\rm in} - F\, L_{\rm in}$$

with $F = 2\pi r\, (\gamma_{\rm SA} - \gamma_{\rm SL})$ meniscus forces

Detailed bifurcation diagram

with $\frac{FD^2}{EI} = 30$





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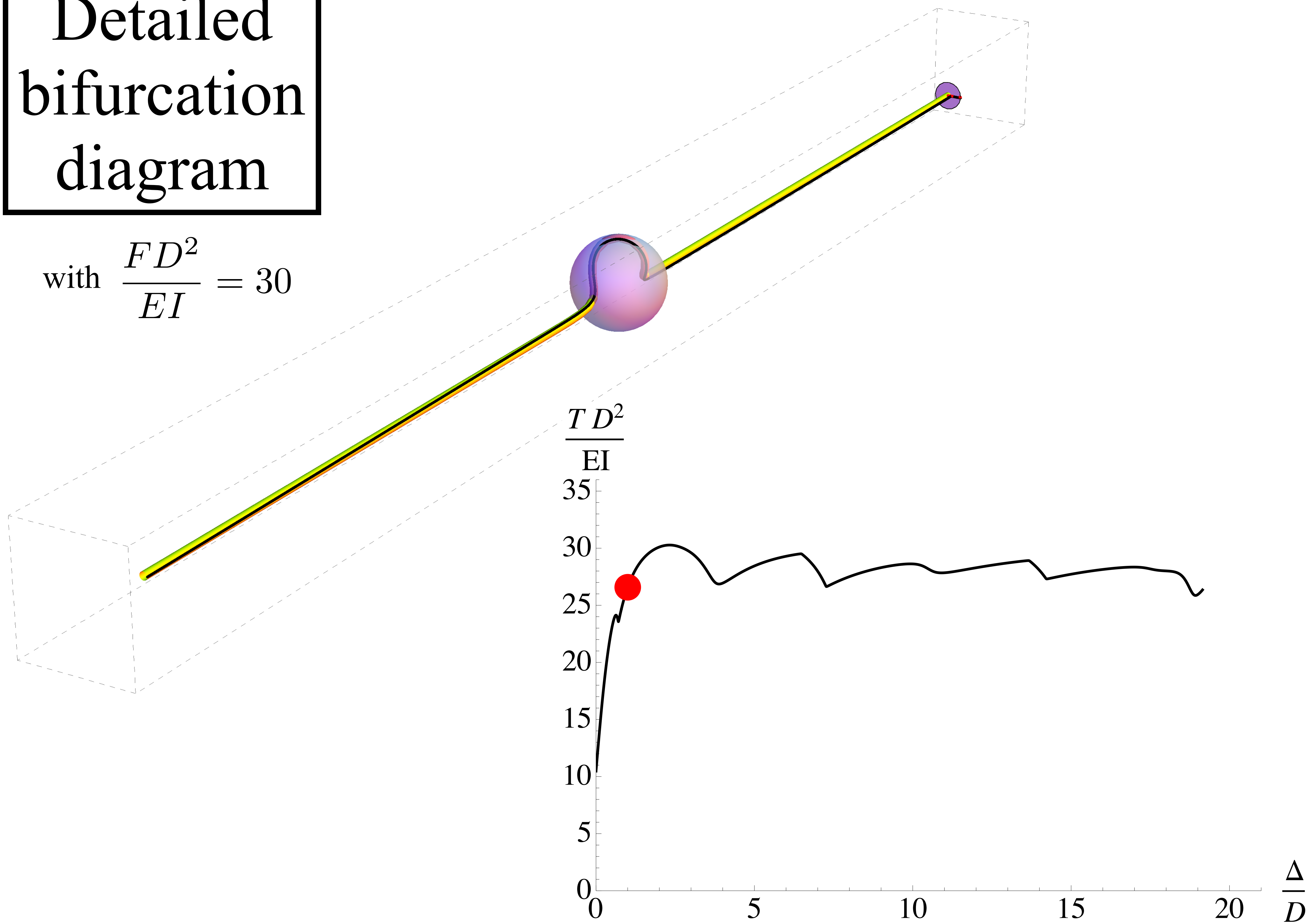

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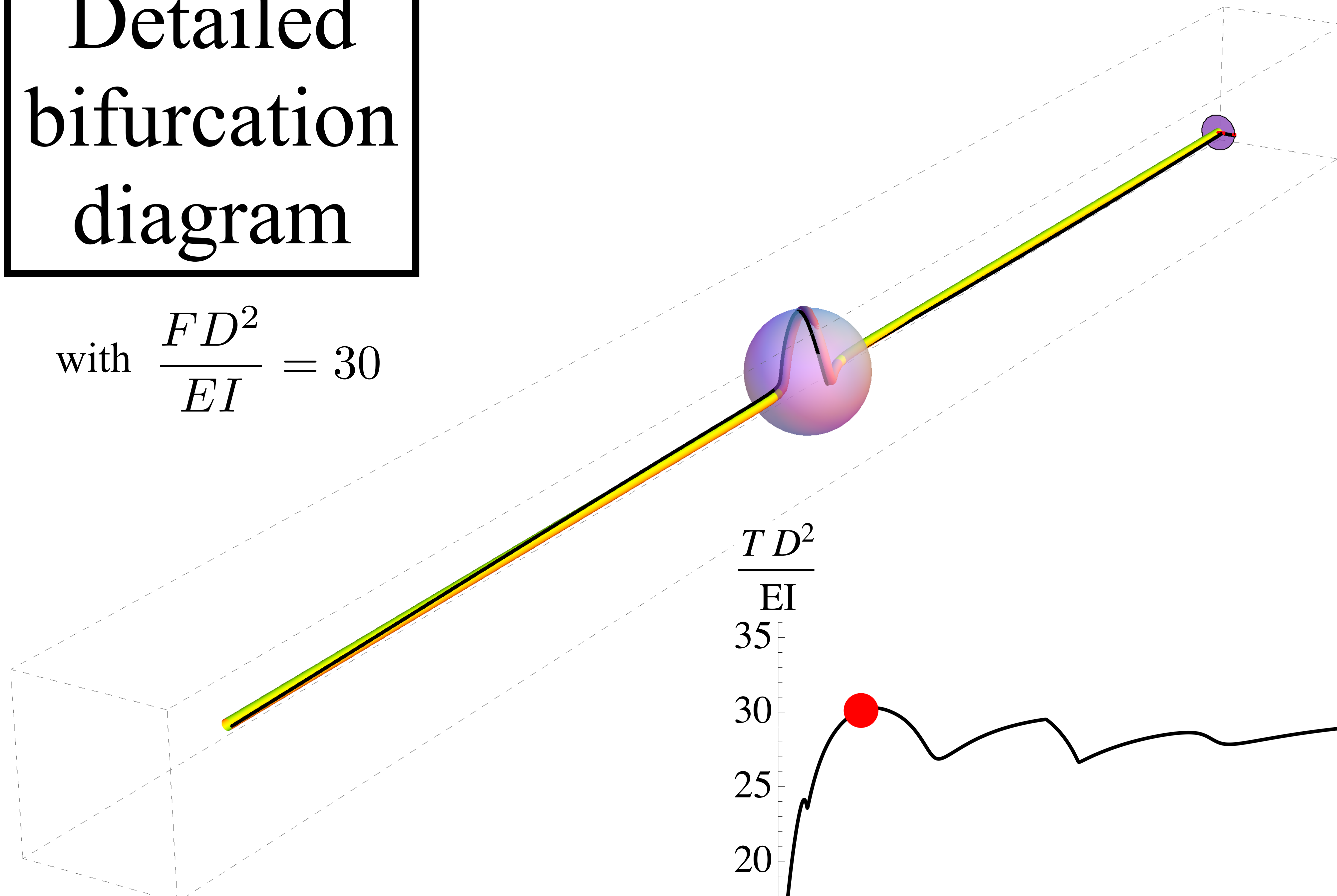

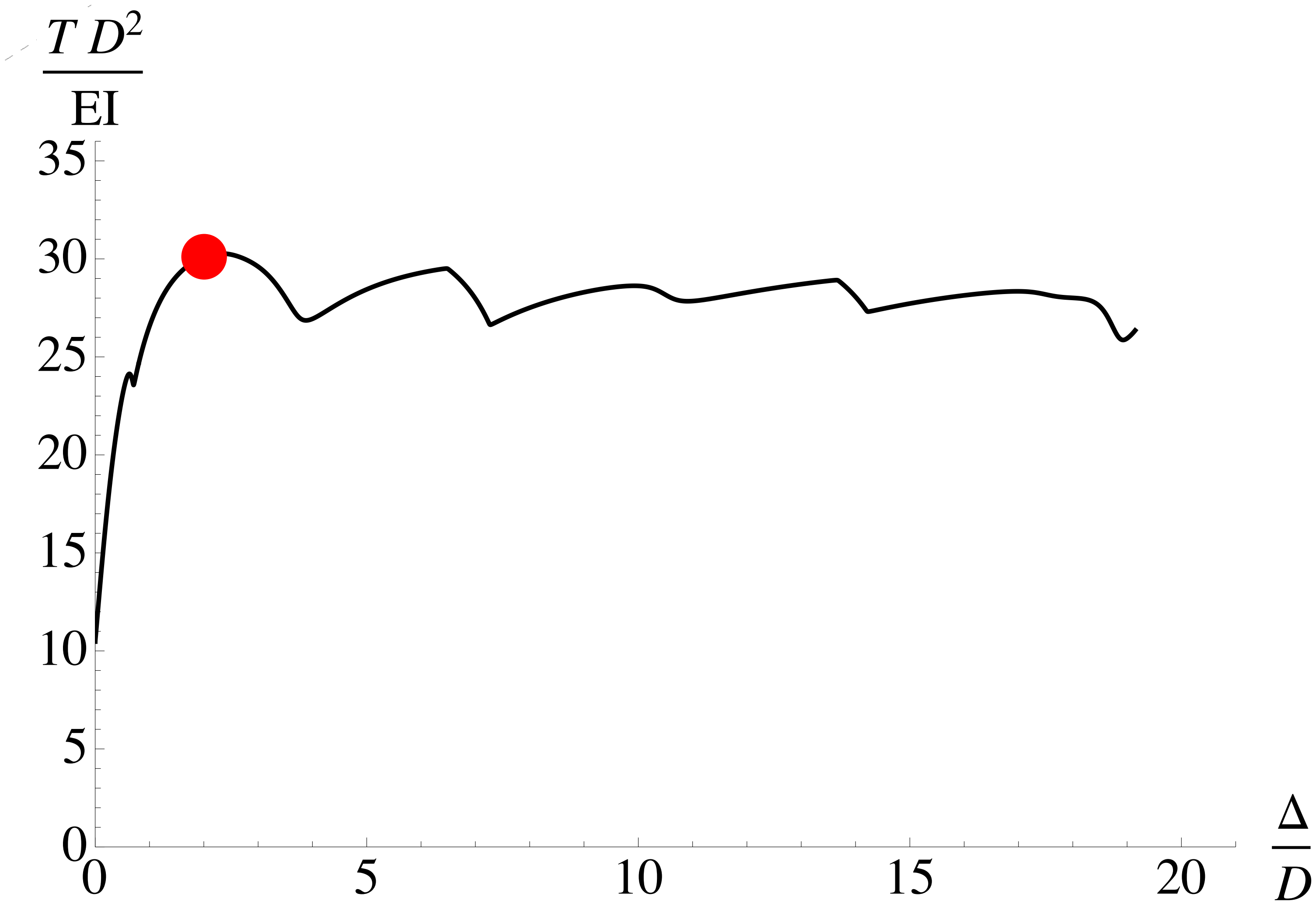

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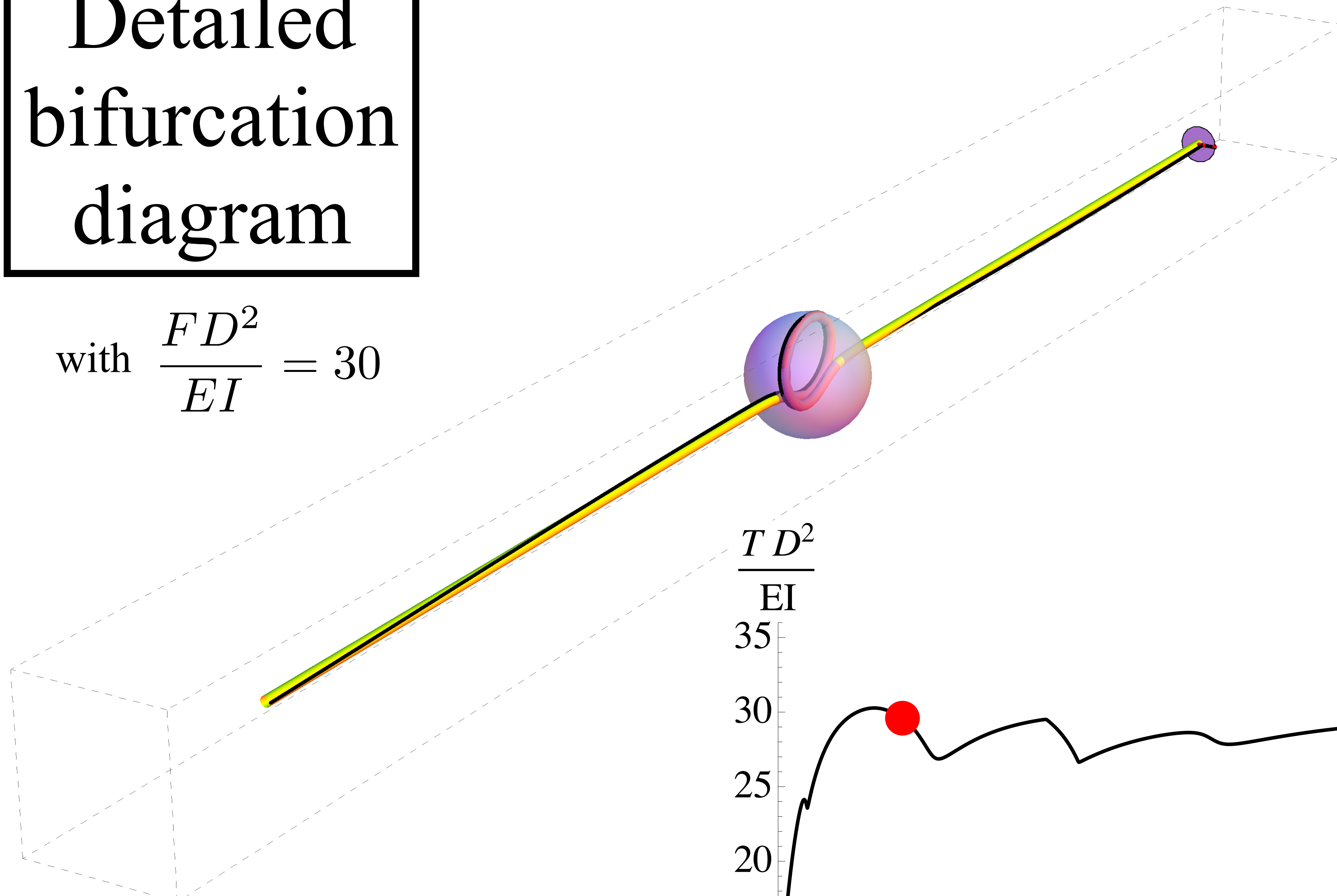

