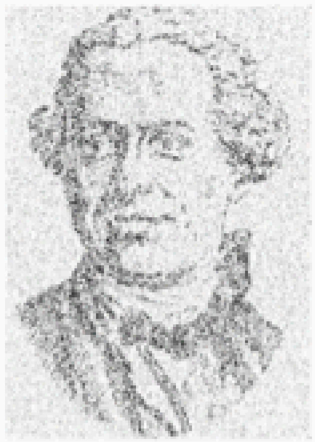


Boundary layer methods in Biomecanics

Simplified set of Navier Stokes Equations:
Applications in Biomecanics

Lagrée Pierre-Yves

Laboratoire de Modélisation en Mécanique
CNRS UMR 7607 -- Université Paris VI
Jussieu

Institut Jean le Rond d'Alembert

www.lmm.jussieu.fr

Aim

- simplification of Navier Stokes equations
- thanks to asymptotic theory:

  "Boundary Layer"

Aim

- simplification of Navier Stokes equations
- thanks to asymptotic theory:

  "Boundary Layer"

Starting from Navier Stokes (Axi)

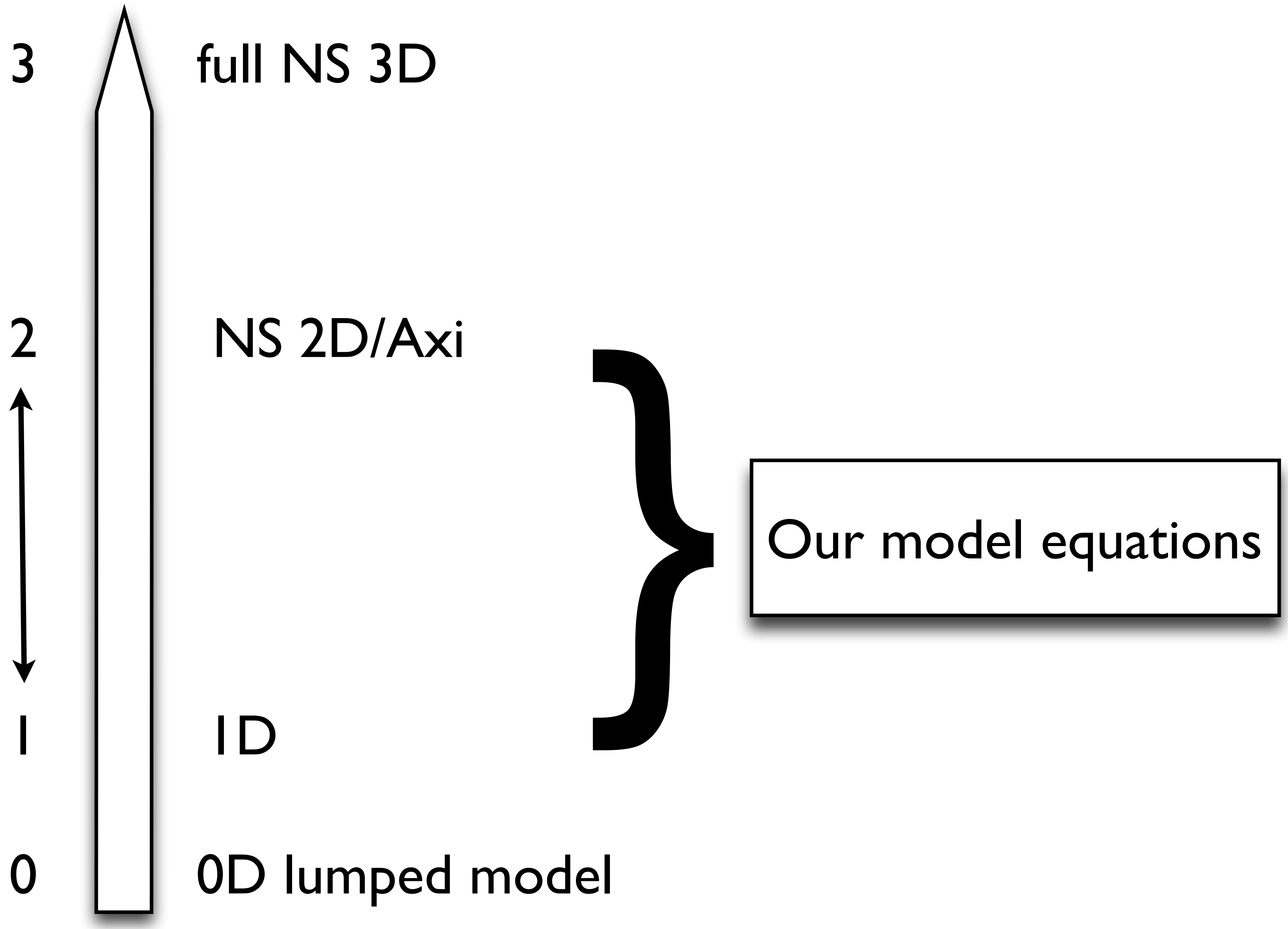
- we simplify NS to a Reduced set of equations
  - which contains the physical scales,
  - the most important phenomena
- much more simple set of equations: Integral equations (1D)
- cross comparisons in some cases of NS/ RNSP/ Integral

