COMMENTS

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Limit cycles of polynomial Liénard systems

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Recently [H. Giacomini and S. Neukirch, Phys. Rev. E **56**, 3809 (1997)], an algorithm to obtain the number of limit cycles of Liénard systems has been proposed. The quoted paper also includes a method to approximate the eventual limit cycles and a conjecture on the behavior of the algorithm. The algorithm is reviewed and some examples, which show that the algorithm is really efficient, are given. However, these examples indicate that the aforementioned conjecture may have been incorrectly stated. A different conjecture is proposed and some open questions are formulated. [S1063-651X(98)16809-5]

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I. INTRODUCTION

In this Comment, we are concerned with the family of Liénard systems

$$\dot{x}=y-F(x),\qquad \dot{y}=-x, \tag{1.1}$$

where $F(x)$ is an odd polynomial. As usual, the dot denotes a derivative with respect to the time t. Obviously, these systems have only one equilibrium at the origin. As it is well known, systems (1.1) are a particular case of the more general Liénard equation

$$\ddot{x}+f(x)\dot{x}+g(x)=0,$$

for it suffices to take $F(x)=\int_0^x f(s)\,ds$, $g(x)=x$, and $y=\dot{x}+F(x)$.

Recently, Giacomini and Neukirch [2,3] have developed an algorithm to determine the number of limit cycles of system (1.1), along with a method to approximate such limit cycles by means of algebraic curves. It is remarkable that the Giacomini-Neukirch algorithm is nonperturbative and seems to work very well. However, it lacks a firm theoretical basis and so it still needs additional research in order to clarify its possibilities and general scope.

In this paper, we first review the Giacomini-Neukirch algorithm. After that, we give some examples that seem to indicate that a conjecture related with the algorithm should be corrected as indicated below, and formulate some open questions about the algorithm.

II. THE GIACOMINI-NEUKIRCH ALGORITHM

As the quoted algorithm is mainly explained by examples (see [2,3]), in order to be more precise the following result will be useful. We remark that our notation differs slightly from that used in [3]. In what follows, the prime will denote a derivative with respect to the variable x.

Proposition 1. Consider a Liénard system

$$\dot{x}=y-F(x),\qquad \dot{y}=-g(x), \tag{2.1}$$

and, for $k\in\mathbb{N}$, define functions $\varphi_0,\varphi_1,\ldots,\varphi_{2k}$ with the following properties:

$$\begin{aligned}&\varphi_0(x)=1,\\ &\varphi_1'(x)=0,\ \ \varphi_2'(x)=2kg(x),\ \ \varphi_3'(x)=F(x)\varphi_2'(x),\\ &\varphi_j'(x)=F(x)\varphi_{j-1}'(x)+(2k-j+2)g(x)\varphi_{j-2}(x),\\ &\qquad j=4,5,\ldots,2k.\end{aligned} \tag{2.2}$$

Then the function

$$V_k(x,y)=\sum_{j=0}^{2k}\varphi_j(x)y^{2k-j}$$

verifies

$$\dot{V}_k(x,y)=-R_k(x),$$

where

$$R_k(x)=F(x)\varphi_{2k}'(x)+g(x)\varphi_{2k-1}(x).$$

Proof. It suffices to consider the expression

$$\dot{V}_k(x,y)=[y-F(x)]\sum_{j=0}^{2k}\varphi_j'(x)y^{2k-j}$$

$$-g(x)\sum_{j=0}^{2k}(2k-j)\varphi_j(x)y^{2k-j-1},$$

and do straightforward manipulations.

The crucial point in the above proposition is that the Lyapunov-like function V_k has an orbital derivative that does not depend at all on y. When $g(x)=x$ and $F(x)$ is an odd polynomial, the functions in Eqs. (2.2) can be obtained as even or odd polynomials accordingly with the evenness or oddness of the index j. In such a case, the R_k polynomial turns out to be even. Clearly, it is possible to formulate the following algorithm.

Algorithm (Giacomini and Neukirch). For system (1.1), take $k\in\mathbb{N}$ and define the polynomials $\varphi_j(x)$, $j=0,1,\ldots,2k$, such that

$$\varphi_0(x)=1,$$

$$\varphi_1(x)=0,$$

$$\varphi_2(x)=kx^2,$$

$$\varphi_3'(x)=2kxF(x),\quad \varphi_3(0)=0,$$

$$\varphi_j'(x)=F(x)\varphi_{j-1}'(x)+(2k-j+2)x\varphi_{j-2}(x),$$

$$\varphi_j(0)=0,\quad j=4,5,\ldots,2k.$$

Compute the even polynomial

$$R_k(x)=F(x)\varphi_{2k}'(x)+x\varphi_{2k-1}(x)$$

and determine the number of positive roots of R_k that have odd multiplicity. Let r_k be this number.

If $\deg F(x)=m$, it is easy to conclude that $\deg R_k(x)\leqslant(2k-1)m+1$.

As will be seen, Giacomini and Neukirch postulate that r_k has a direct relation with the number of limit cycles of system (1.1). In fact, it is not difficult to arrive at the following result (see [1,3]).

Proposition 2. If there exists $k\in\mathbb{N}$ such that $r_k=0$, then the system (1.1) has no limit cycles.