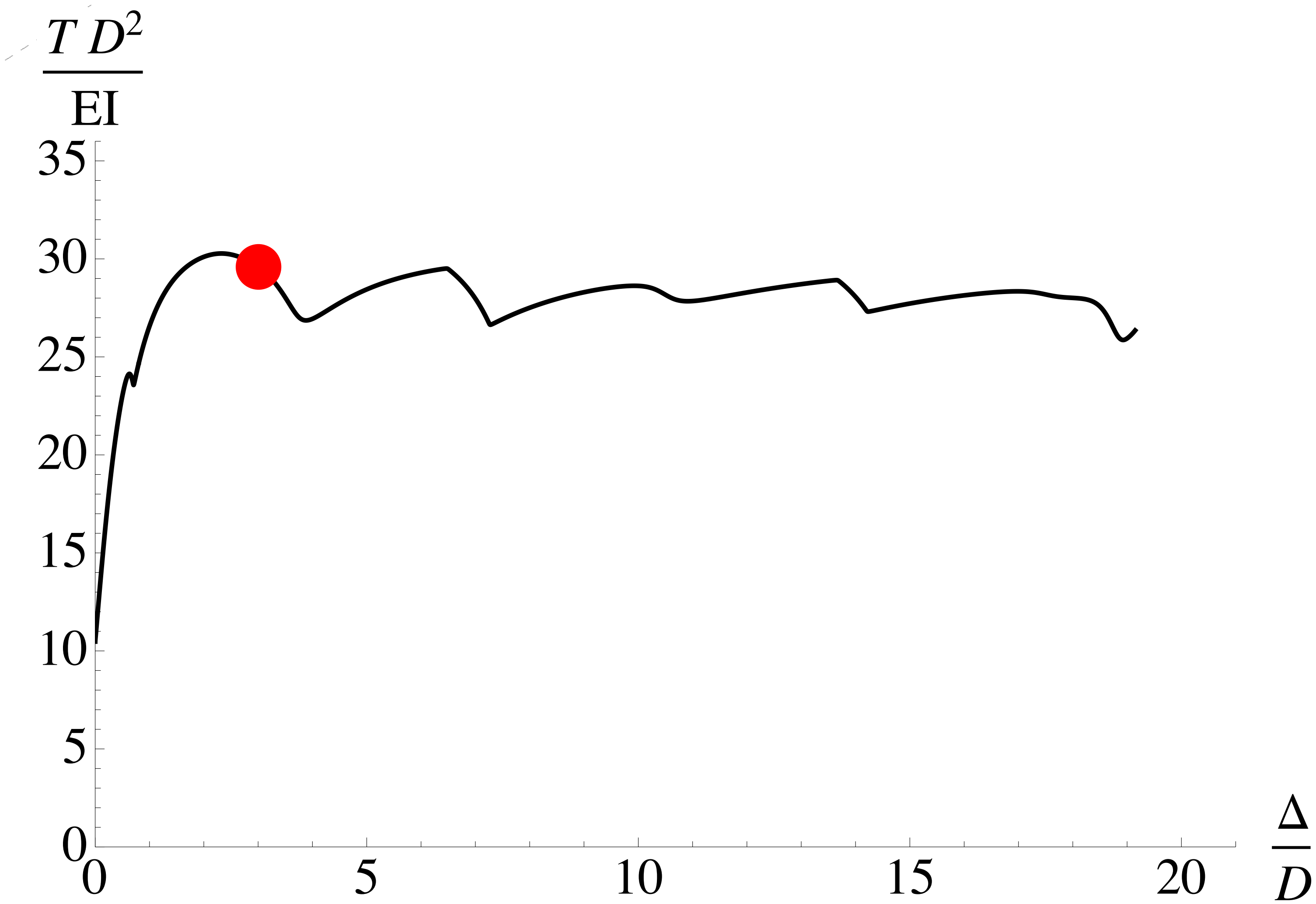

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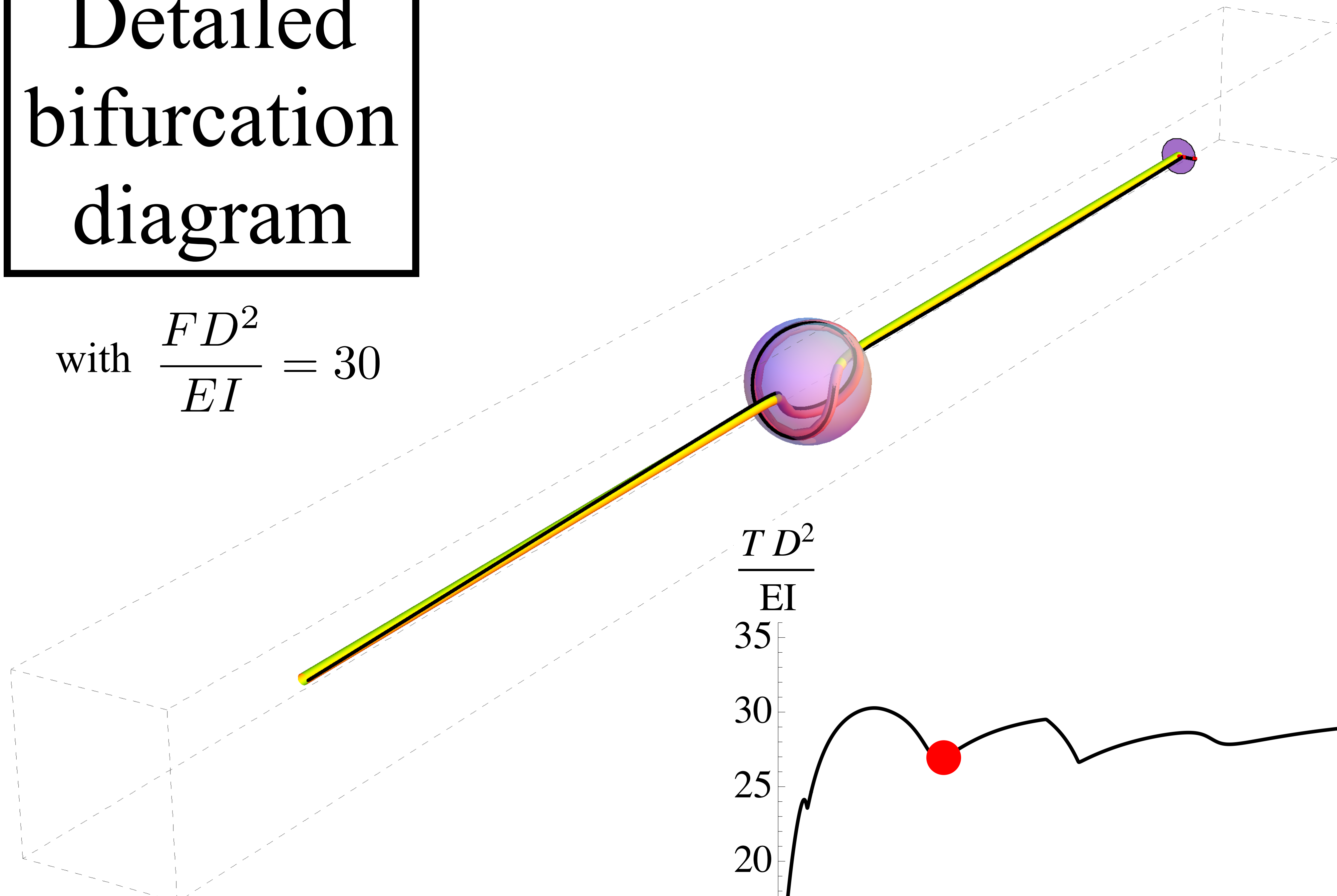

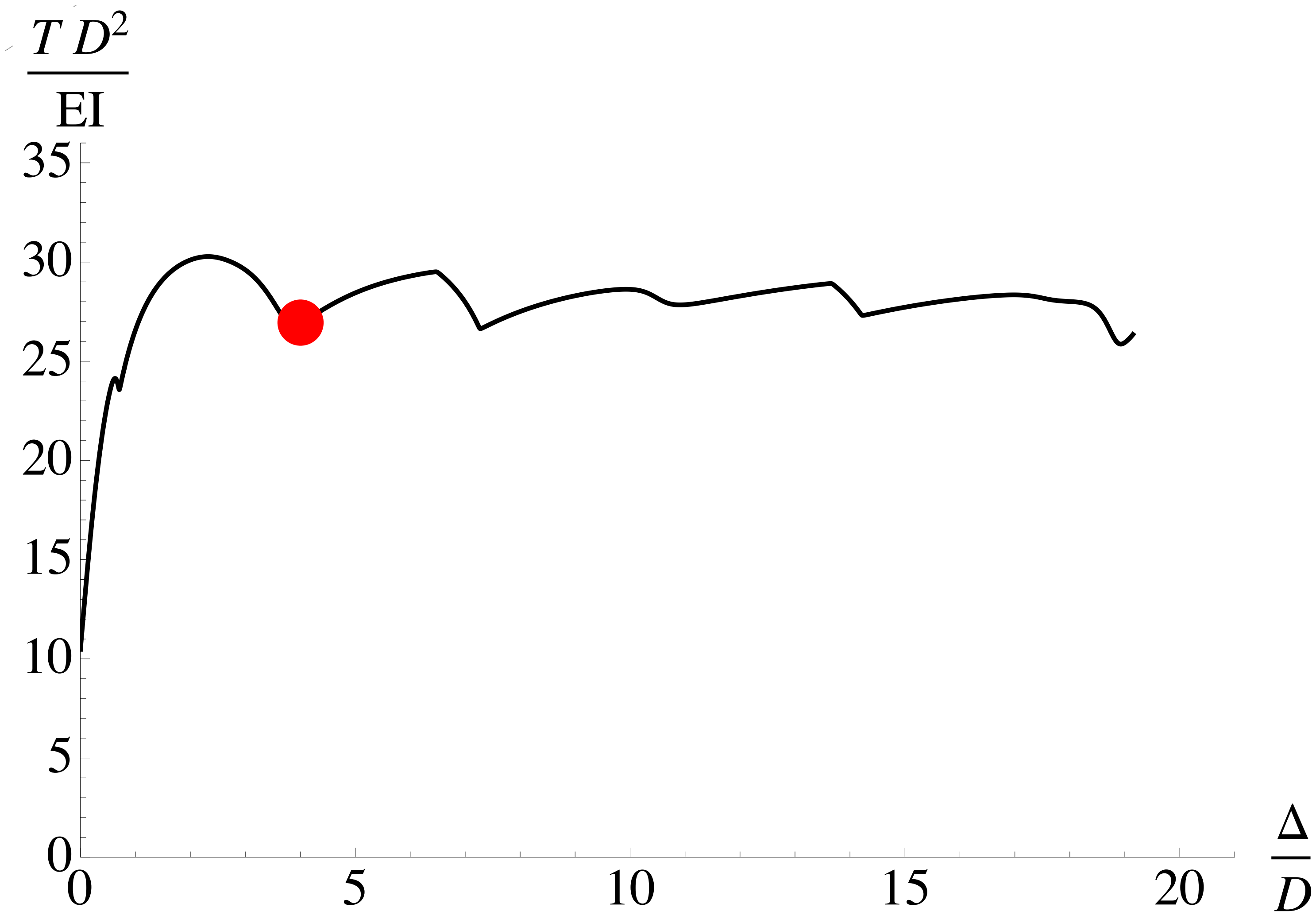

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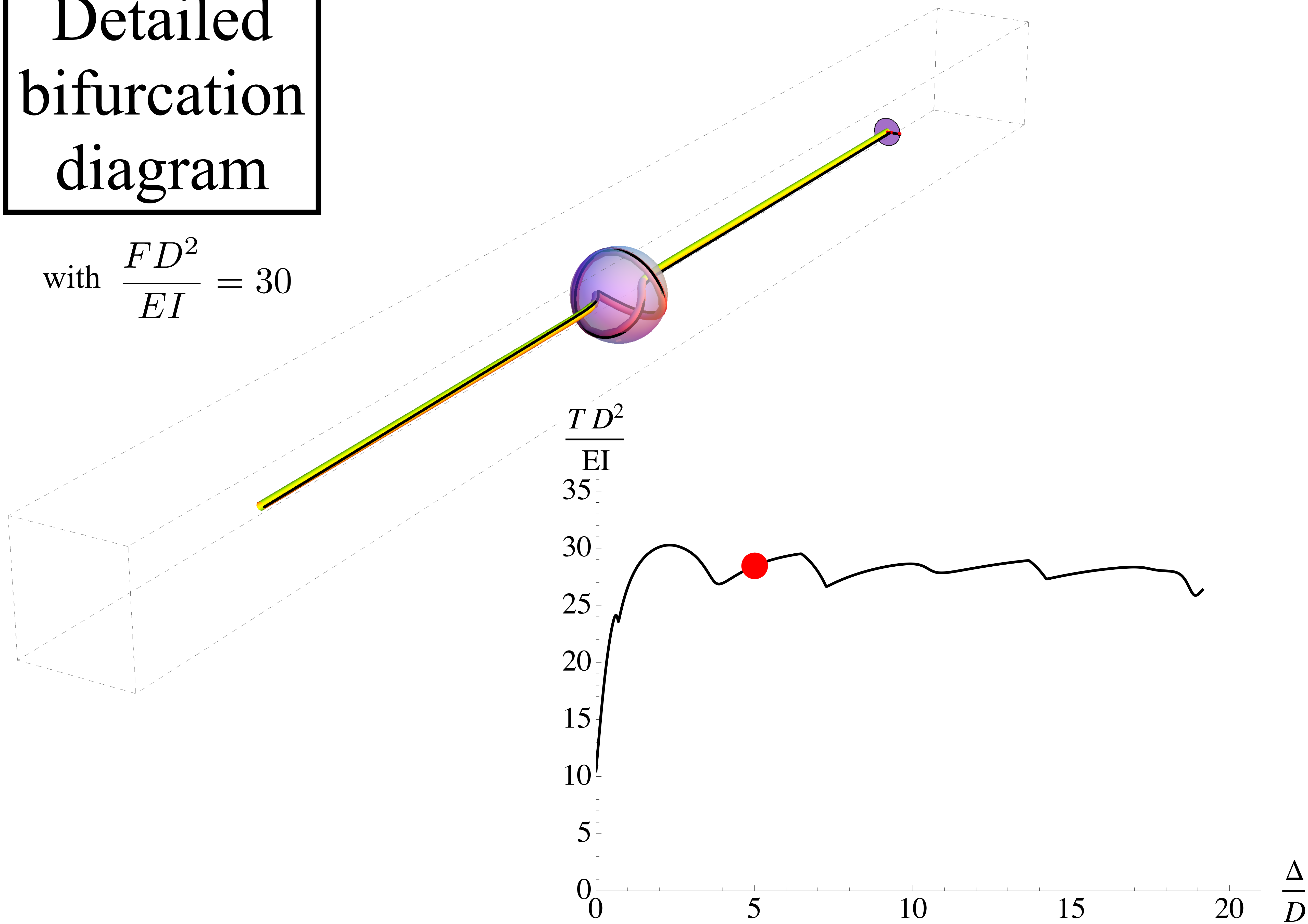

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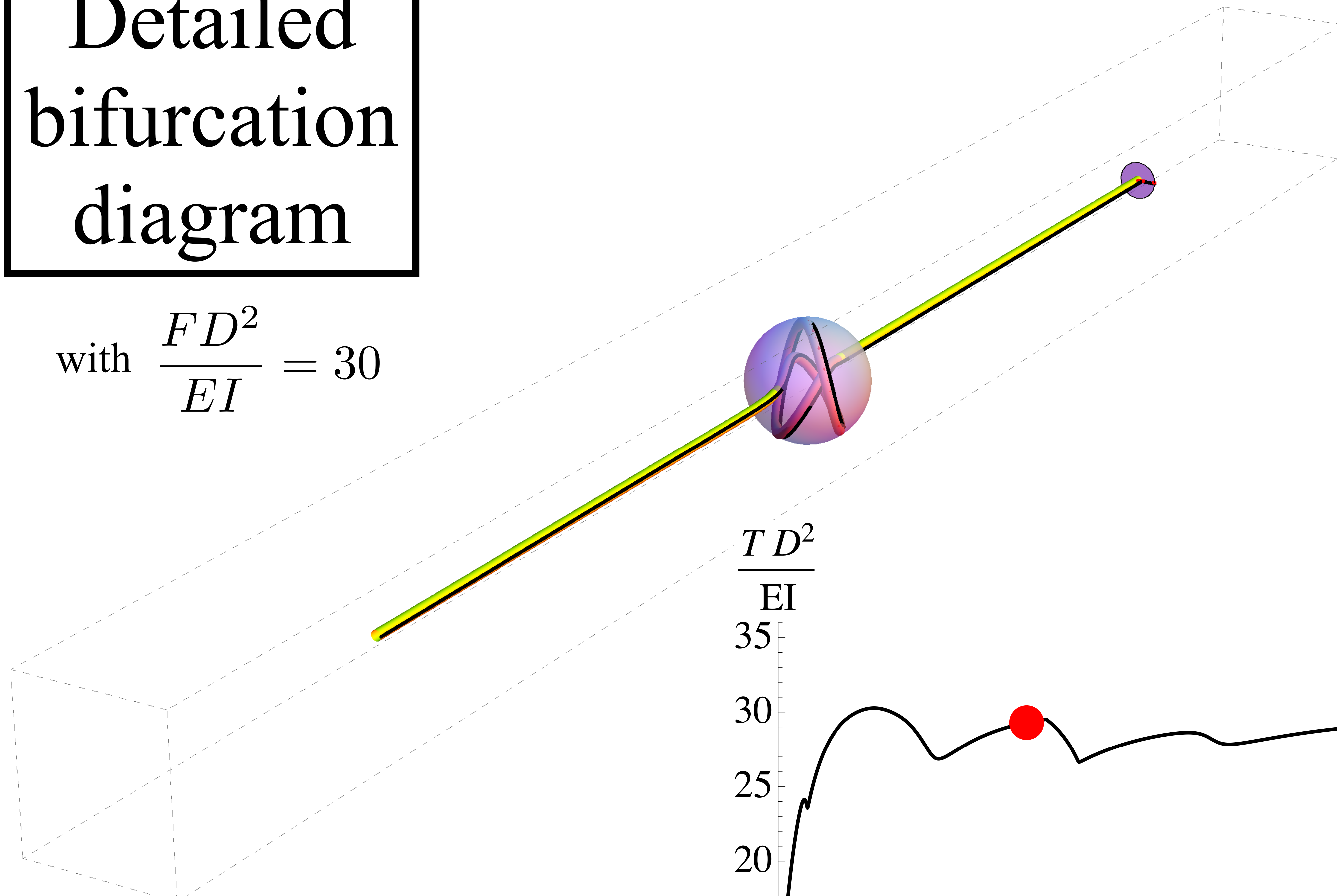