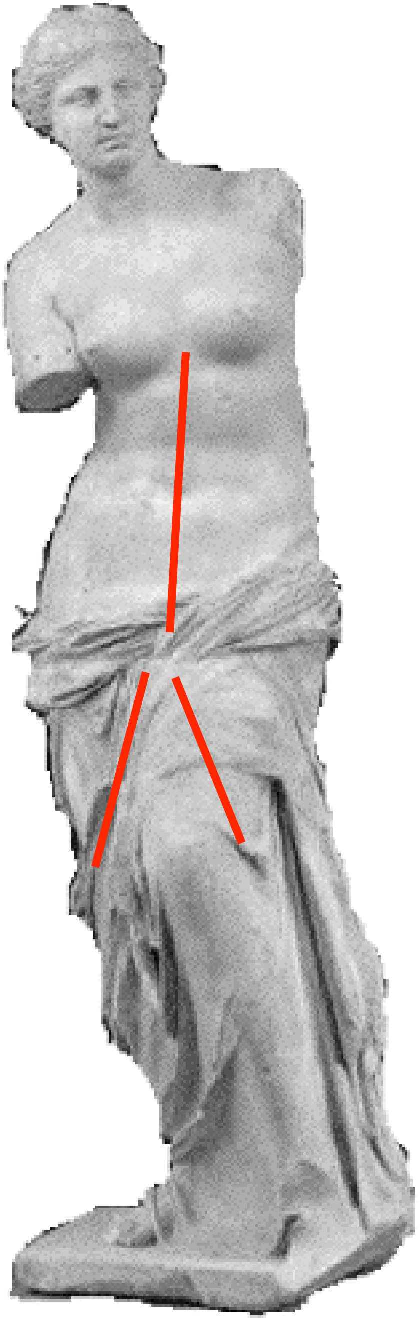
3 full NS 3D

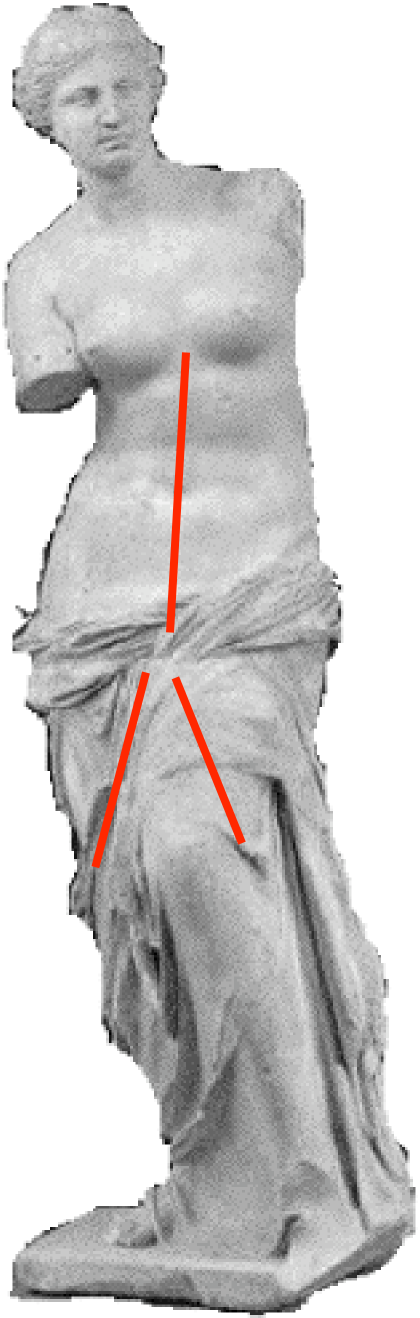
2 NS 2D/Axi

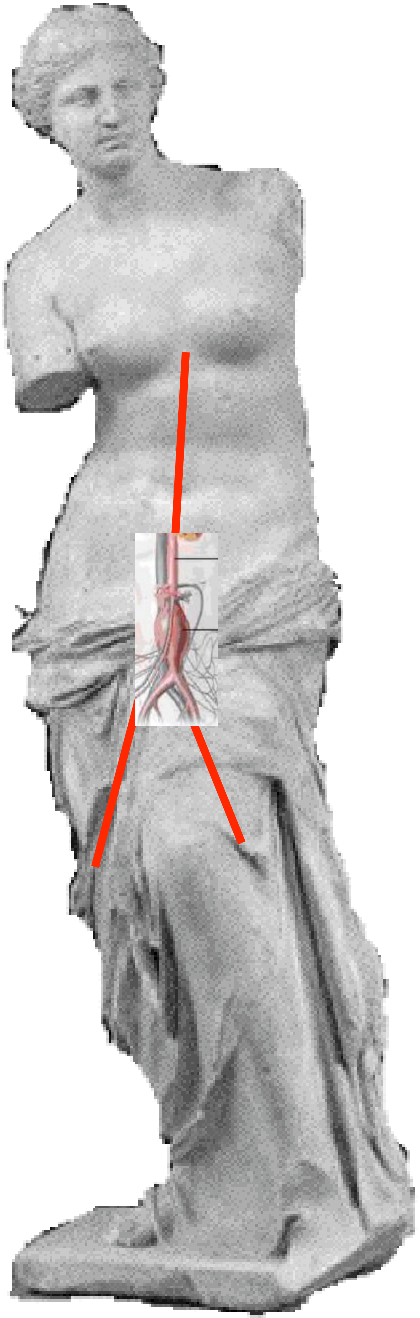
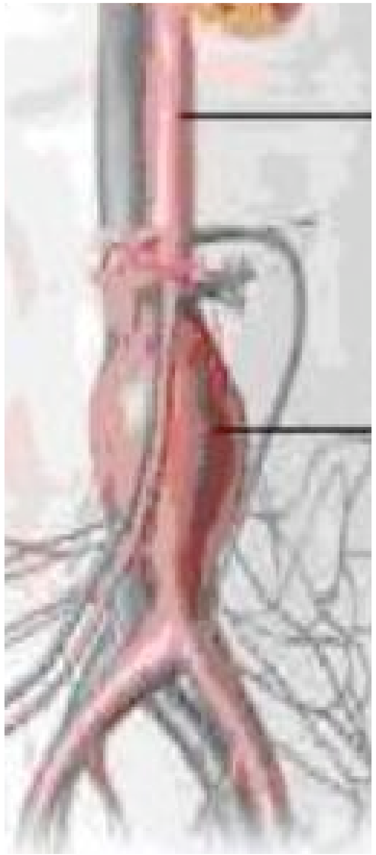
1 1D