Proof. Suppose that the system (1.1) has a limit cycle of period T, and let $(x(t),y(t))$ be a parametrization of this cycle, for $t\in[0,T]$. Integrating with respect to the time t the function $\dot{V}_k(x,y)$ along the cycle, we get

$$\int_0^T\dot{V}_k(x(t),y(t))\,dt=V_k(x(T),y(T))-V_k(x(0),y(0))=0,$$

but, since $\dot{V}_k(x,y)=-R_k(x)$, we should have the contradiction

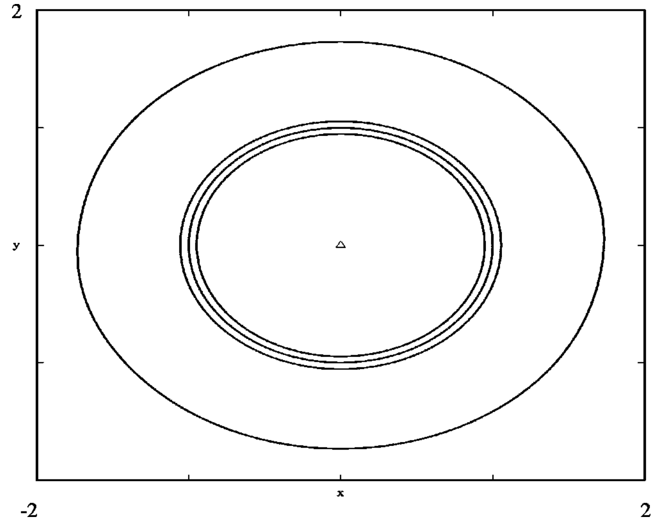

FIG. 1. The four limit cycles numerically obtained for the first counterexample, with $F(x)$ given in Eq. (3.1).

$$\int_0^T R_k(x(t))\,dt\neq 0,$$

since R_k does not change its sign.

III. CONJECTURES ON THE NUMBER OF LIMIT CYCLES

In [2,3], the following conjecture appears.

Conjecture 1. Let L be the number of limit cycles of system (1.1). Then, by applying the above algorithm, the following statements hold.

(i) $L\leqslant r_k$ for all $k\in\mathbb{N}$.

(ii) If $k_1>k_2$, then $r_{k_2}-r_{k_1}\geqslant 0$ and even.

So, it is claimed that the sequence $\{r_k\}$ is decreasing and lower bounded by L. In fact, they analyze a number of different examples getting that the sequence $\{r_k\}$ ultimately equals L for moderated values of k.

In the quoted works, it is also implicitly conjectured that the functions $V_k(x,y)$ are local Lyapunov functions at the origin, so that the curves defined by $V_k(x,y)=C$, $C>0$, are always closed. In fact, they propose the following scheme to approximate the different limit cycles of system (1.1).

(i) For every odd positive root $x^\star$ of $R_k(x)$, compute the value $K^\star$ such that the curve $V_k(x,y)=K^\star$ has the maximun value of x equal to $x^\star$.

(ii) The curve $V_k(x,y)=K^\star$ gives an algebraic curve that is close to a limit cycle.

TABLE I. The number of odd positive roots r_k of the even polynomial R_k for the first five steps of the Giacomini-Neukirch algorithm for the system (1.1) with $F(x)$ given in Eq. (3.1).

k	$\deg R_k$	r_k
1	10	2
2	28	2
3	46	2
4	64	2
5	82	4

(iii) The above approximations improve as k increases.

We have investigated other examples and the above algorithm essentially works, but the conjecture is not true, as the following counterexample demonstrates.

Consider the system (1.1) with

$$F(x)=\varepsilon\left[\frac{x^9}{63}-\frac{3}{35}x^7+\frac{1199}{8000}x^5-\frac{83}{800}x^3+\frac{297}{12\,800}x\right], \tag{3.1}$$

which for $\varepsilon>0$ and sufficiently small has four limit cycles (see [4], p. 263). We have taken $\varepsilon=1$ and confirmed numerically that the four limit cycles persist (see Fig. 1). We have computed some steps of the algorithm and the results obtained appear in Table I. For $k>5$, r_k seems to be always equal to 4. From Table I, both statements in conjecture 1 are not true.

In fact, we have found another counterexample with

$$F(x)=\frac{x^9}{63}-\frac{3}{70}x^7+\frac{307}{8000}x^5-\frac{193}{16\,000}x^3+\frac{11}{16\,000}x$$

(see [4], p. 262), which also have four limit cycles. We obtain $r_1=2$ and $r_2=4$.

In view of these counterexamples we propose a different conjecture.

Conjecture 2. Let L be the number of limit cycles of system (1.1). Then, by applying the algorithm of Giacomini and Neukirch, there exists $k_0\in\mathbb{N}$ such that $L=r_k$ for all $k\geqslant k_0$.

We end by suggesting some open questions that deserve additional investigation.

(i) Conjecture 1 could be true if k is restricted to some values greater than a certain constant related to $m=\deg F(x)$.

(ii) Are the functions $V_k(x,y)$ local Lyapunov functions with simple closed level curves for all k?

(iii) In view of proposition 1, there is some freedom in the determination of φ_j with j even. Possibly, using this idea one can get better algebraic approximations to limit cycles.

(iv) The algorithm also could work for systems (2.1) with a more general function g (maybe even for the case with more than one equilibrium).

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