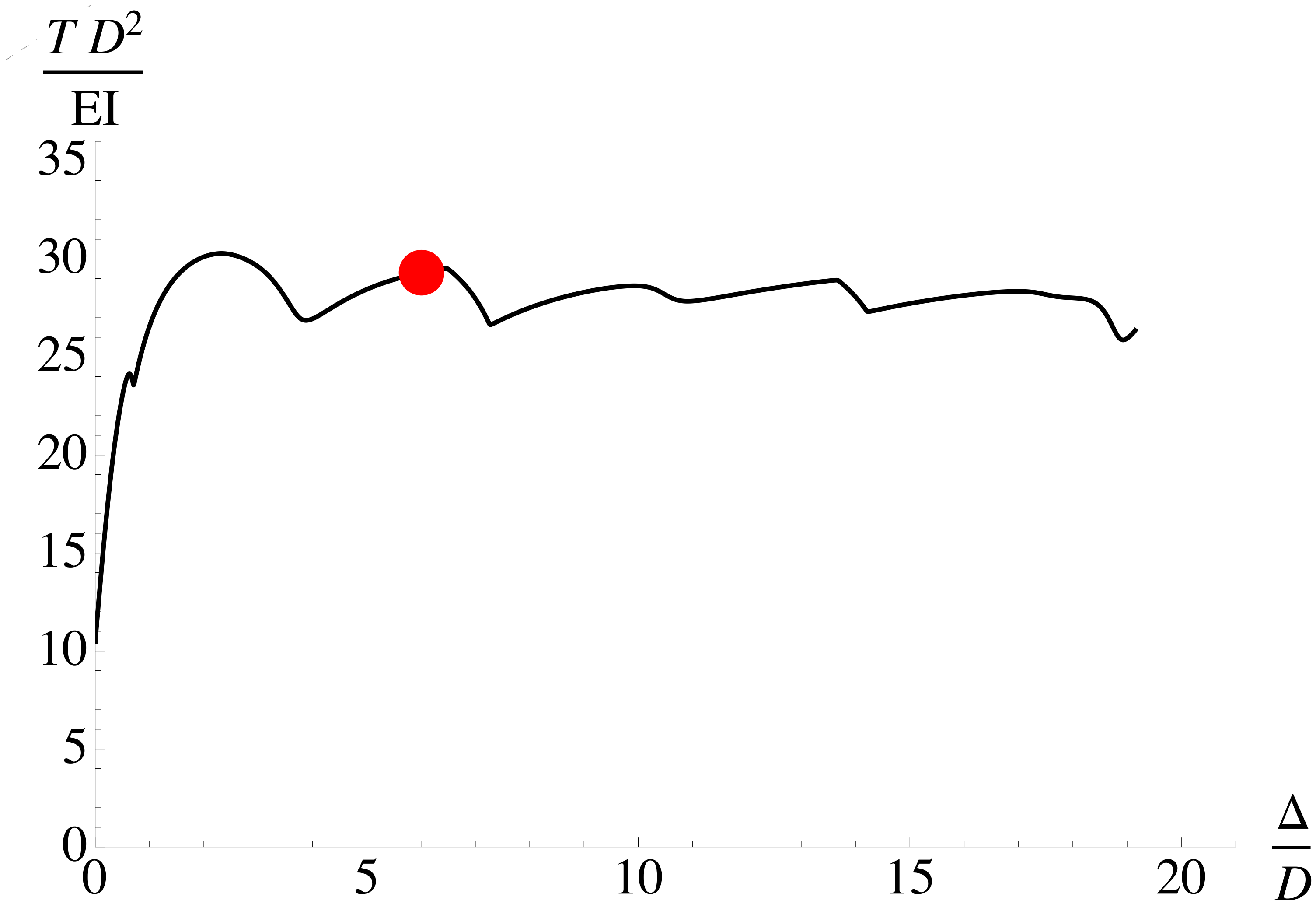

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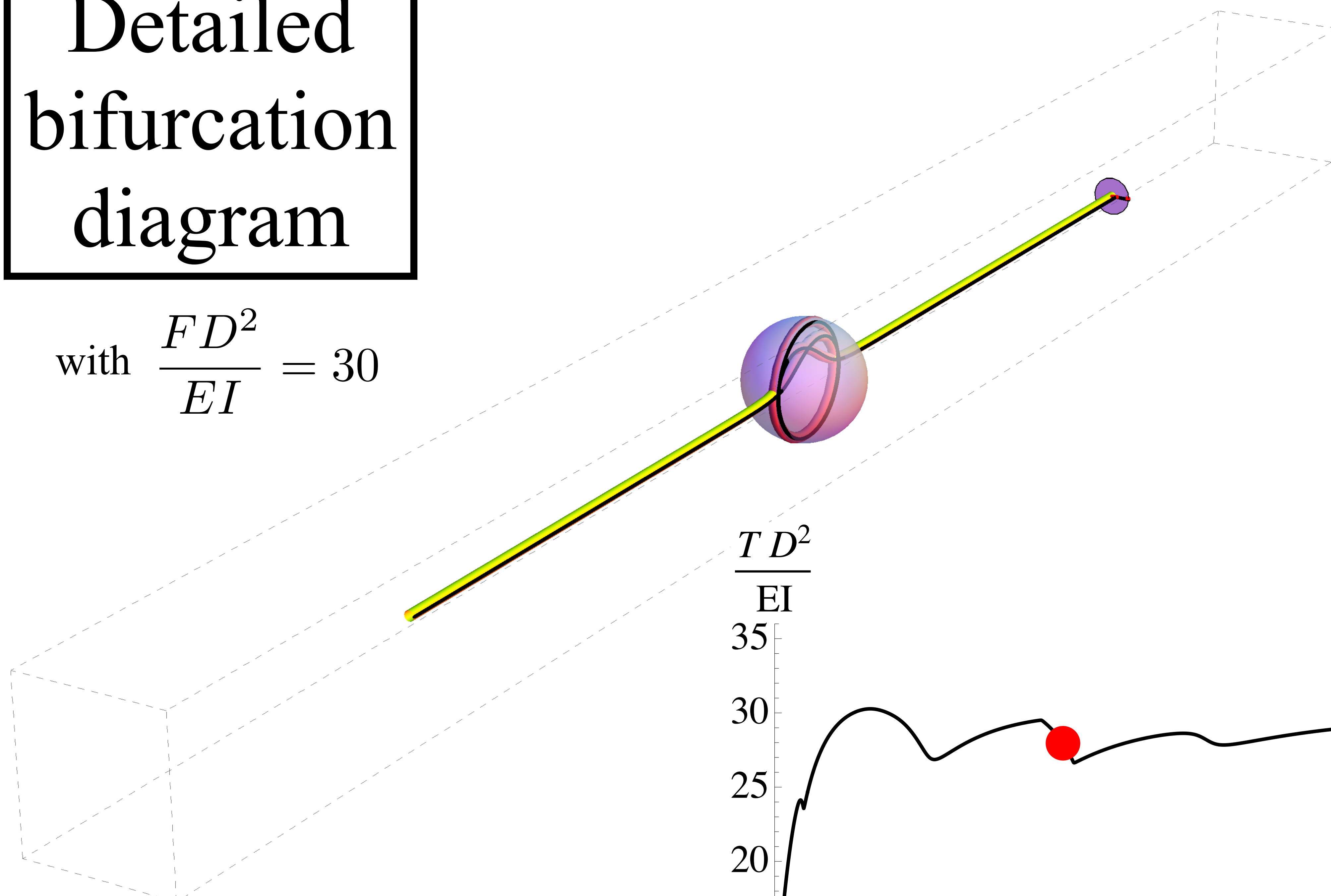

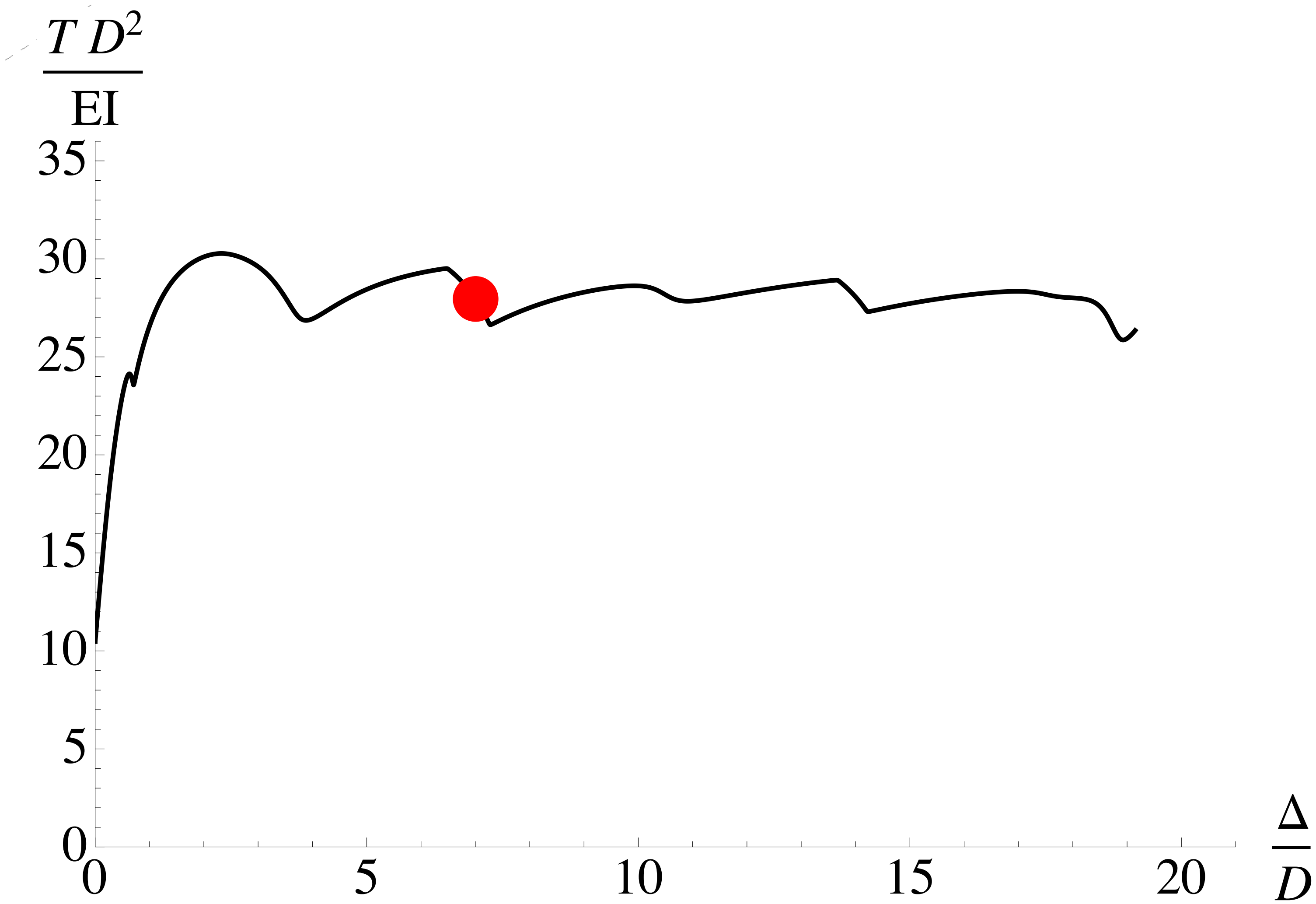

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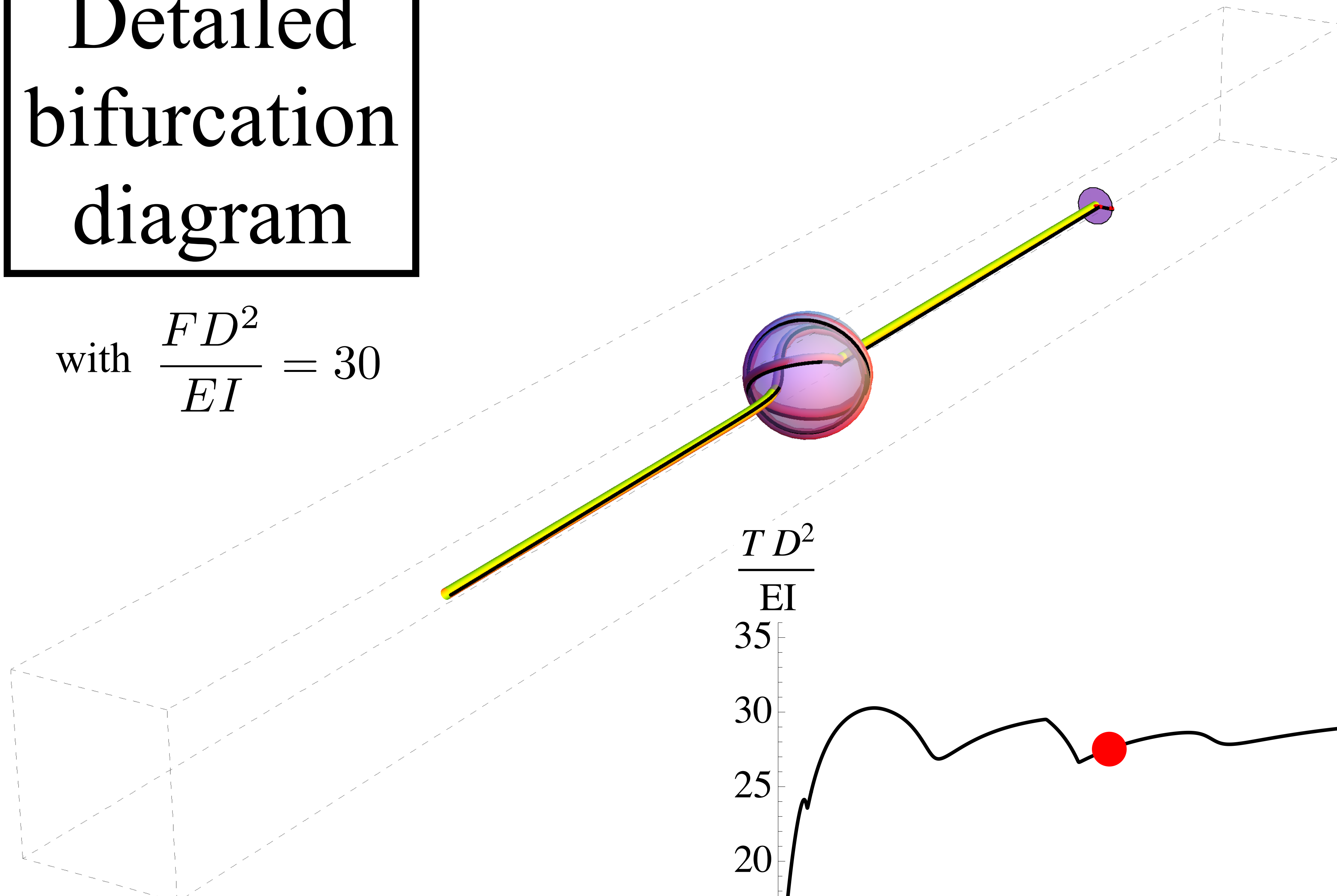