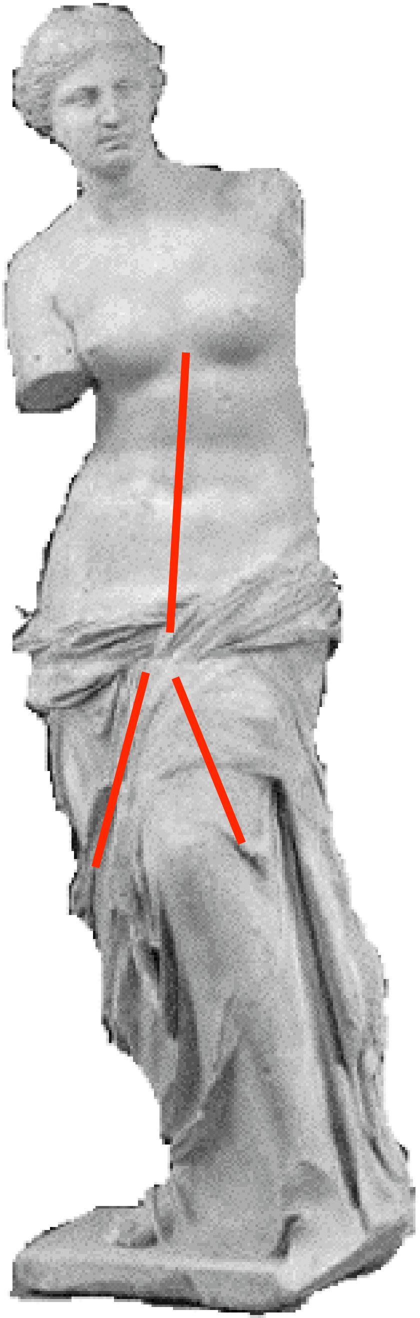
0 0D lumped model

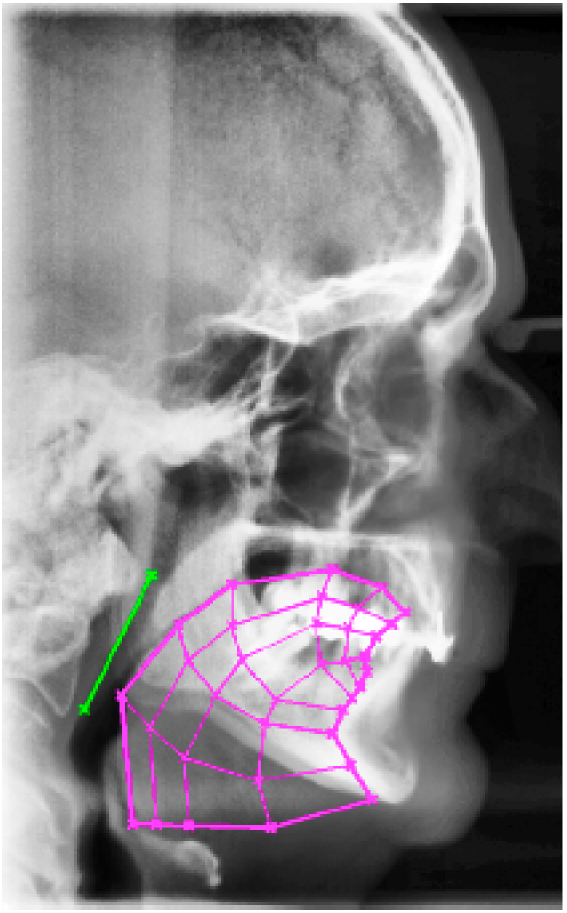

3
full NS 3D
2
NS 2D/Axi
1
1D
0
0D lumped model
Our model equations

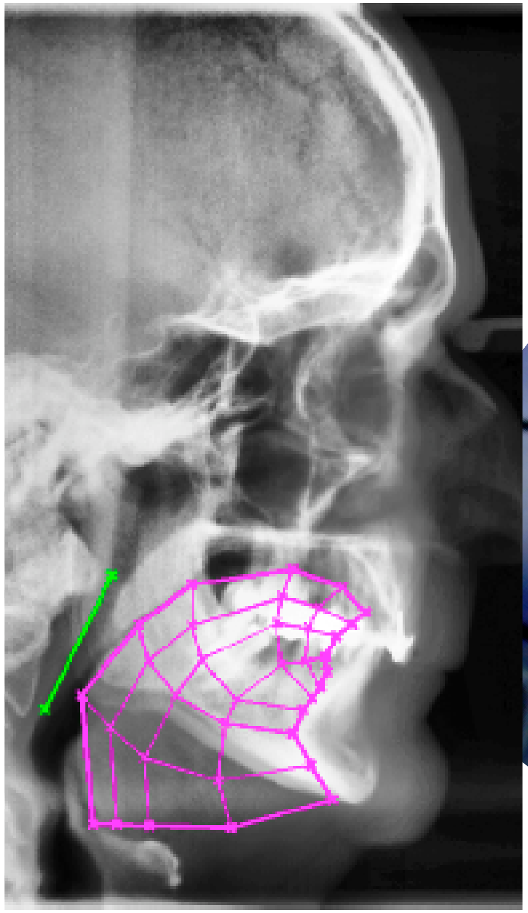
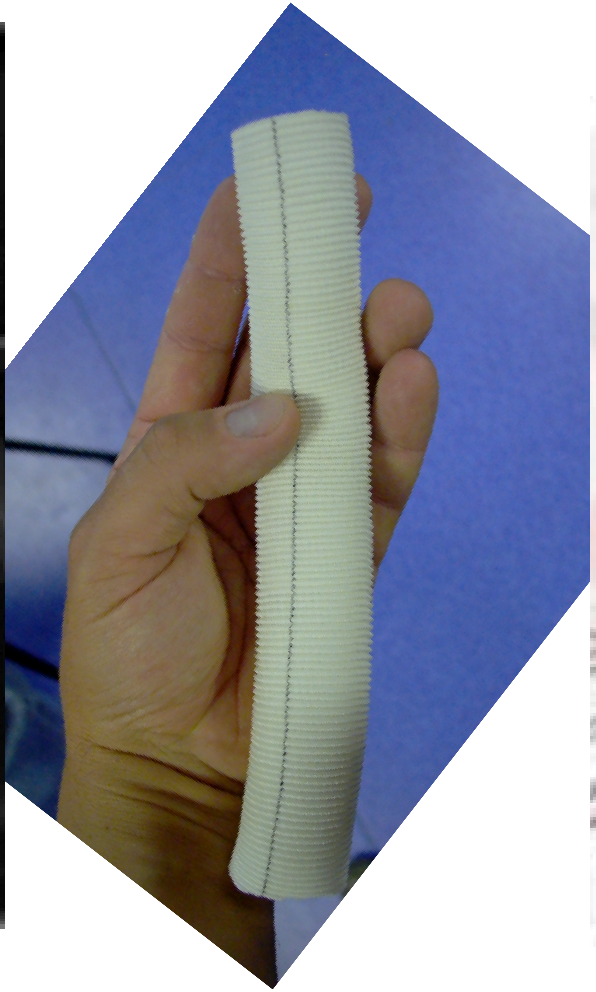
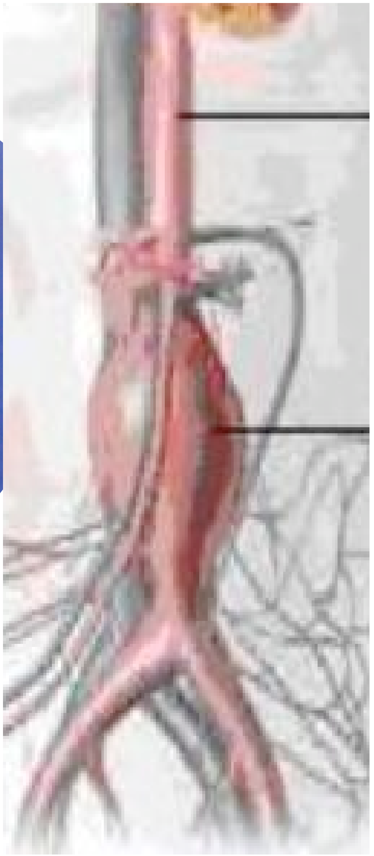


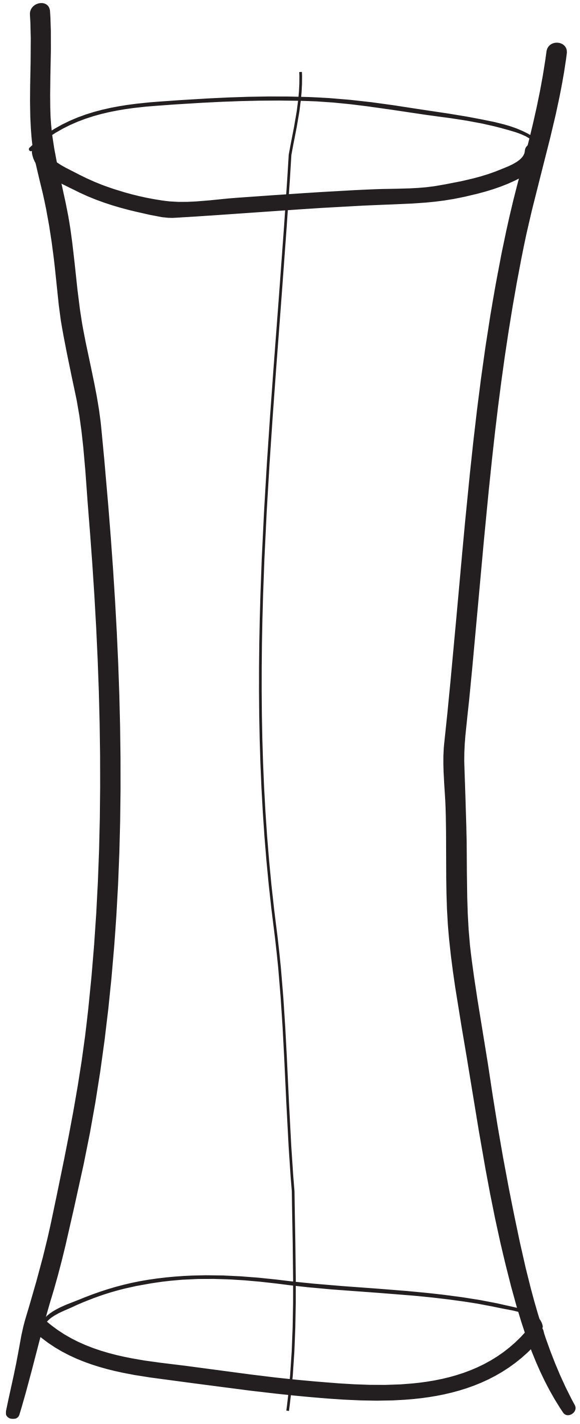


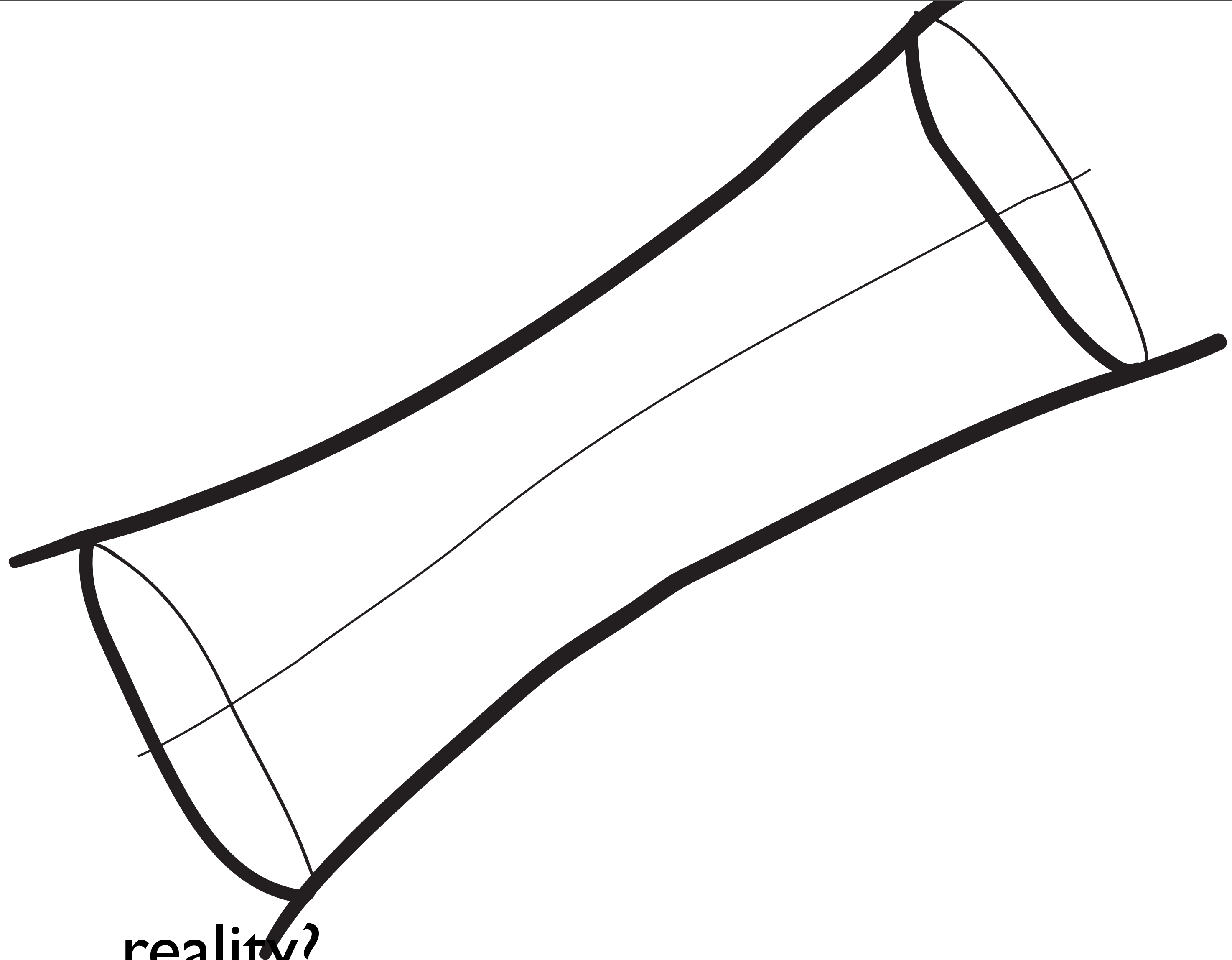


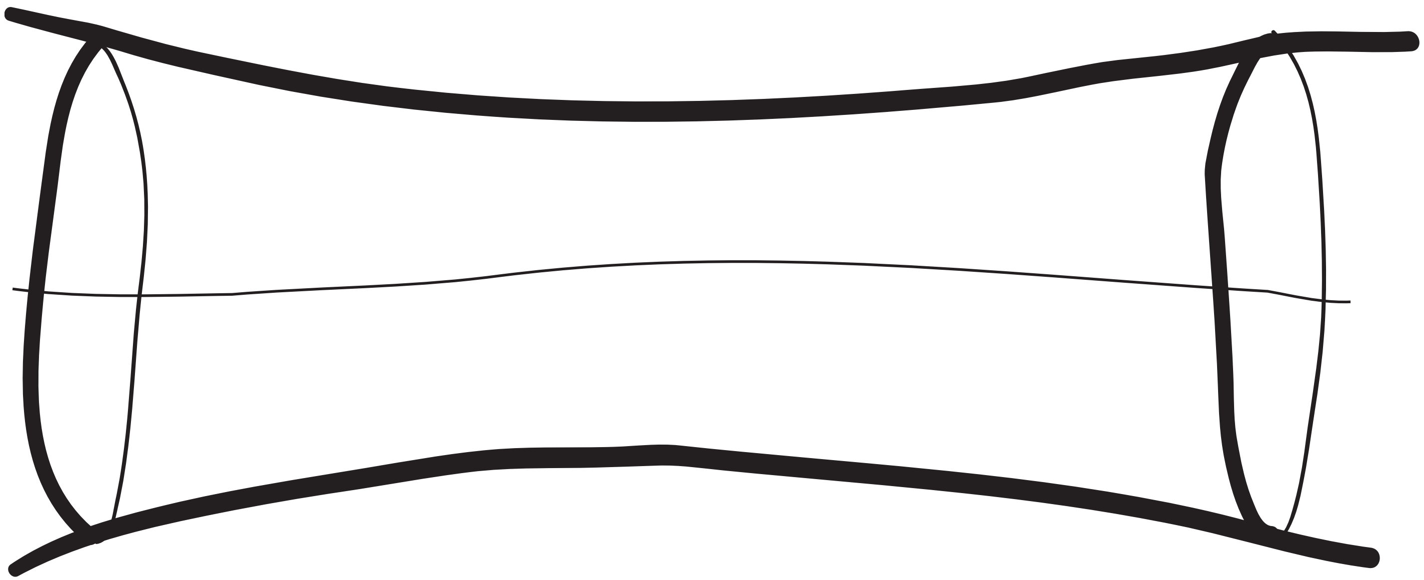







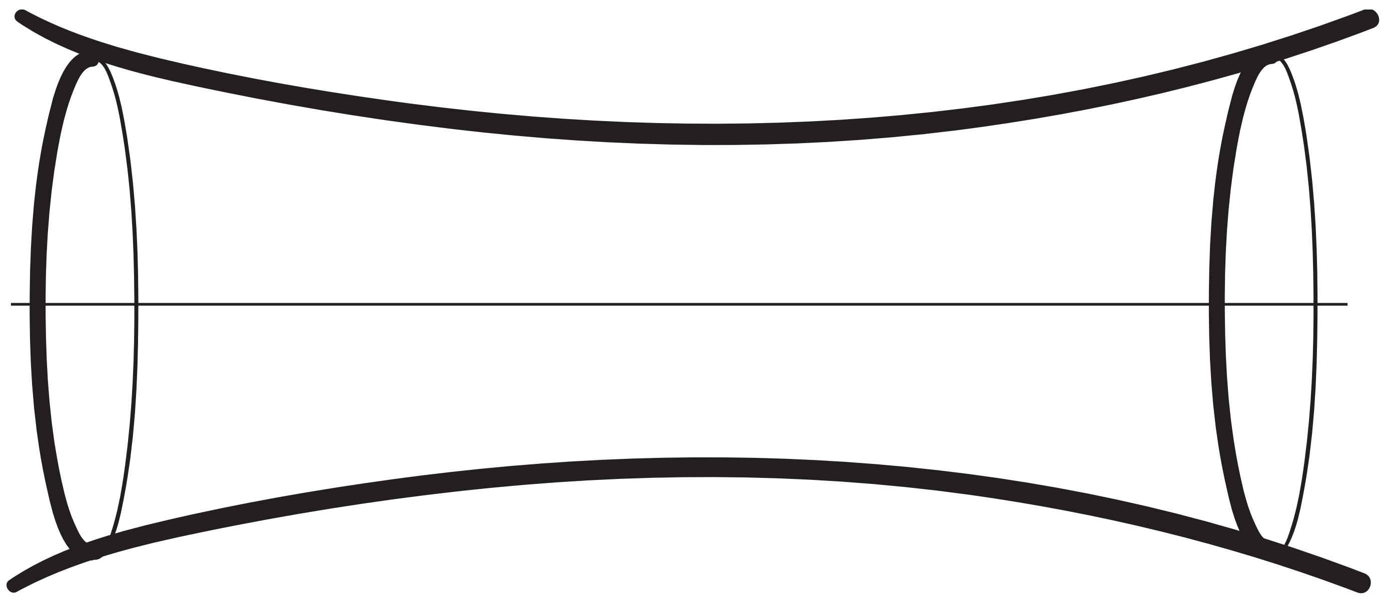
reality?



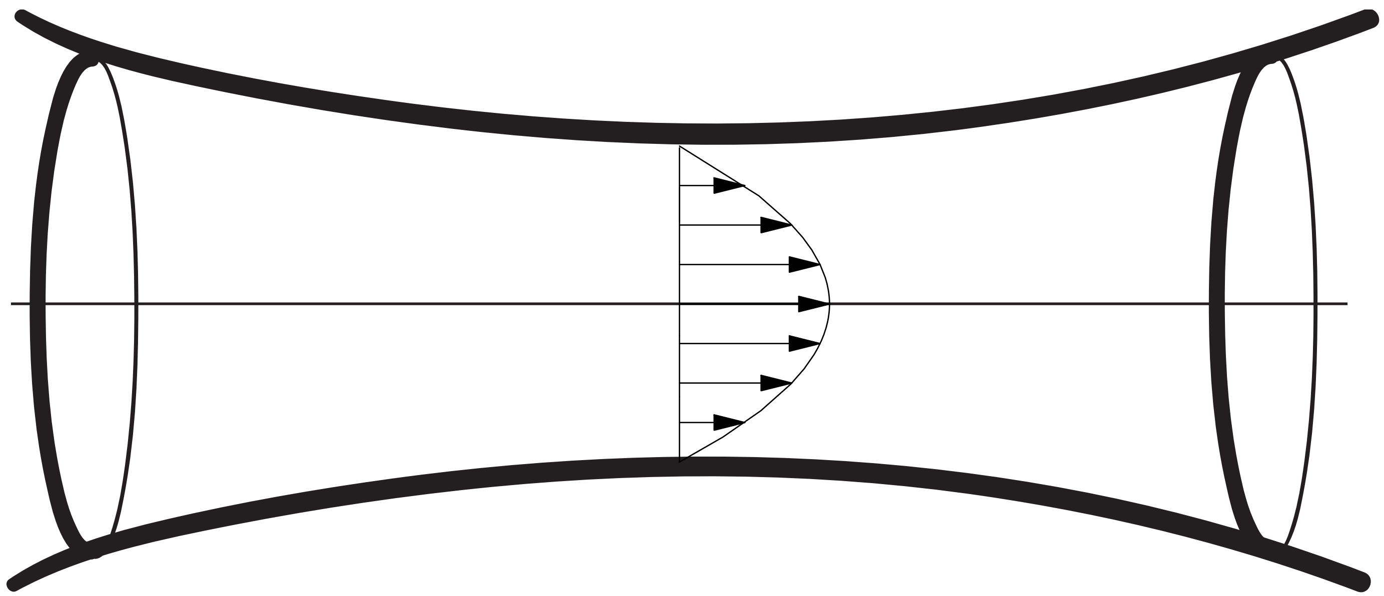
reality?