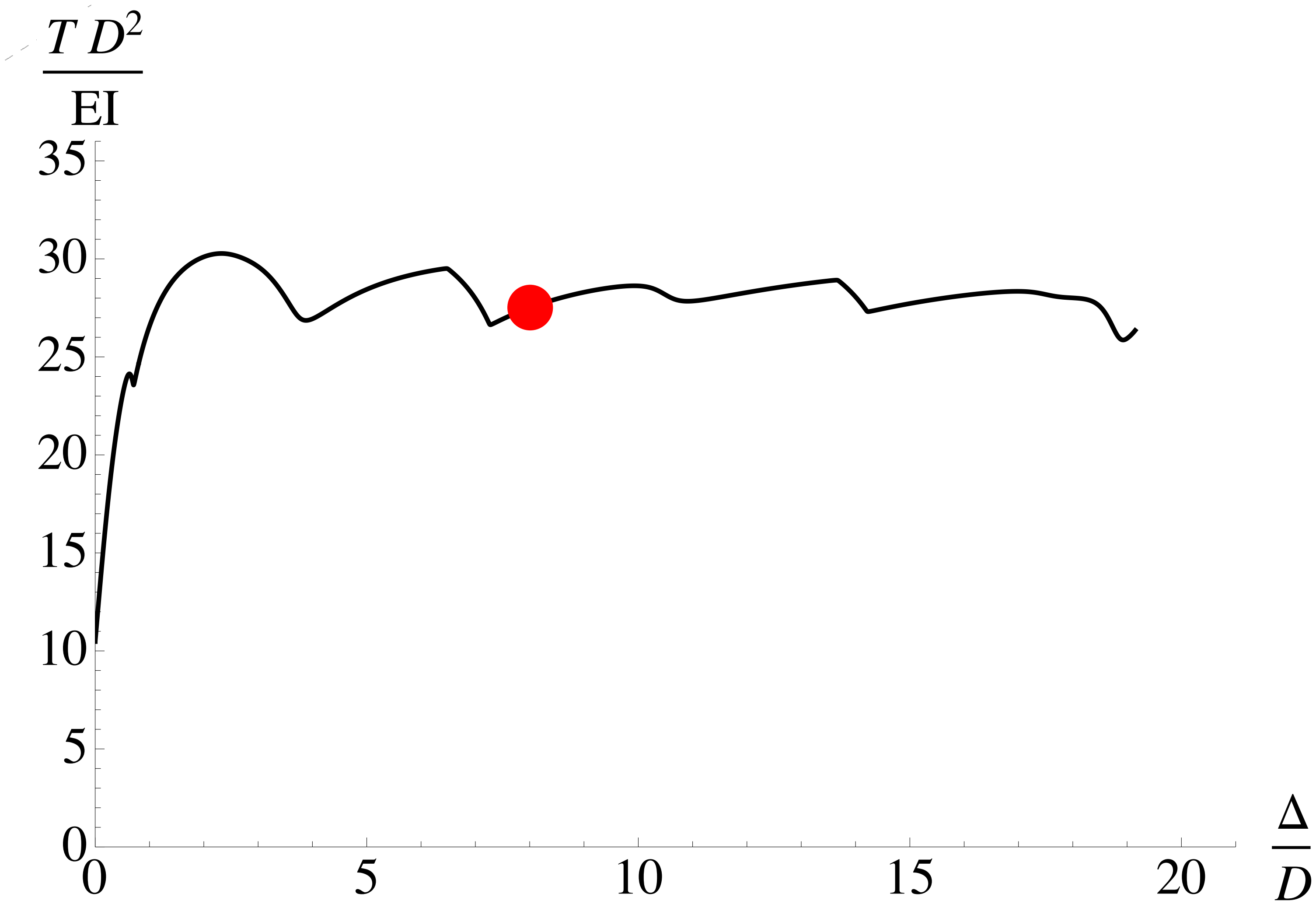

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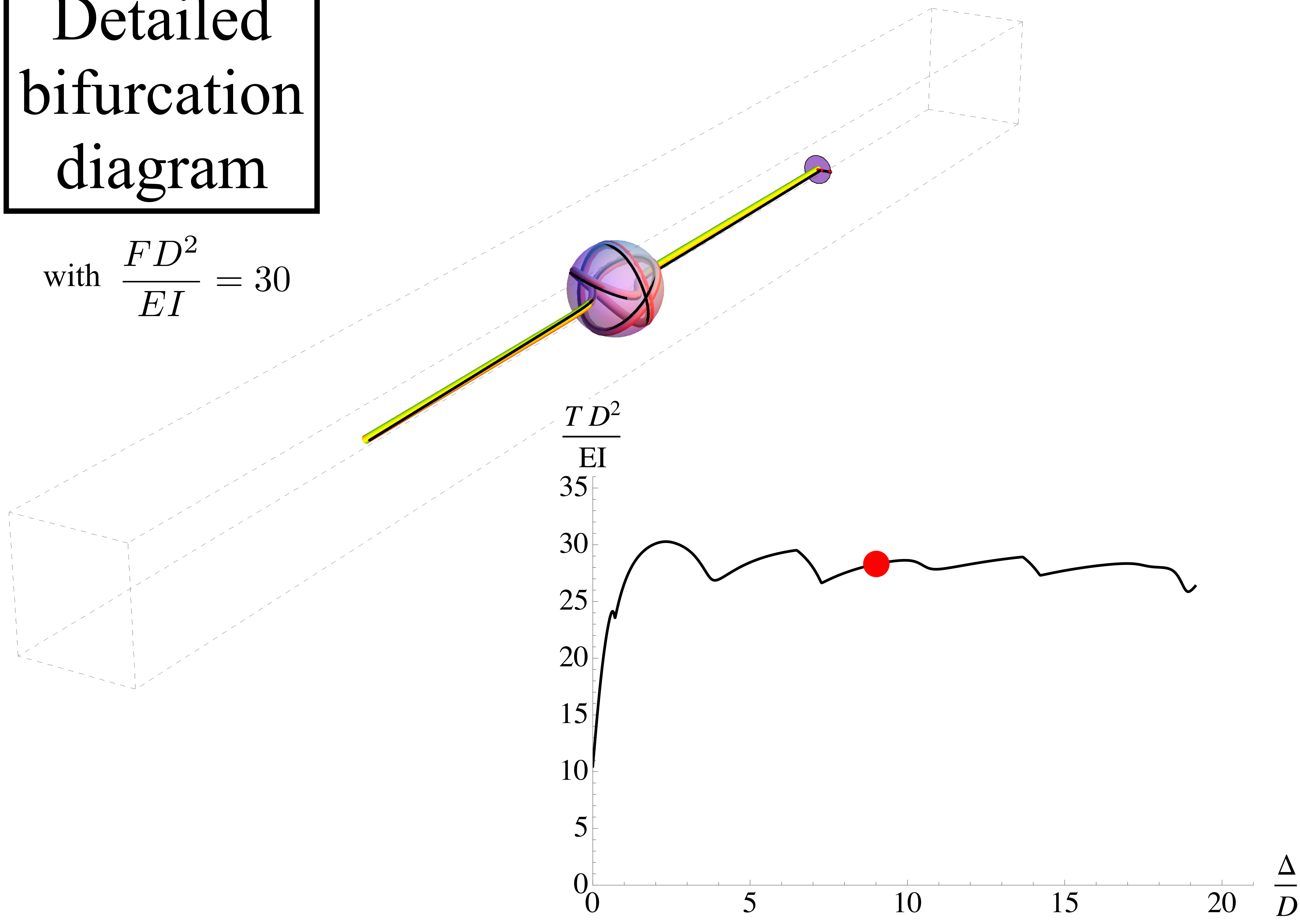

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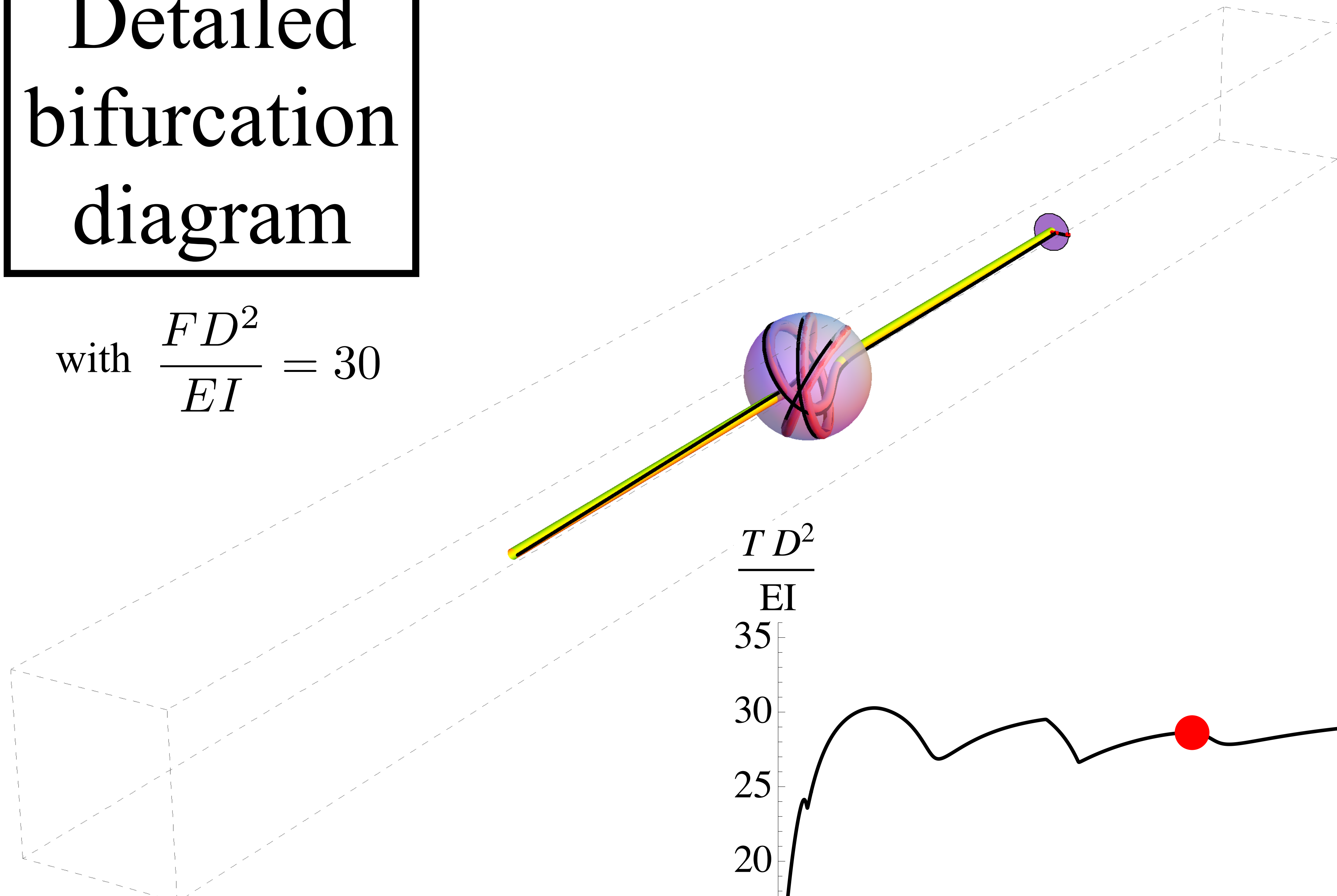