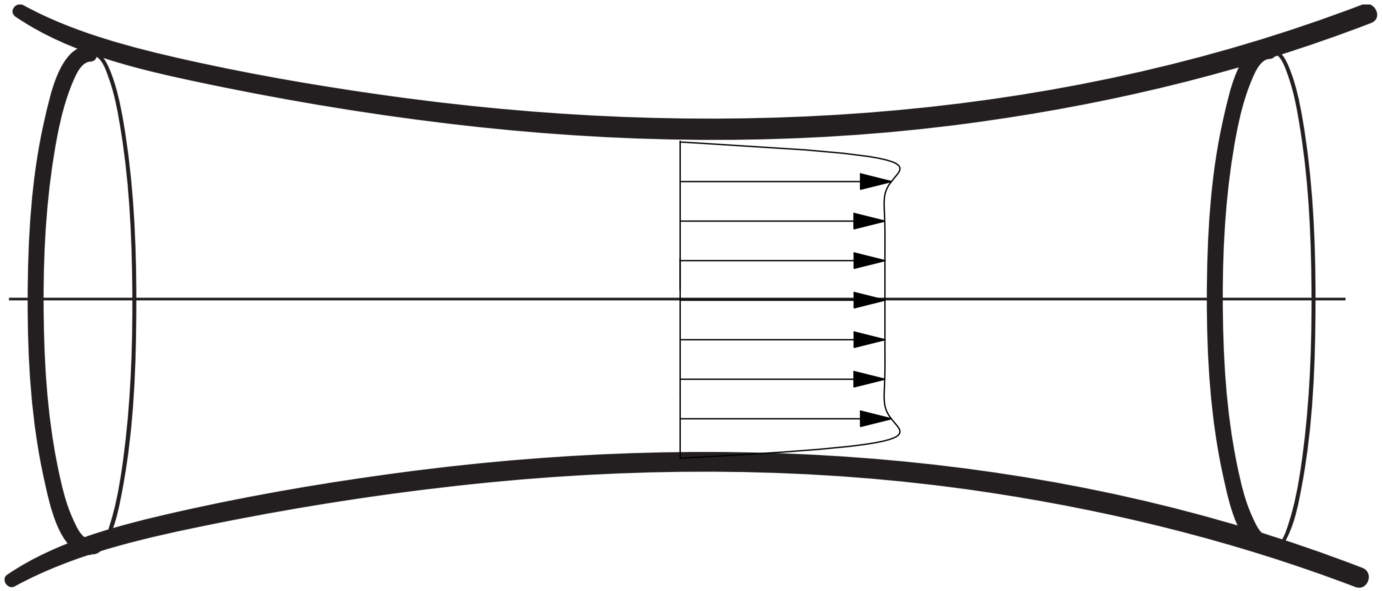
reality?



straight pipe, smooth walls, symmetry

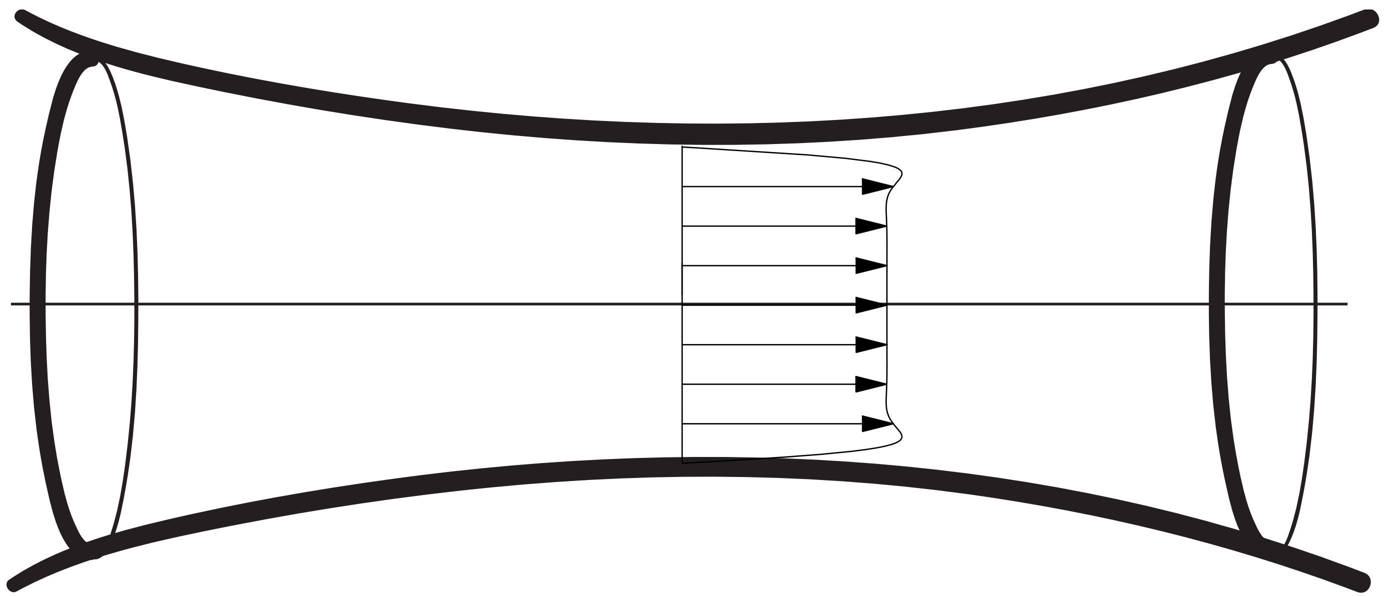


velocity profile

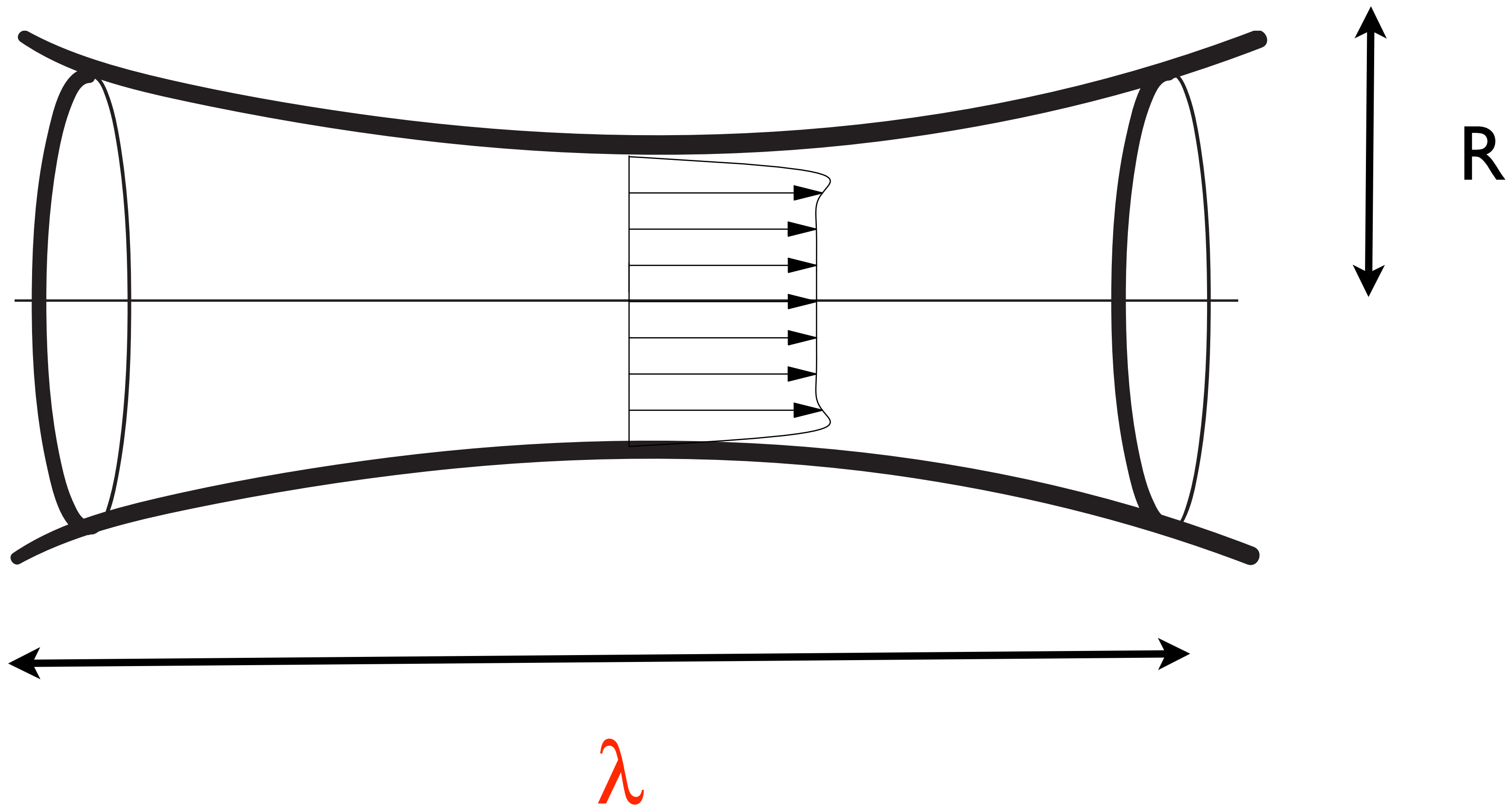


velocity profile

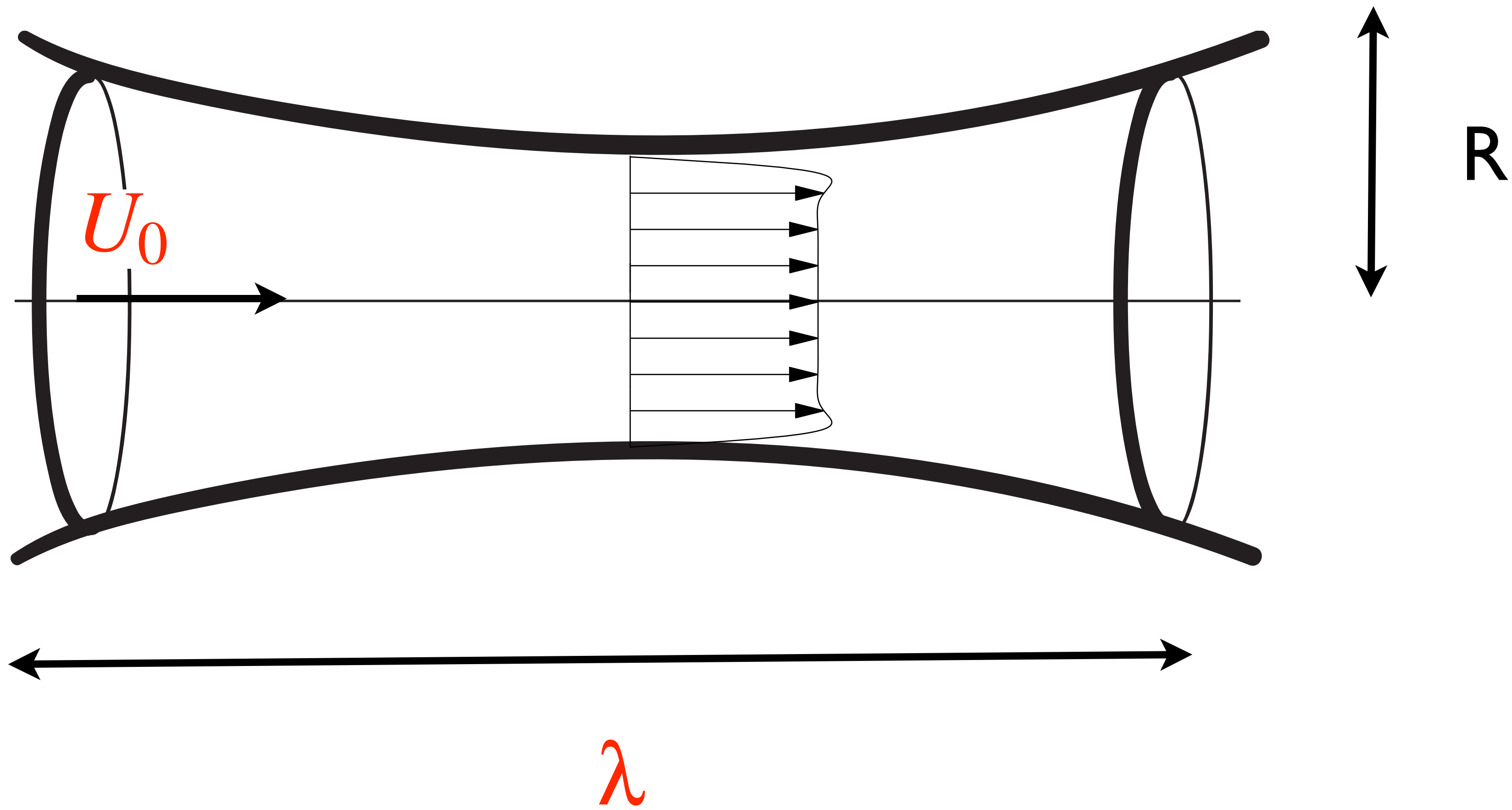
Equations



- simplified set
- deduced from orders of magnitude

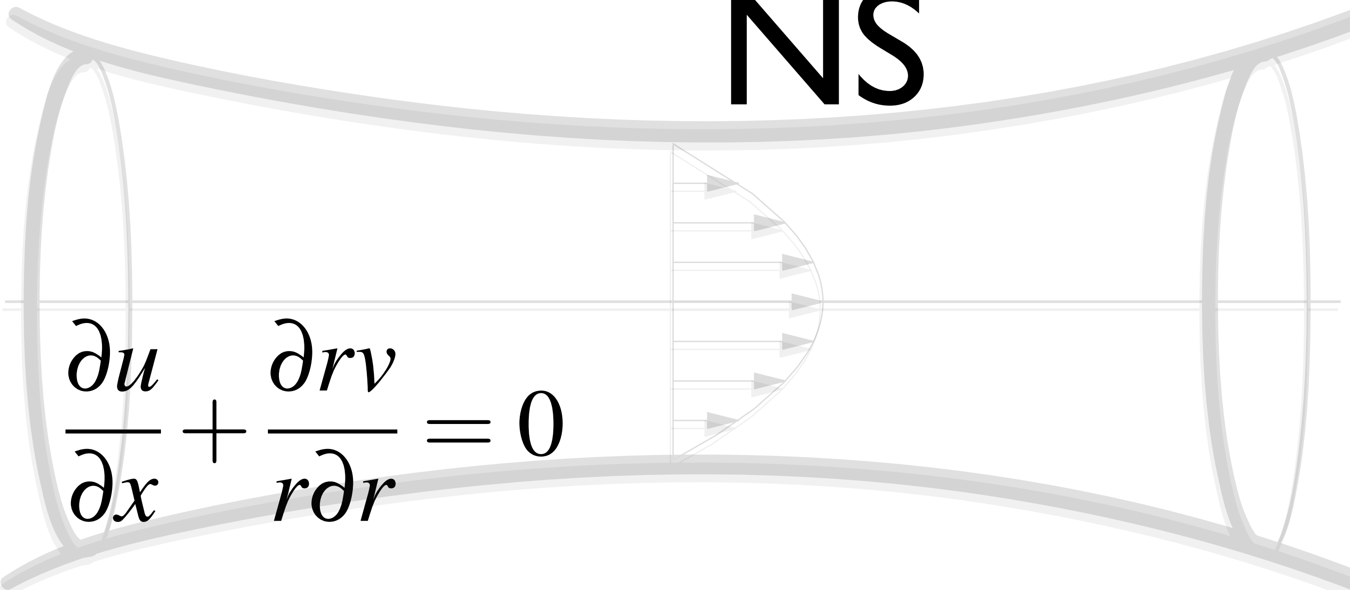


$$R << \lambda$$



U_0
R
λ

NS

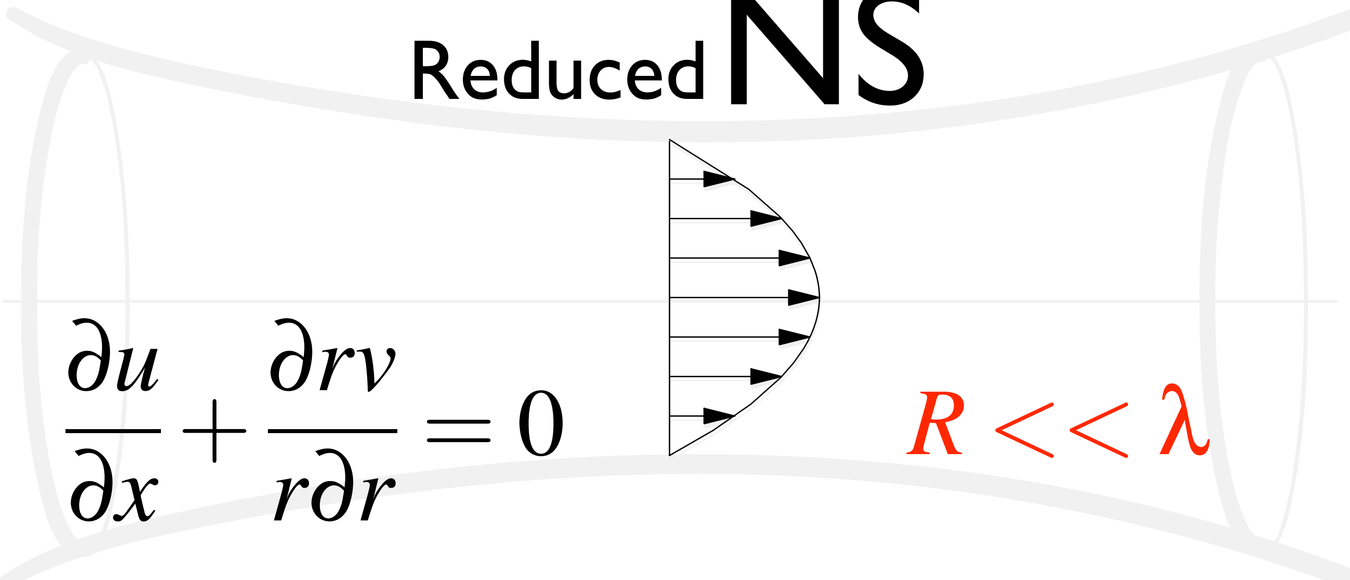


$$\frac{\partial u}{\partial x} + \frac{\partial r v}{r \partial r} = 0$$

$$\frac{\partial u}{\partial t} + u\frac{\partial}{\partial x}u + v\frac{\partial}{\partial r}u = -\frac{\partial p}{\rho \partial x} + \nu\frac{\partial^2}{\partial x^2}u + \nu\frac{\partial}{r\partial r}r\frac{\partial u}{\partial r}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial}{\partial x}v + v\frac{\partial}{\partial r}v = -\frac{\partial p}{\rho \partial r} + \nu\frac{\partial^2}{\partial x^2}v + \nu\frac{\partial}{r\partial r}r\frac{\partial v}{\partial r}$$

Reduced NS



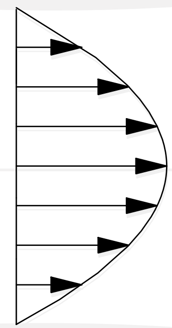
$$\frac{\partial u}{\partial x} + \frac{\partial r v}{r \partial r} = 0$$

$R << \lambda$

$$\frac{\partial u}{\partial t} + u\frac{\partial}{\partial x}u + v\frac{\partial}{\partial r}u = -\frac{\partial p}{\rho \partial x} + \nu \cancel{\frac{\partial^2}{\partial x^2}} u + \nu \frac{\partial}{r \partial r} r \frac{\partial u}{\partial r}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial}{\partial x}v + v\frac{\partial}{\partial r}v = -\frac{\partial p}{\rho \partial r} + \nu \cancel{\frac{\partial^2}{\partial x^2}} v + \nu \frac{\partial}{r \partial r} r \frac{\partial v}{\partial r}$$