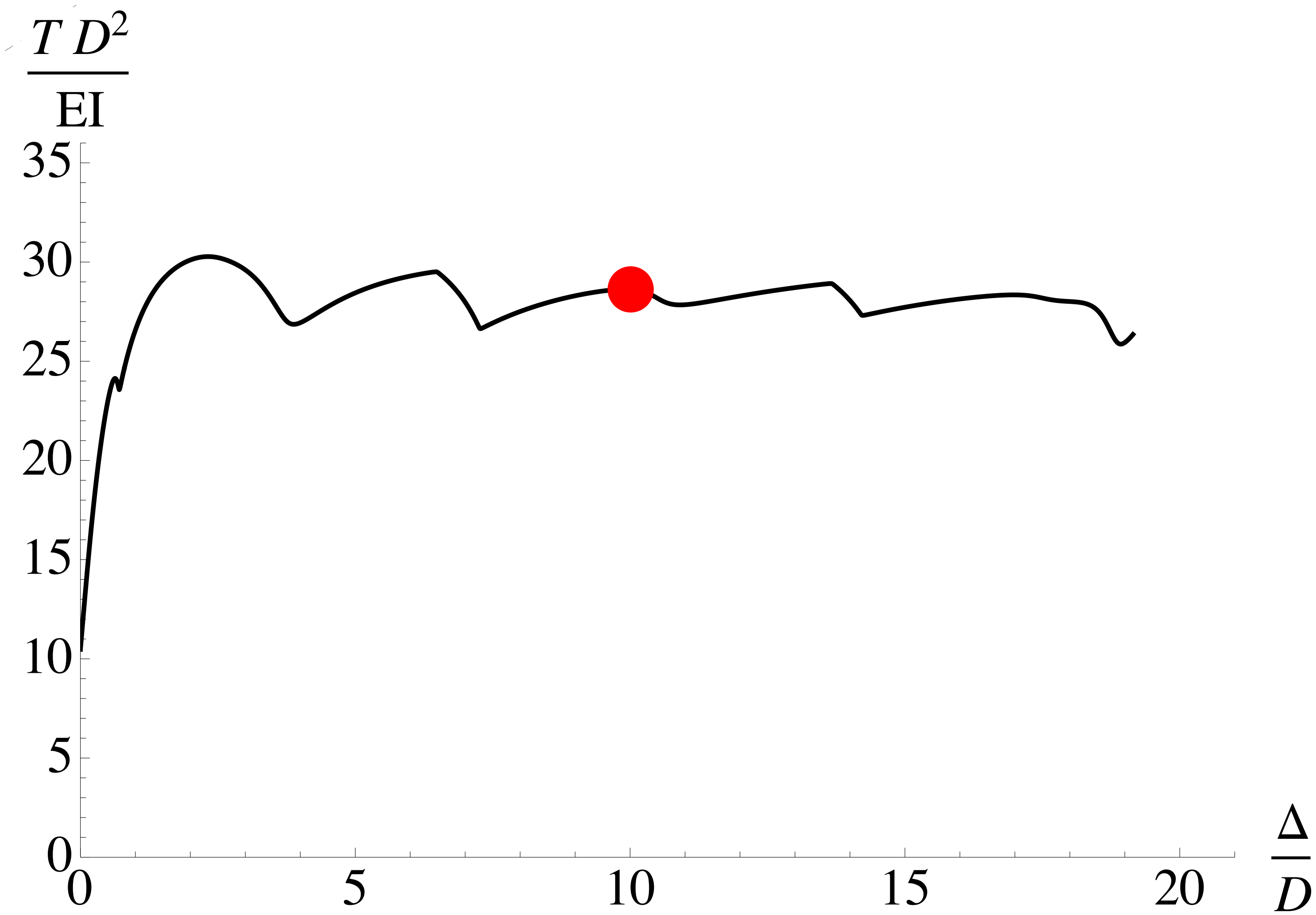

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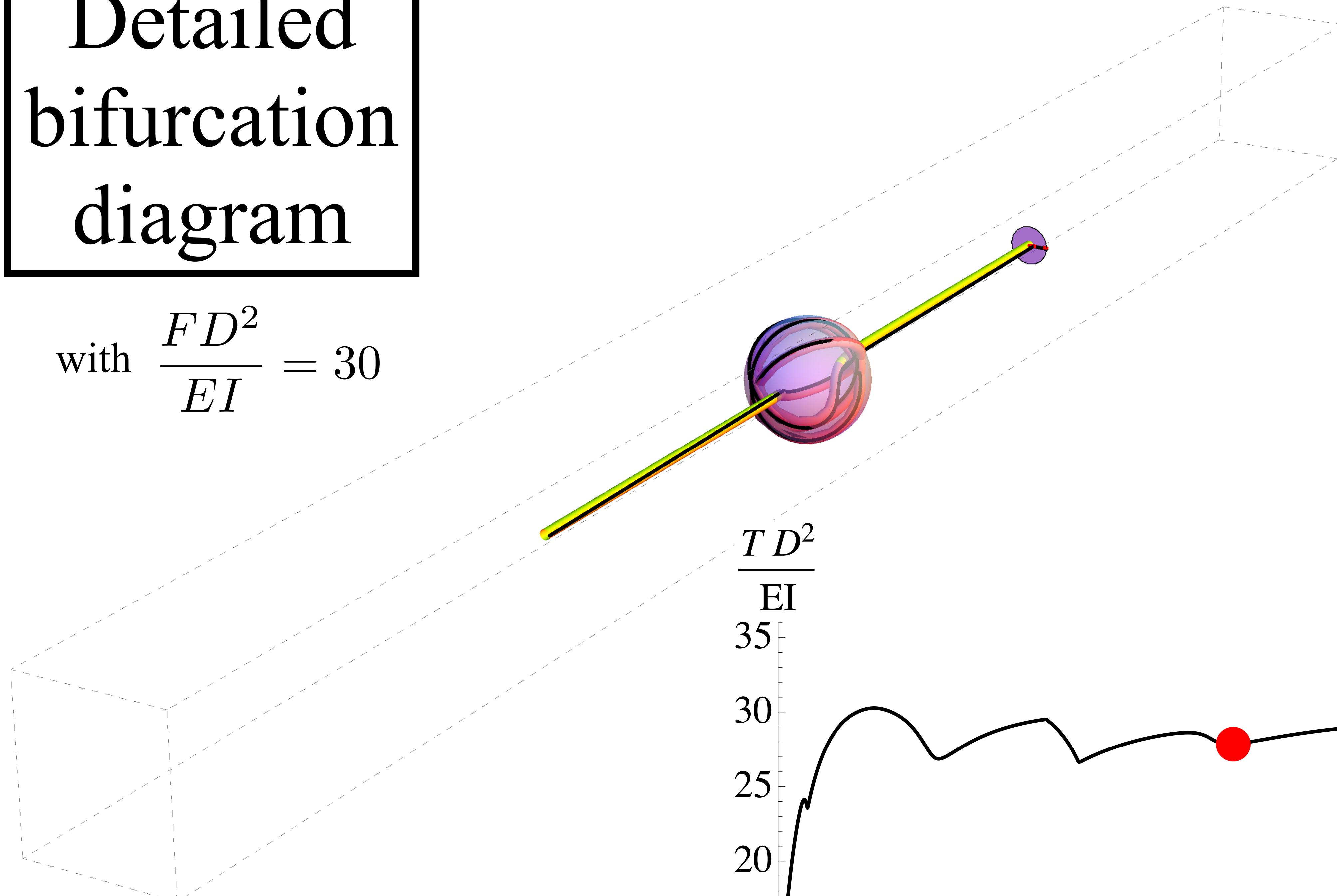

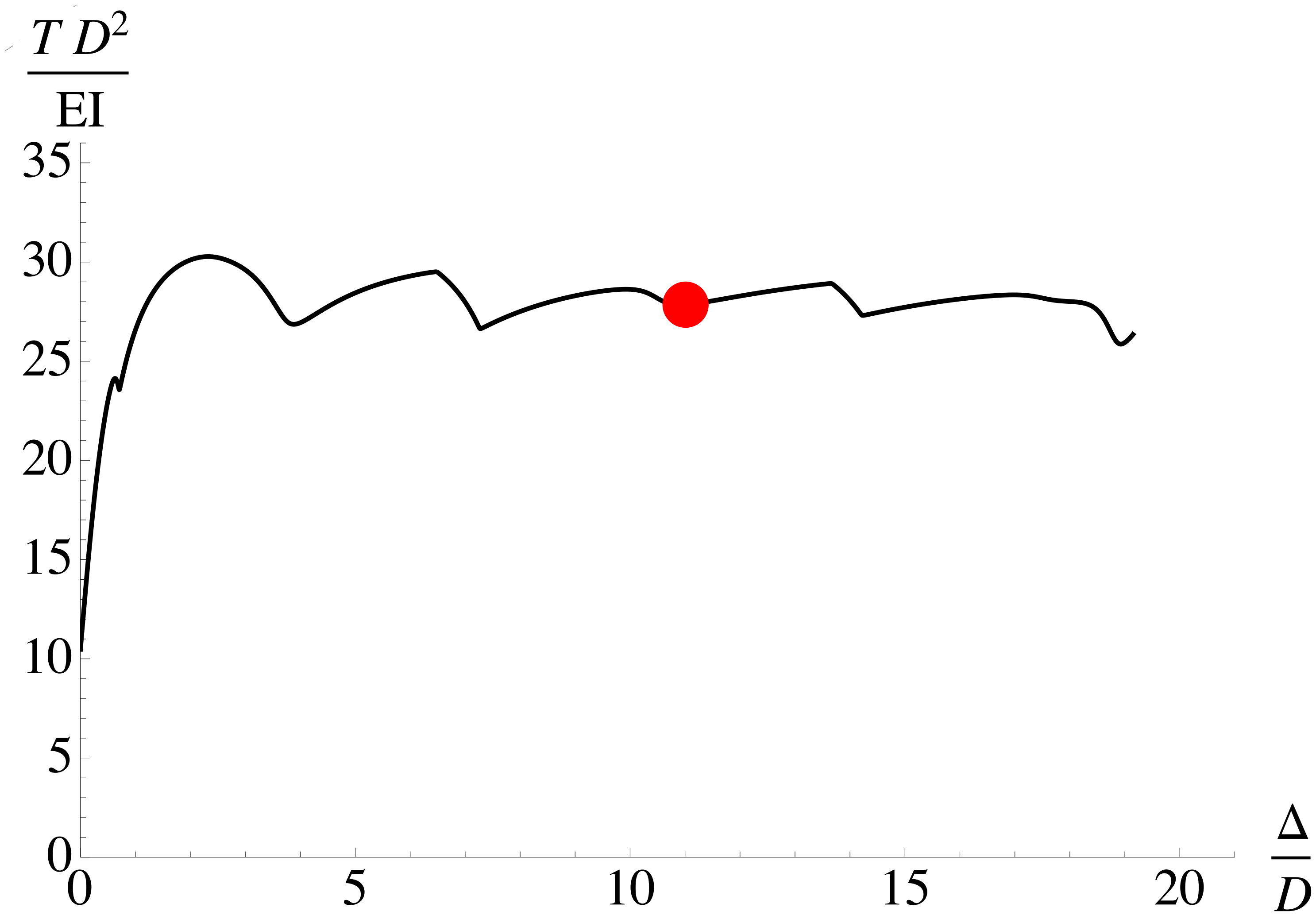

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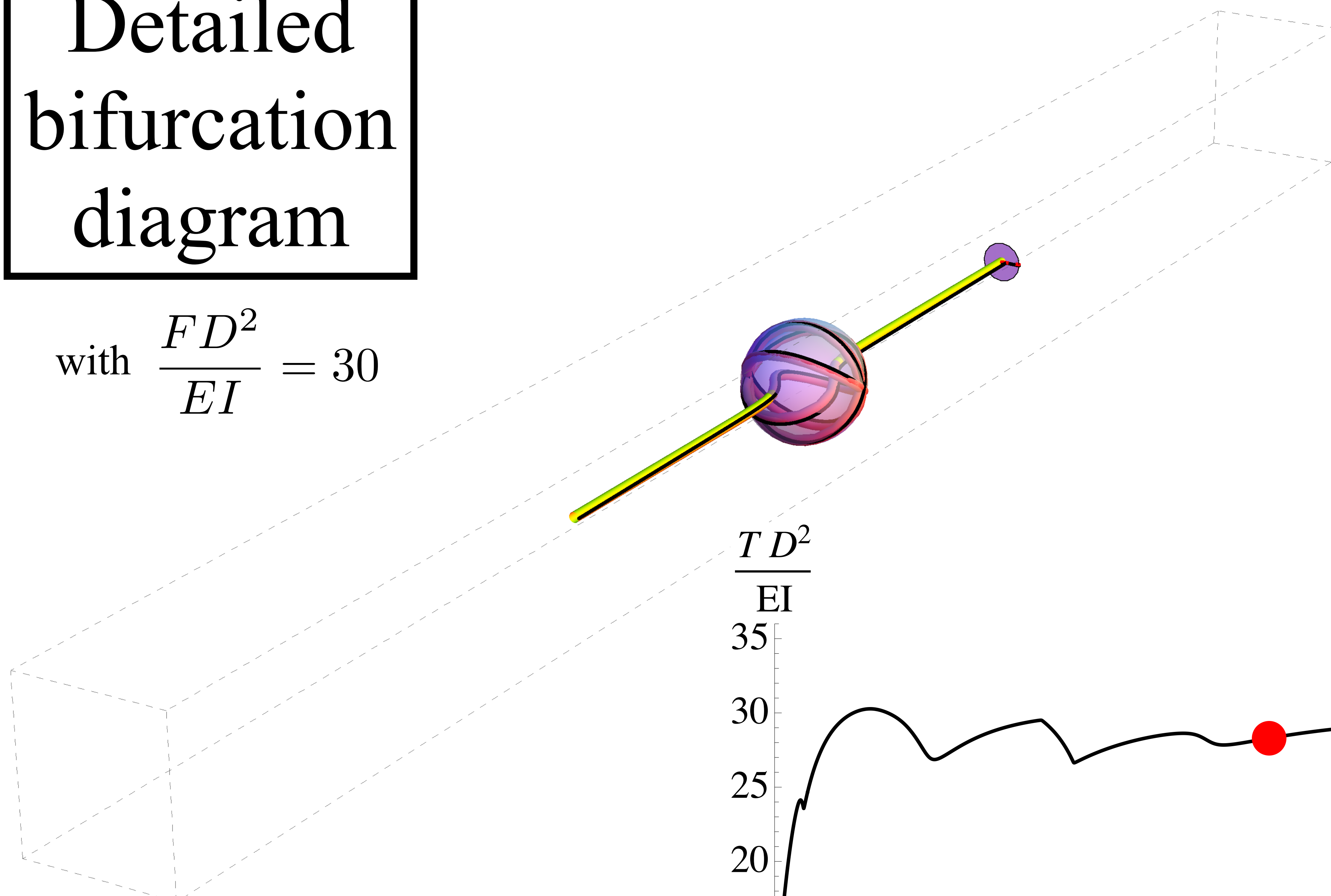

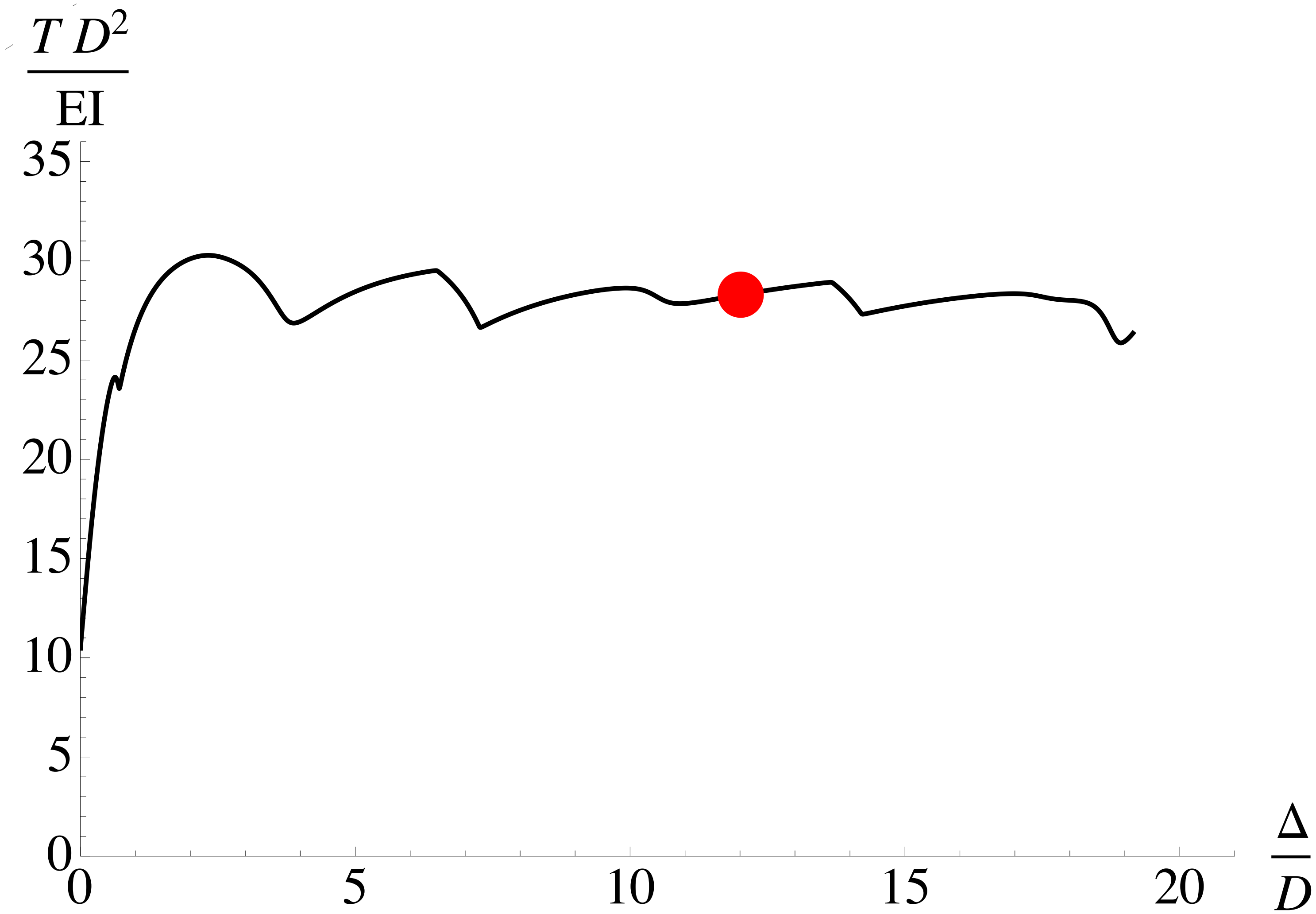

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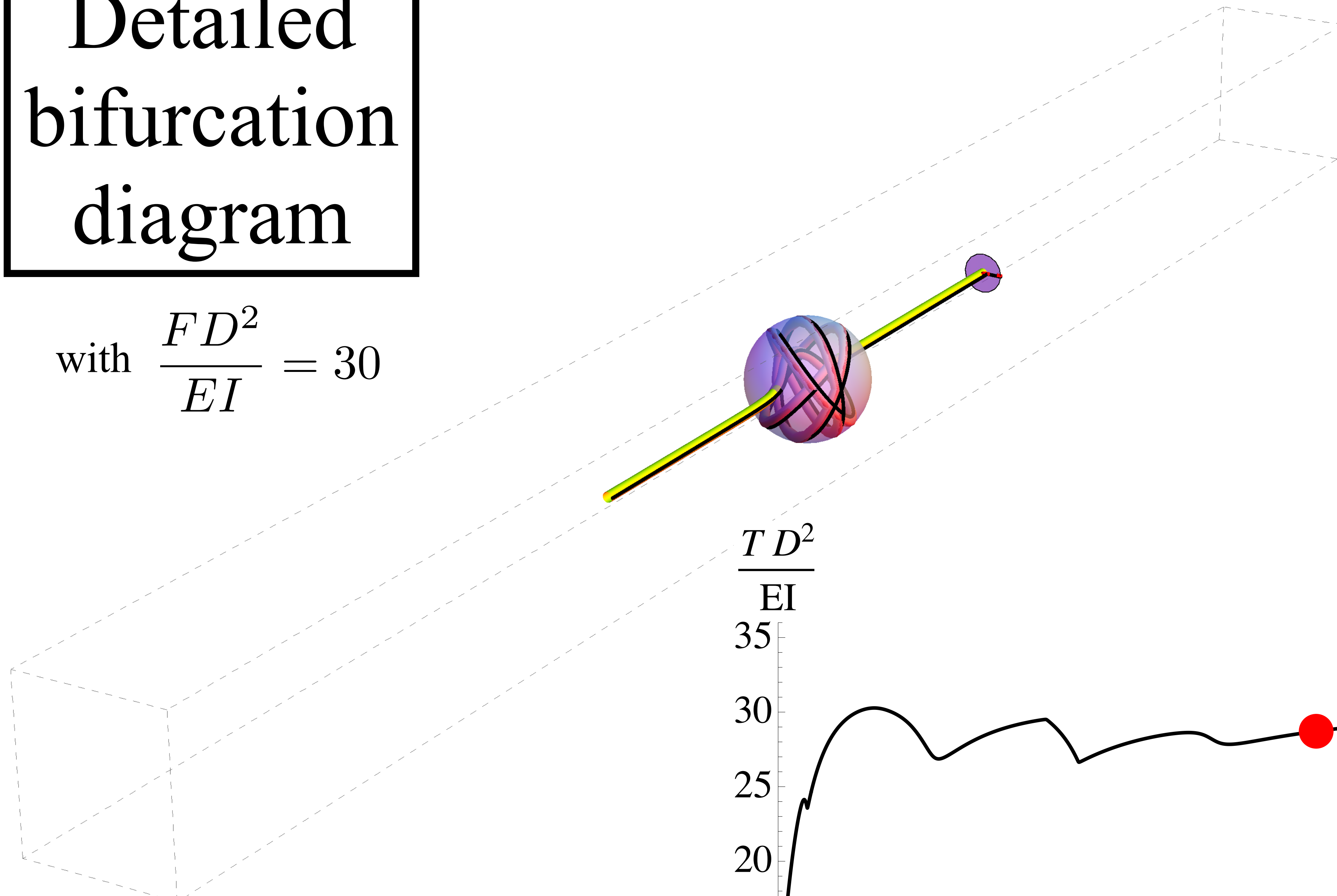