RNS/P



$$\frac{\partial u}{\partial x} + \frac{\partial r v}{r \partial r} = 0$$

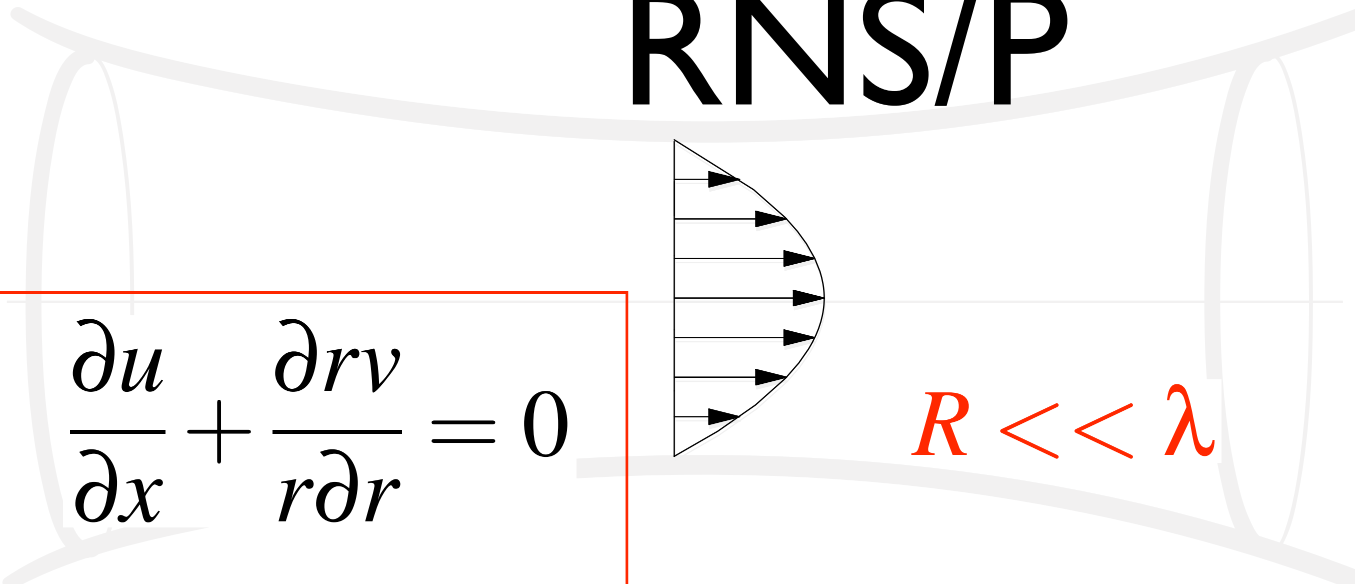
$R << \lambda$

$V \sim U_0 \frac{R}{\lambda}$

$$\frac{\partial u}{\partial t} + u\frac{\partial}{\partial x}u + v\frac{\partial}{\partial r}u = -\frac{\partial p}{\rho \partial x} + \nu \cancel{\frac{\partial^2}{\partial x^2}} u + \nu \frac{\partial}{r\partial r} r \frac{\partial u}{\partial r}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial}{\partial x}v + v\frac{\partial}{\partial r}v = -\frac{\partial p}{\rho \partial r} + \nu \cancel{\frac{\partial^2}{\partial x^2}} v + \nu \frac{\partial}{r\partial r} r \frac{\partial v}{\partial r}$$

RNS/P



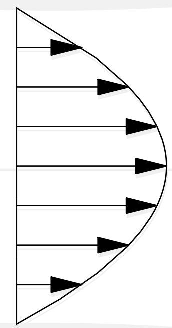
$$\frac{\partial u}{\partial x} + \frac{\partial rv}{r \partial r} = 0$$

$$R << \lambda \qquad V \sim U_0 \frac{R}{\lambda}$$

$$\frac{\partial u}{\partial t} + u \frac{\partial}{\partial x} u + v \frac{\partial}{\partial r} u = -\frac{\partial p}{\rho \partial x} + \nu \cancel{\frac{\partial^2}{\partial x^2}} u + \nu \frac{\partial}{r \partial r} r \frac{\partial u}{\partial r}$$

$$\frac{\partial v}{\partial t} + u \frac{\partial}{\partial x} v + v \frac{\partial}{\partial r} v = -\frac{\partial p}{\rho \partial r} + \nu \cancel{\frac{\partial^2}{\partial x^2}} v + \nu \frac{\partial}{r \partial r} r \frac{\partial v}{\partial r}$$