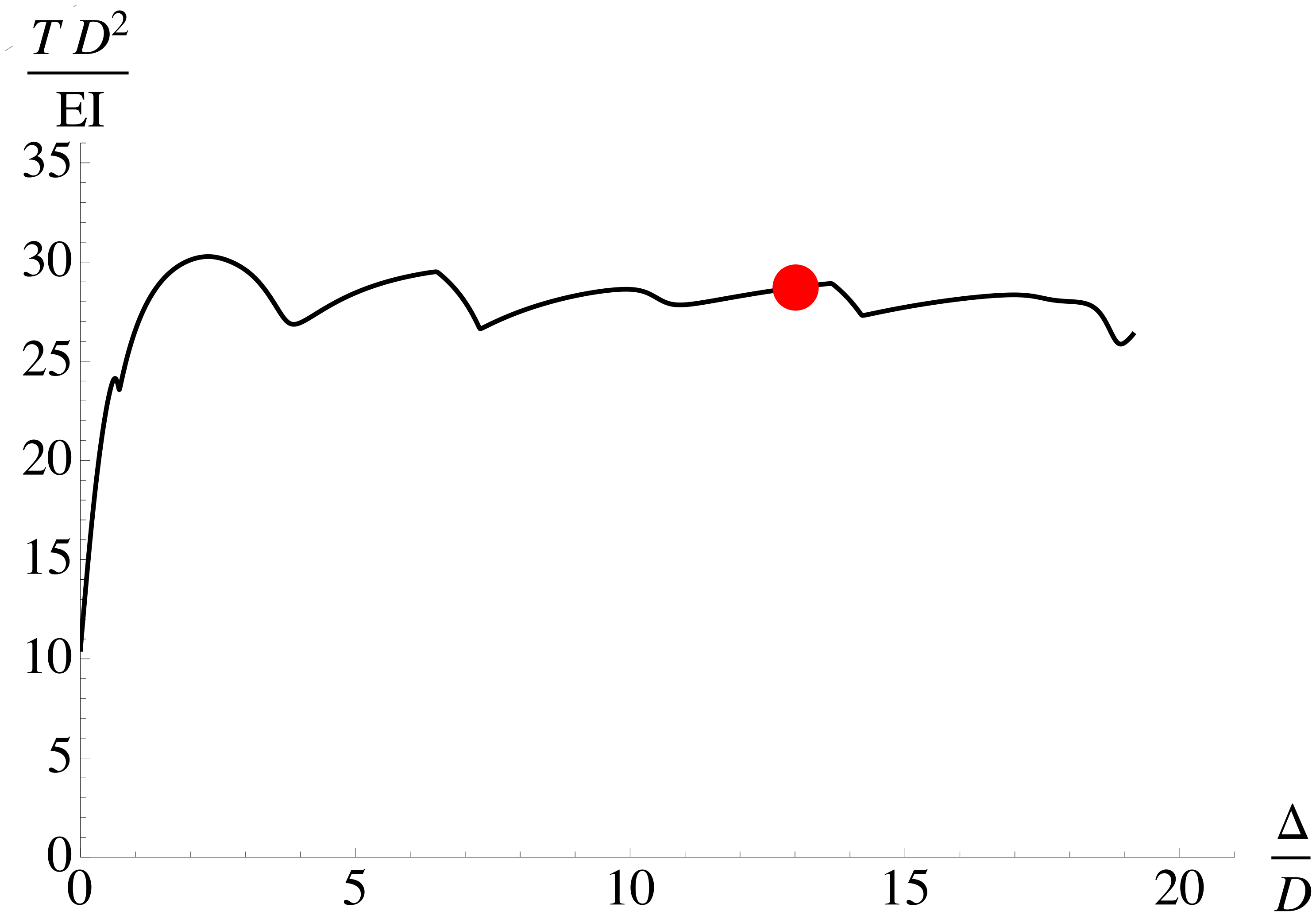

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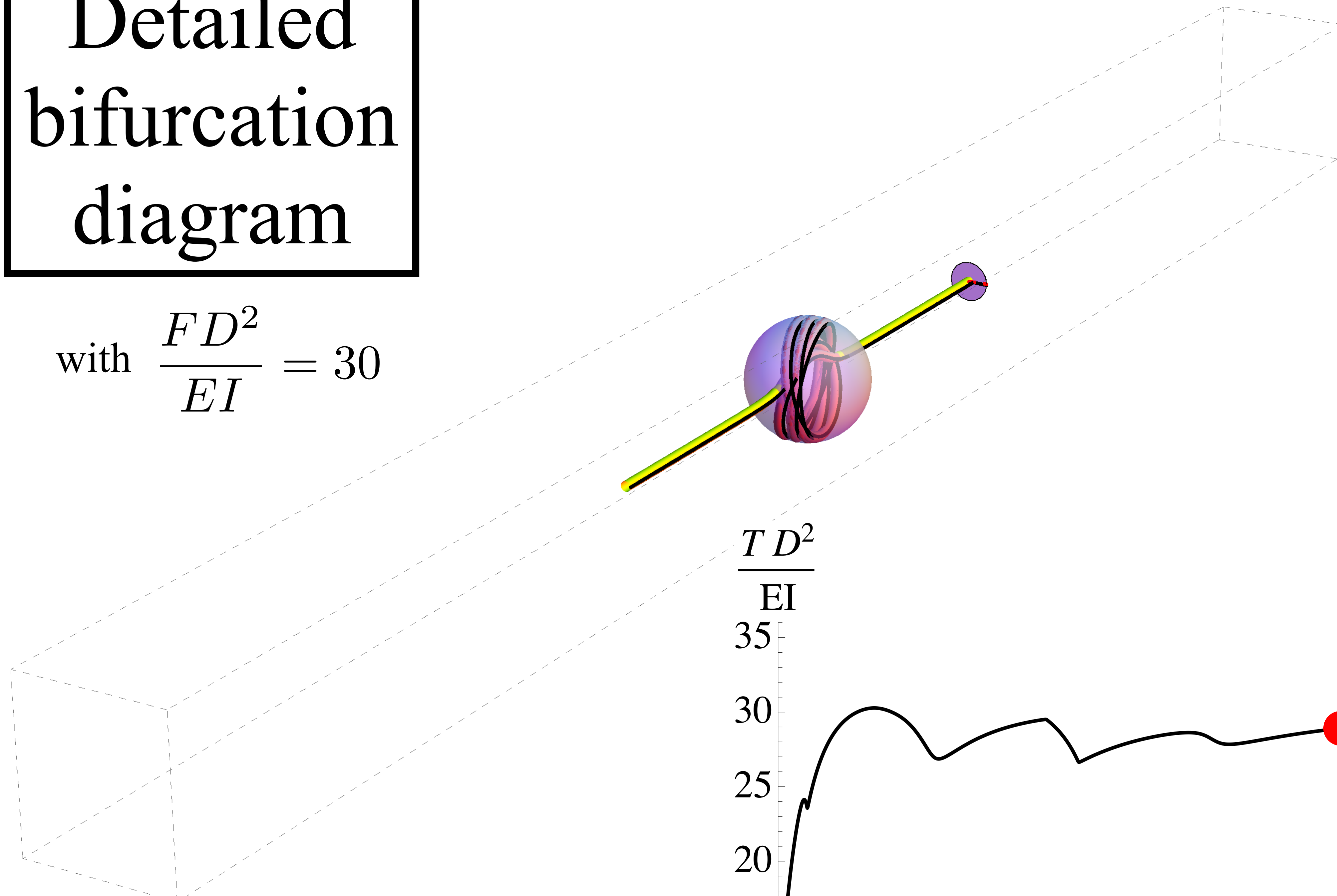

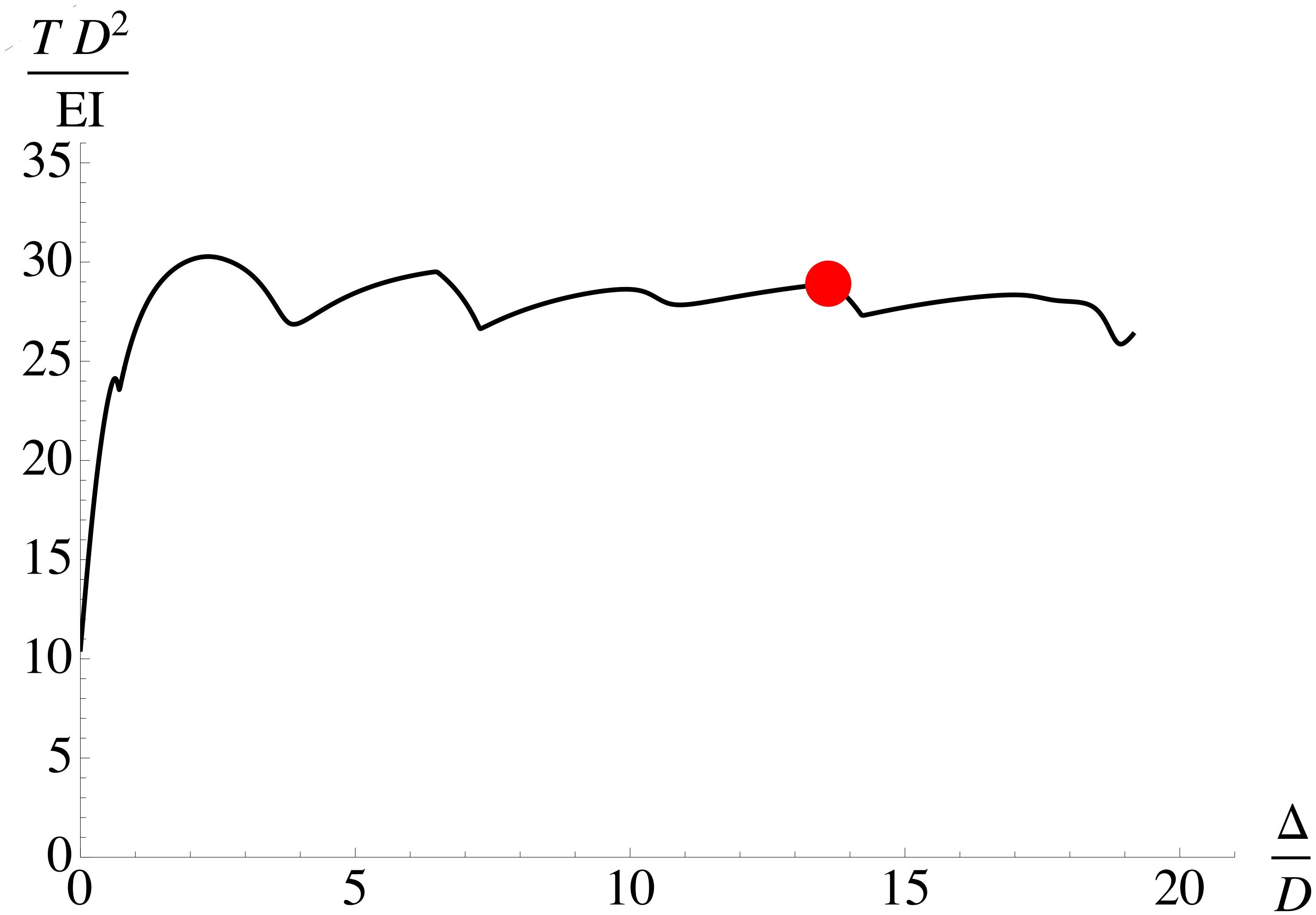

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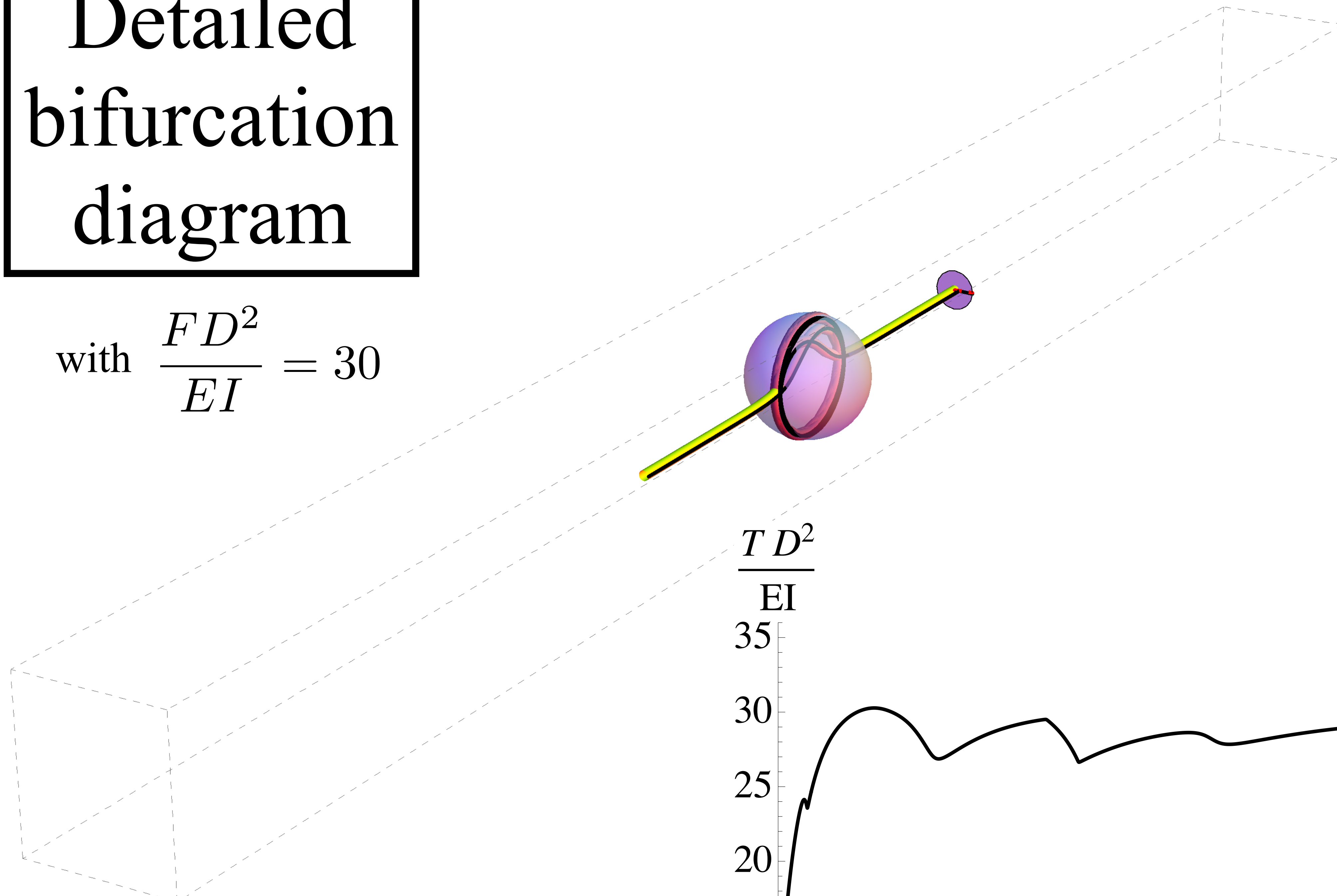
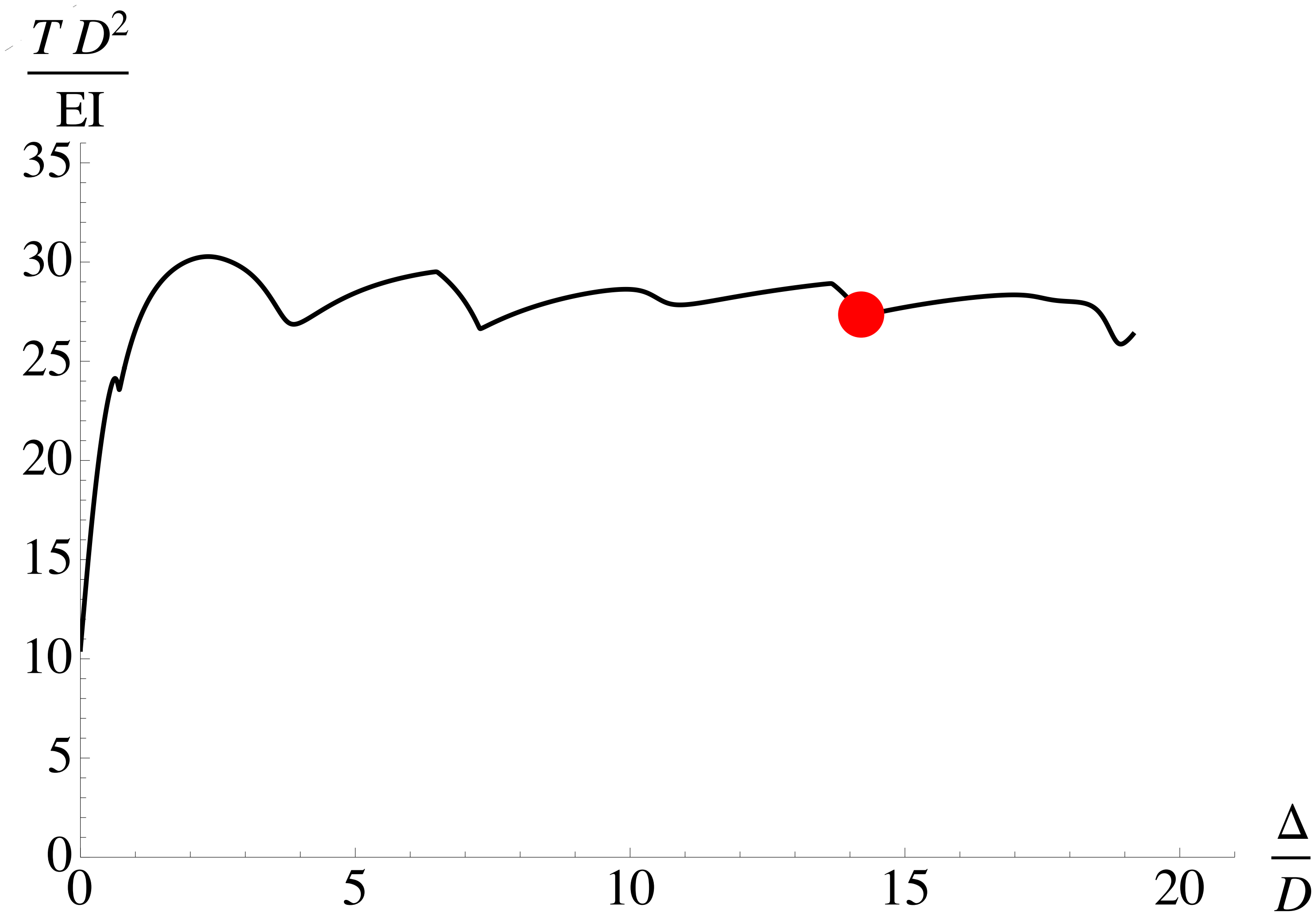
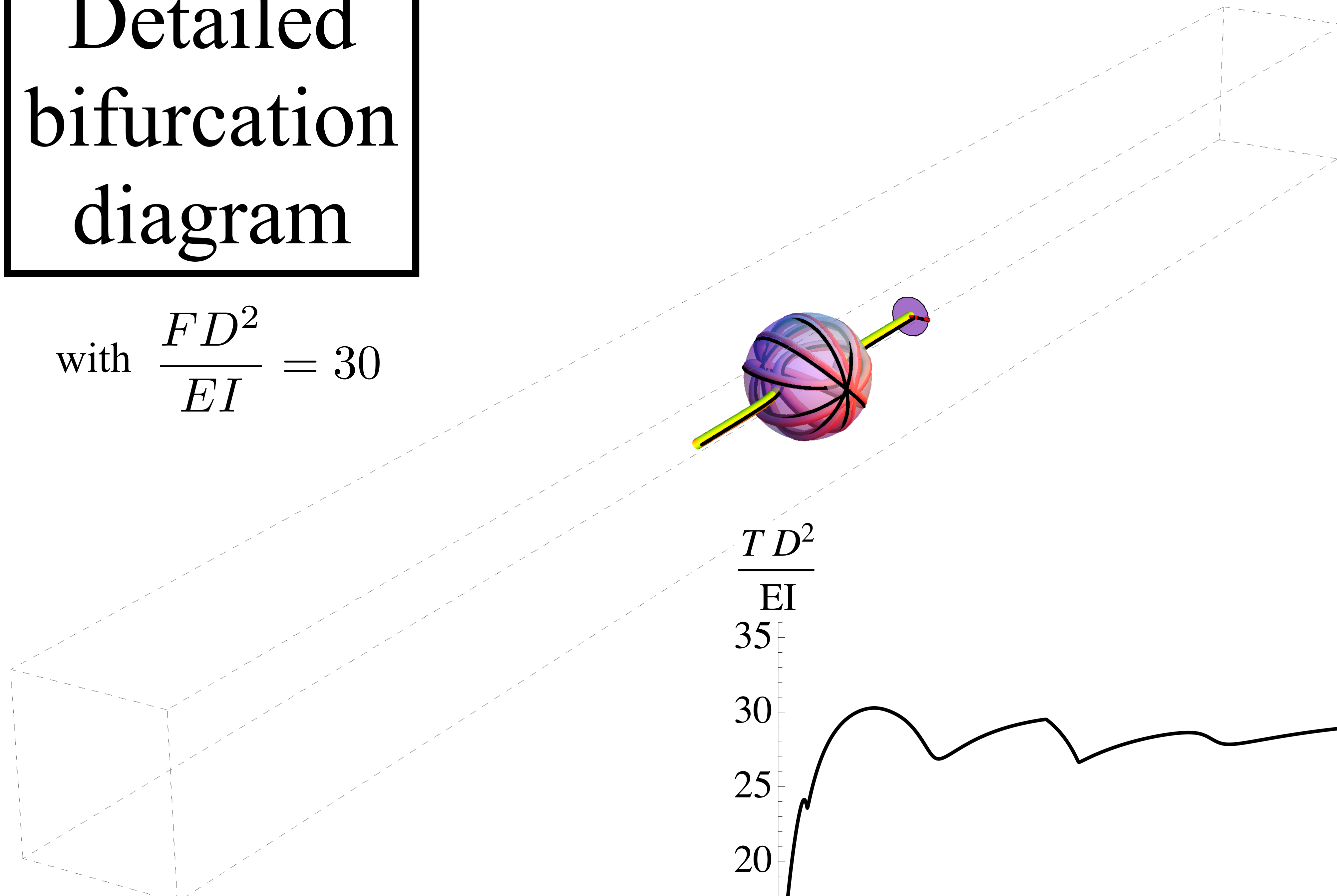
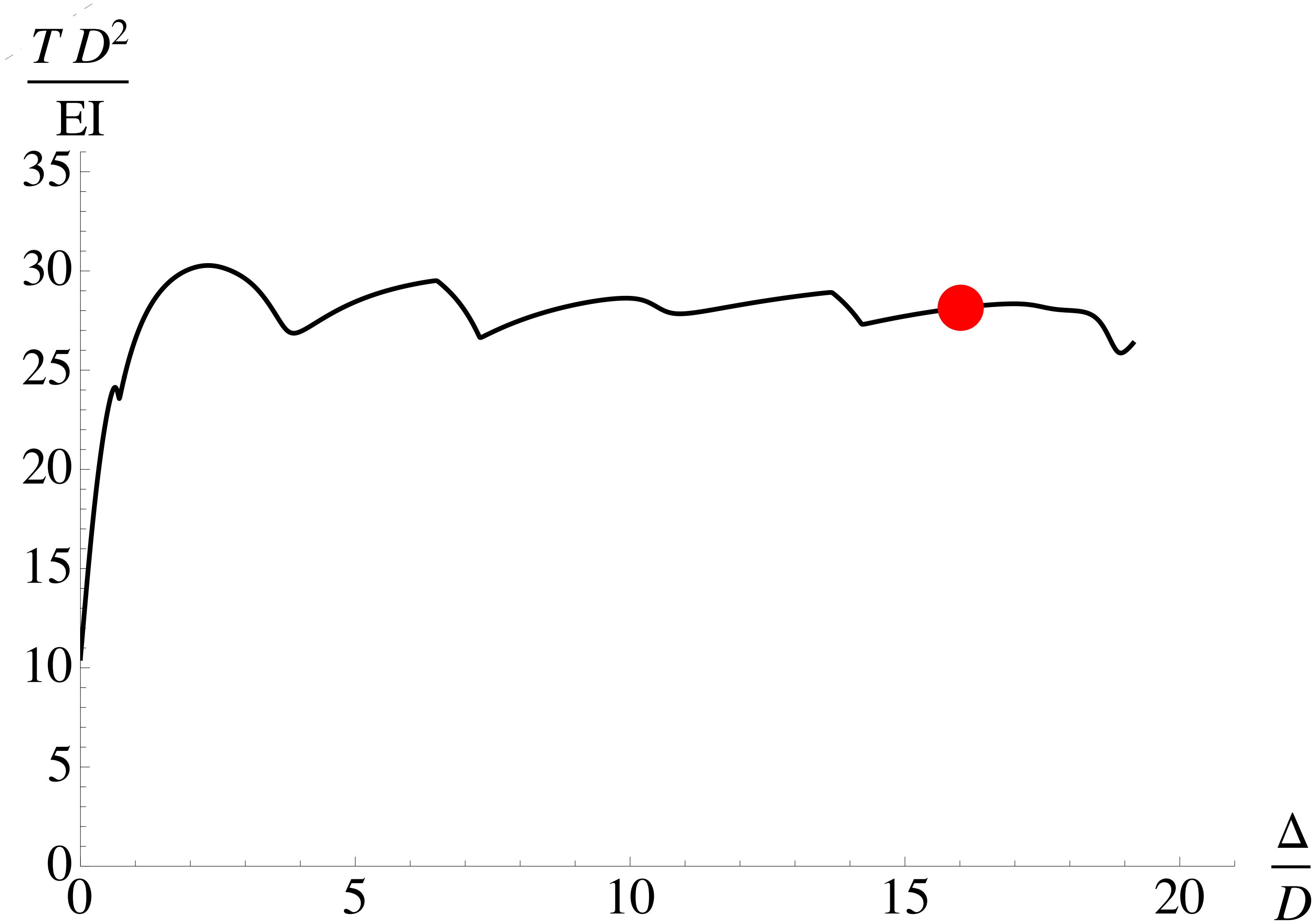




Detailed bifurcation diagram

with $\frac{FD^2}{EI} = 30$





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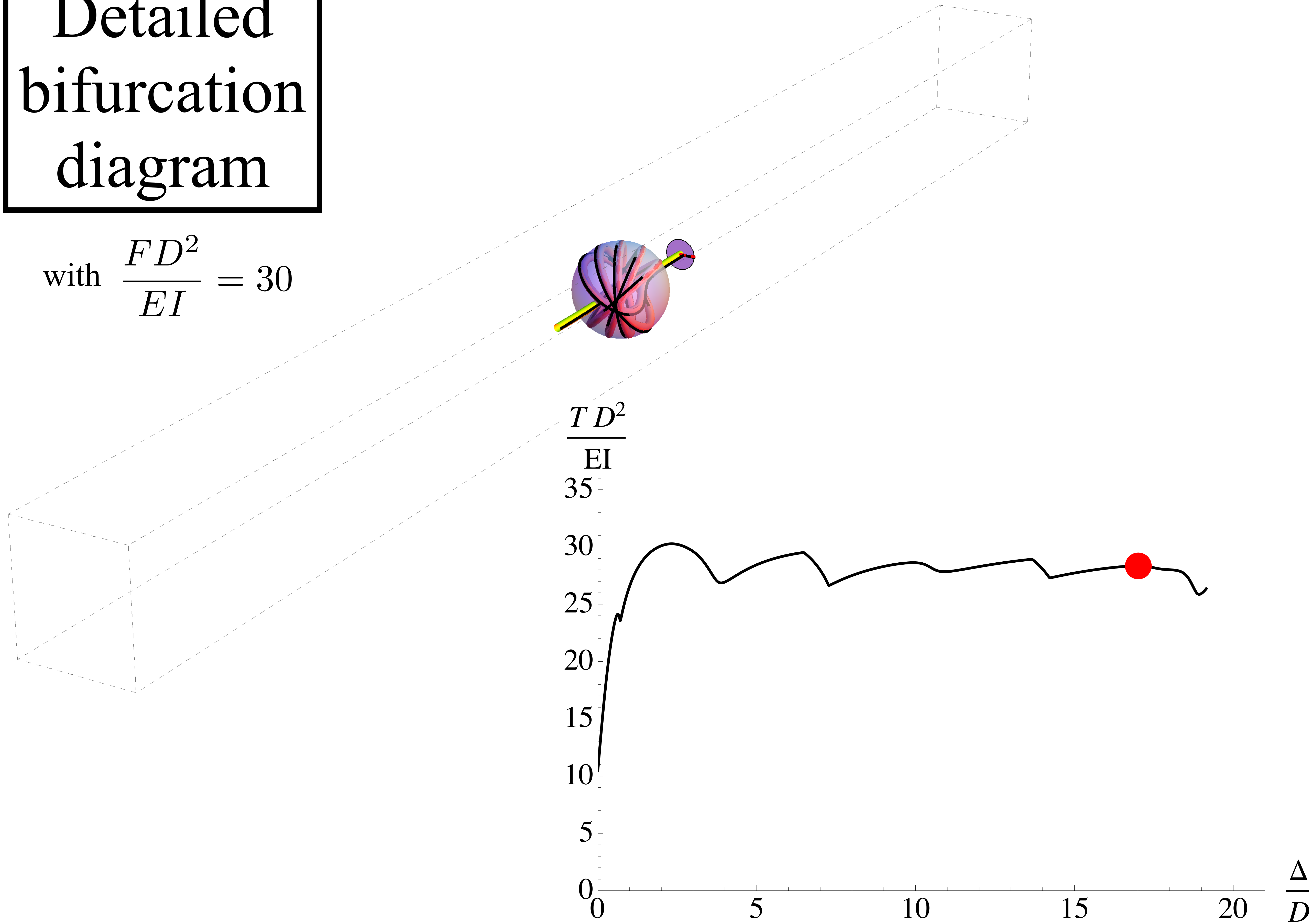

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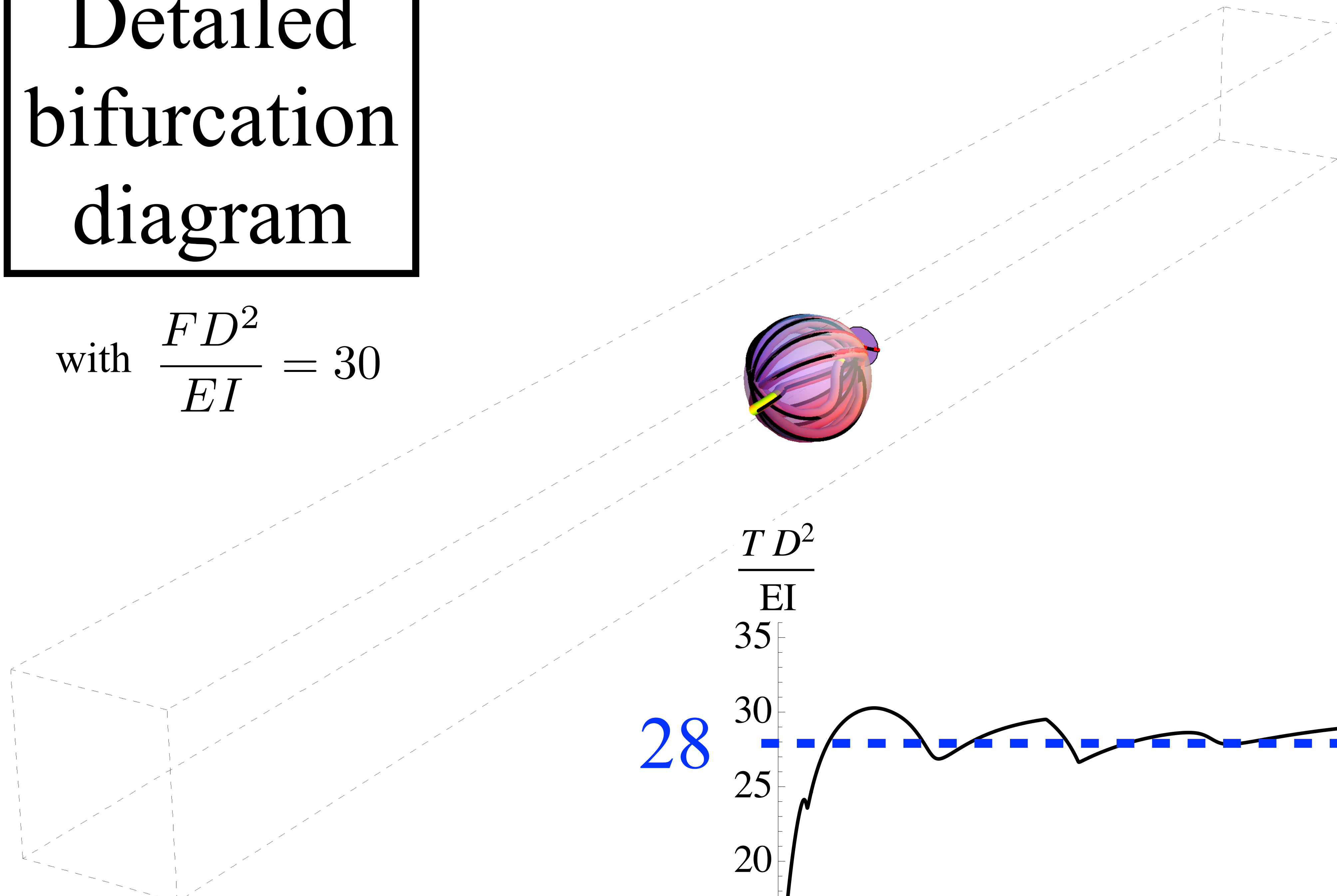
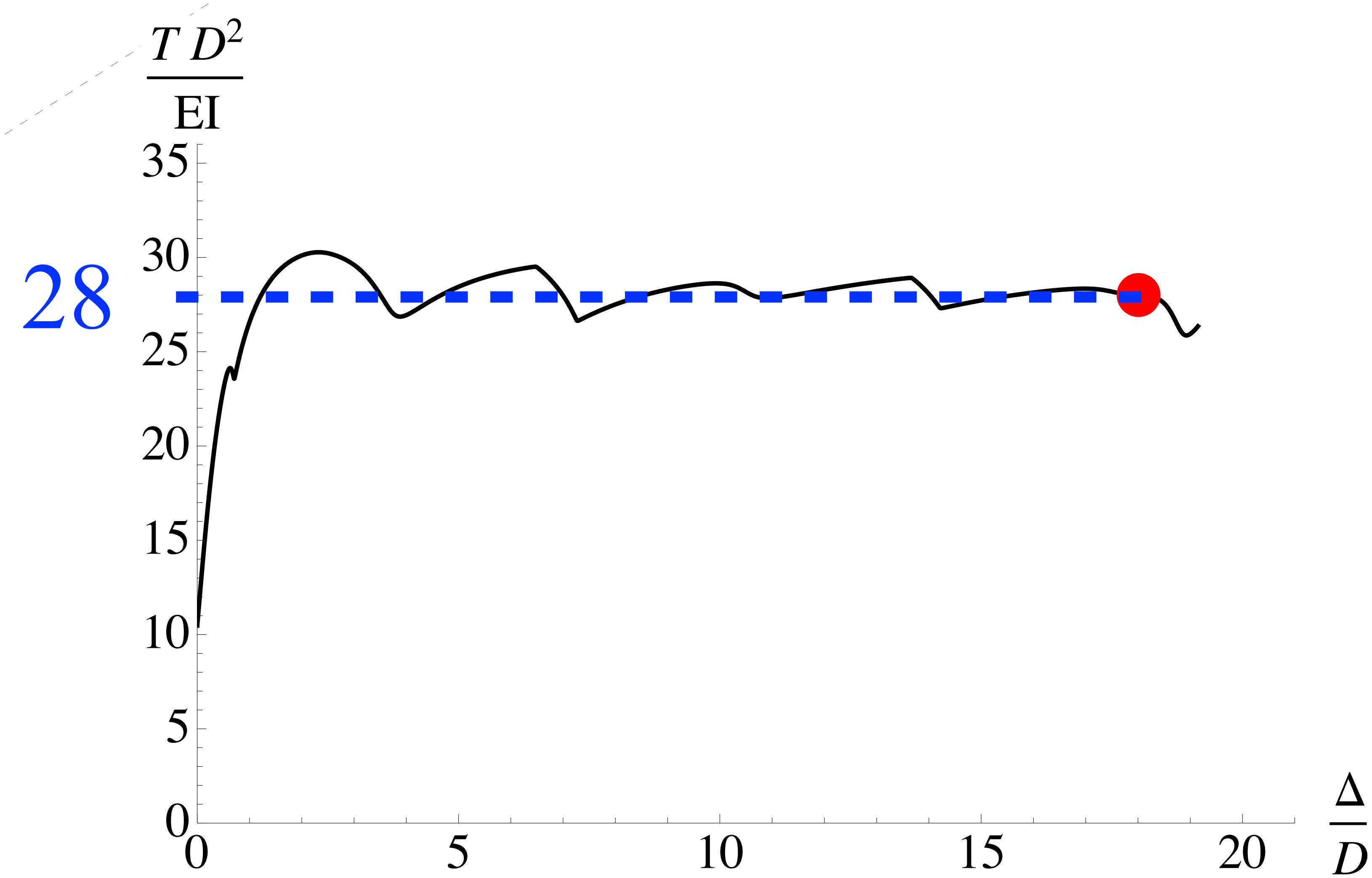
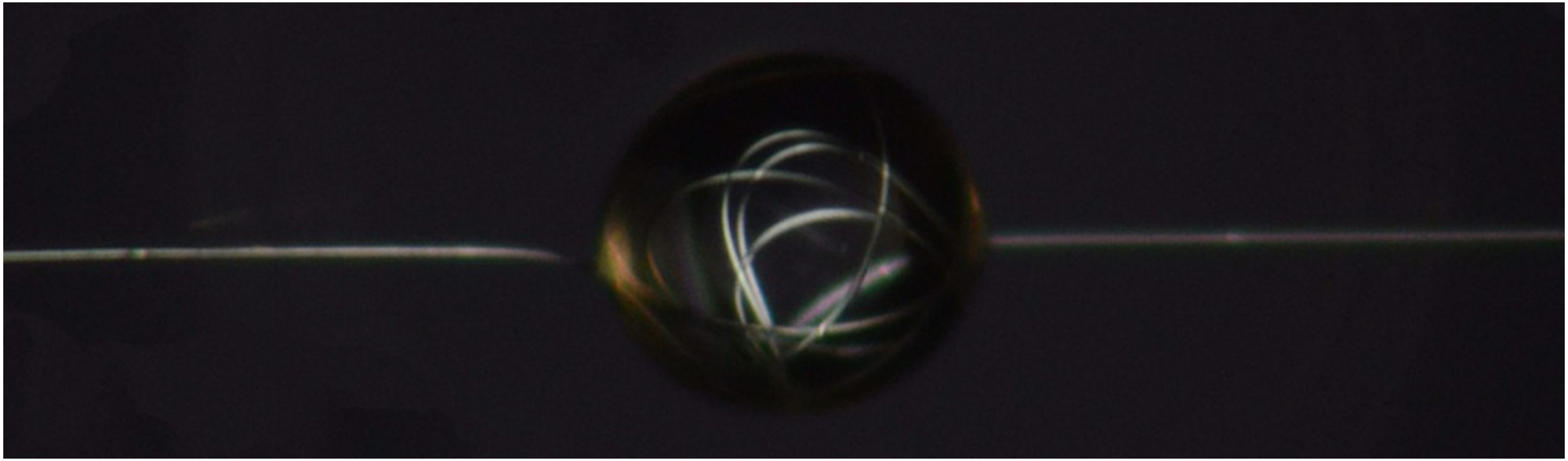




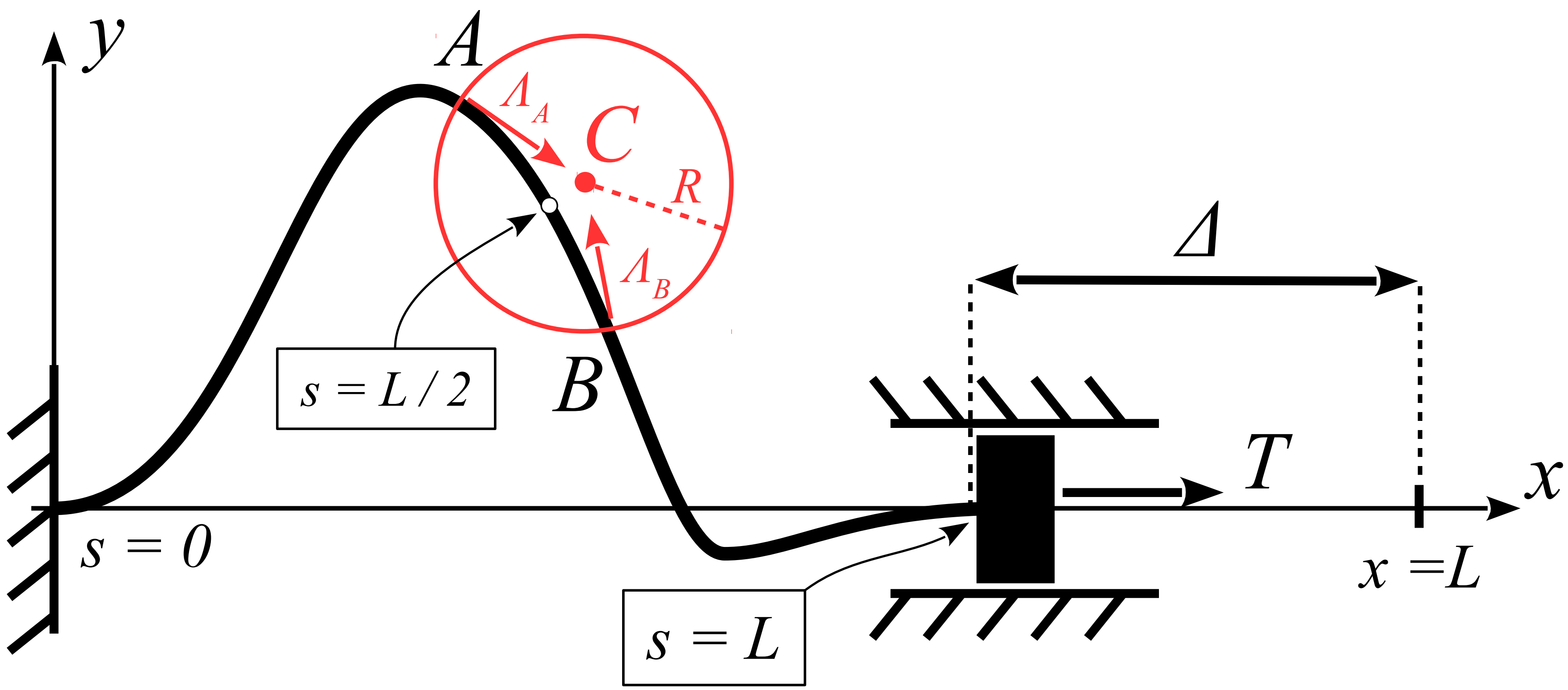
Conclusions

+ beam coiling in a liquid drop
+ capillary forces are large enough to buckle a beam
+ buckling is sub-critical
+ force plateau in the far post-buckling regime



Fin

Detailed equilibrium equations

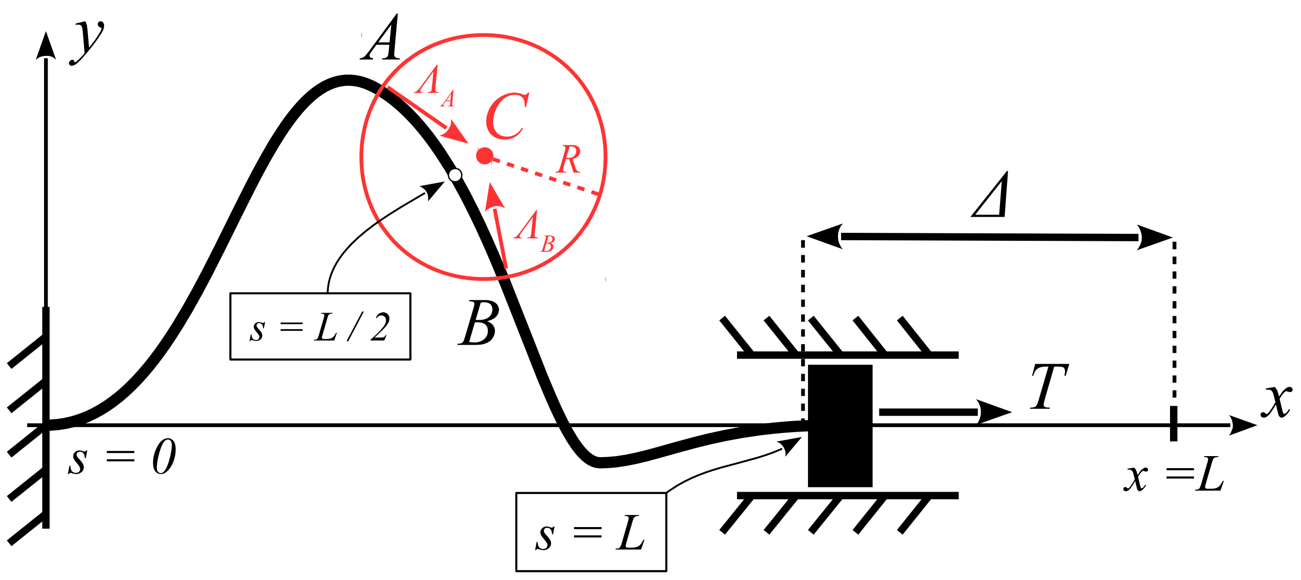


soft wall repulsion potential

$$V(X,Y) = \frac{V_0}{1 + \rho - (1/R)\sqrt{(X - X_C)^2 + (Y - Y_C)^2}}$$

R. S. Manning and G. B. Bulman, *Stability of an elastic rod buckling into a soft wall*, Proc Roy Soc A 2005

Detailed equilibrium equations



$$E_\kappa = \frac{1}{2} EI \int_0^{S_A} \kappa_1^2 \, dS + \frac{1}{2} EI \int_{S_A}^{S_B} \kappa_2^2 \, dS + \frac{1}{2} EI \int_{S_B}^{L} \kappa_3^2 \, dS$$

$$E_w = \int_{S_A}^{S_B} V(X(S), Y(S), X_C, Y_C) \, dS$$

$$E_\gamma = P \, \gamma_{sa} \, S_A + P \, \gamma_{sl} \, (S_B - S_A) + P \, \gamma_{sa} \, (L - S_B)$$

=> boundary-value problem, solved with AUTO