RNS/P



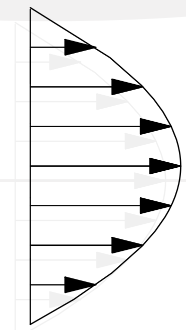
$$\frac{\partial u}{\partial x} + \frac{\partial rv}{r\partial r} = 0$$

$$R << \lambda \qquad V \sim U_0 \frac{R}{\lambda}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial}{\partial x}u + v\frac{\partial}{\partial r}u = -\frac{\partial p}{\rho \partial x} + \nu \cancel{\frac{\partial^2}{\partial x^2}} u + \nu \frac{\partial}{r\partial r} r \frac{\partial u}{\partial r}$$

$$\cancel{\frac{\partial v}{\partial t}} + u\cancel{\frac{\partial}{\partial x}}v + v\cancel{\frac{\partial}{\partial r}}v = -\frac{\partial p}{\rho \partial r} + \nu \cancel{\frac{\partial^2}{\partial x^2}} v + \nu \frac{\partial}{r\partial r} \cancel{r \frac{\partial v}{\partial r}}$$

RNS/P

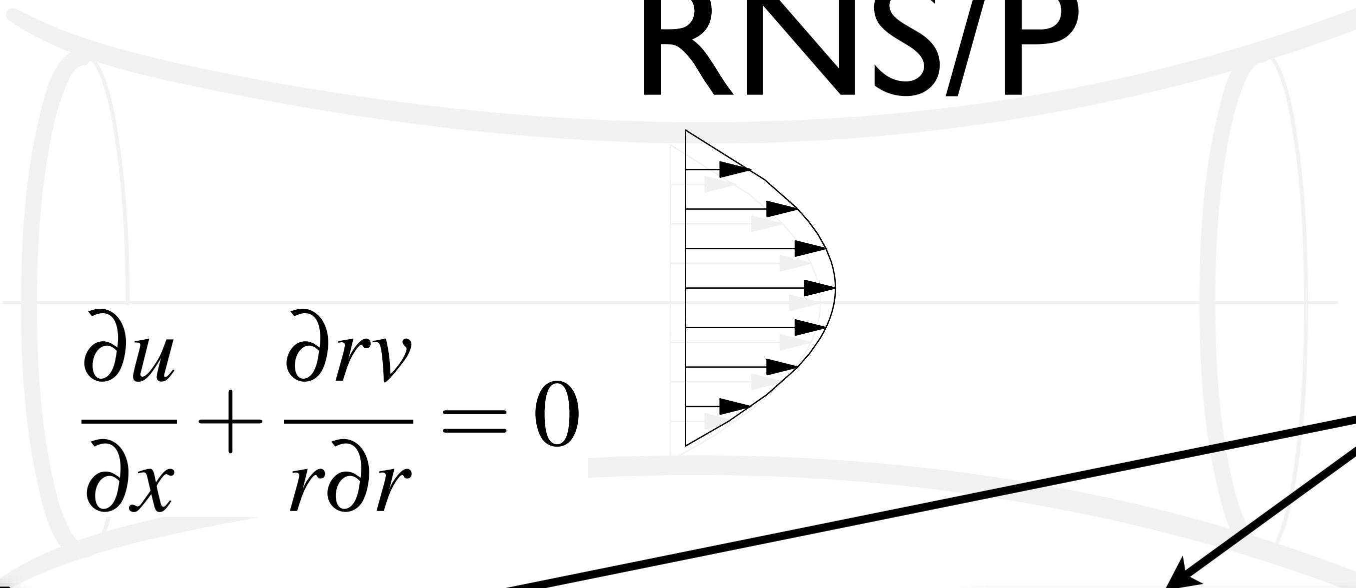


$$\frac{\partial u}{\partial x}+\frac{\partial rv}{r\partial r}=0$$

$$\frac{\partial u}{\partial t}+u\frac{\partial}{\partial x}u+v\frac{\partial}{\partial r}u=-\frac{\partial p}{\rho\partial x}+\nu\frac{\partial}{r\partial r}r\frac{\partial u}{\partial r}$$

$$0=-\frac{\partial p}{\rho\partial r}$$

RNS/P

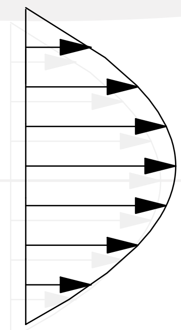


$$\frac{\partial u}{\partial x} + \frac{\partial r v}{r\partial r} = 0$$

$$\frac{\partial u}{\partial t} + u\frac{\partial}{\partial x}u + v\frac{\partial}{\partial r}u = -\frac{\partial p}{\rho\partial x} + \nu\frac{\partial}{r\partial r}r\frac{\partial u}{\partial r}$$

$$0 = -\frac{\partial p}{\rho\partial r}$$

RNS/P



$$\frac{\partial u}{\partial x} + \frac{\partial r v}{r \partial r} = 0$$

$$\alpha = R\sqrt{\frac{\omega}{\nu}}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial}{\partial x}u + v\frac{\partial}{\partial r}u = -\frac{\partial p}{\rho \partial x} + \nu \frac{\partial}{r \partial r} r \frac{\partial u}{\partial r}$$

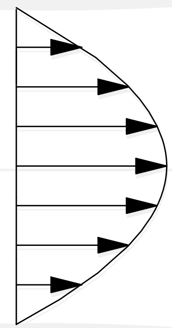
$$0 = -\frac{\partial p}{\rho \partial r}$$

1/(Womersley)^2

final system

RNS/P

Prandtl

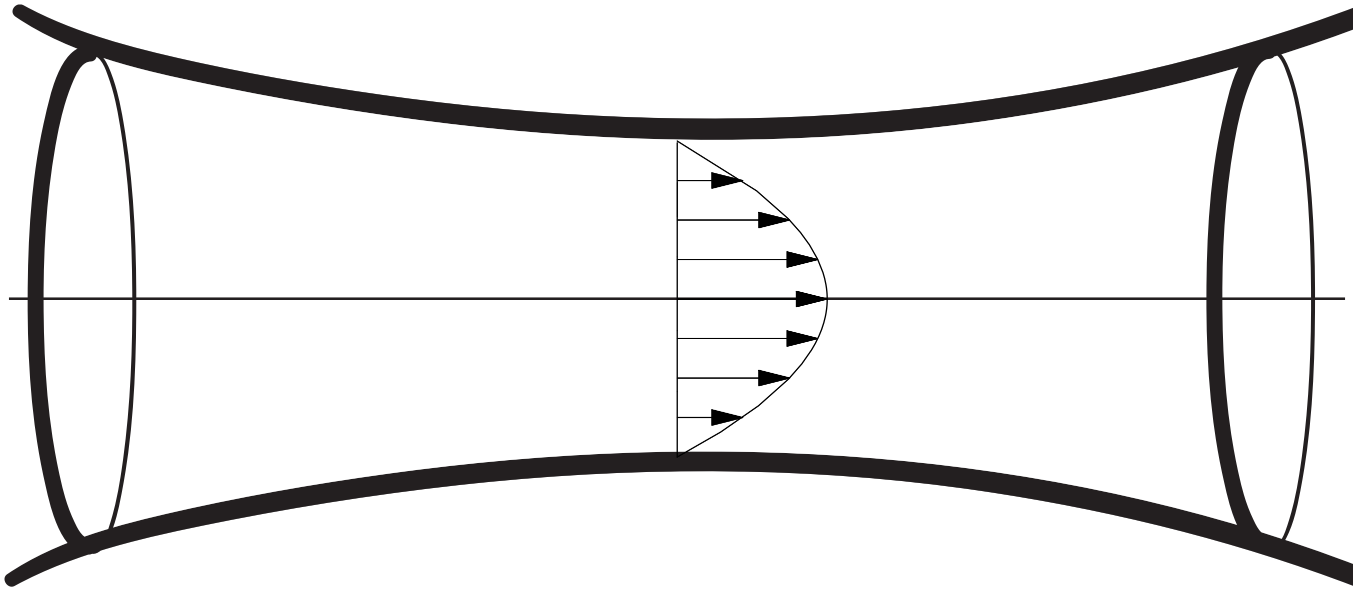


$$\frac{\partial u}{\partial x} + \frac{\partial r v}{r \partial r} = 0$$

$$\frac{\partial u}{\partial t} + u\frac{\partial}{\partial x}u + v\frac{\partial}{\partial r}u = -\frac{\partial p}{\rho \partial x} + \nu \frac{\partial}{r \partial r} r \frac{\partial u}{\partial r}$$

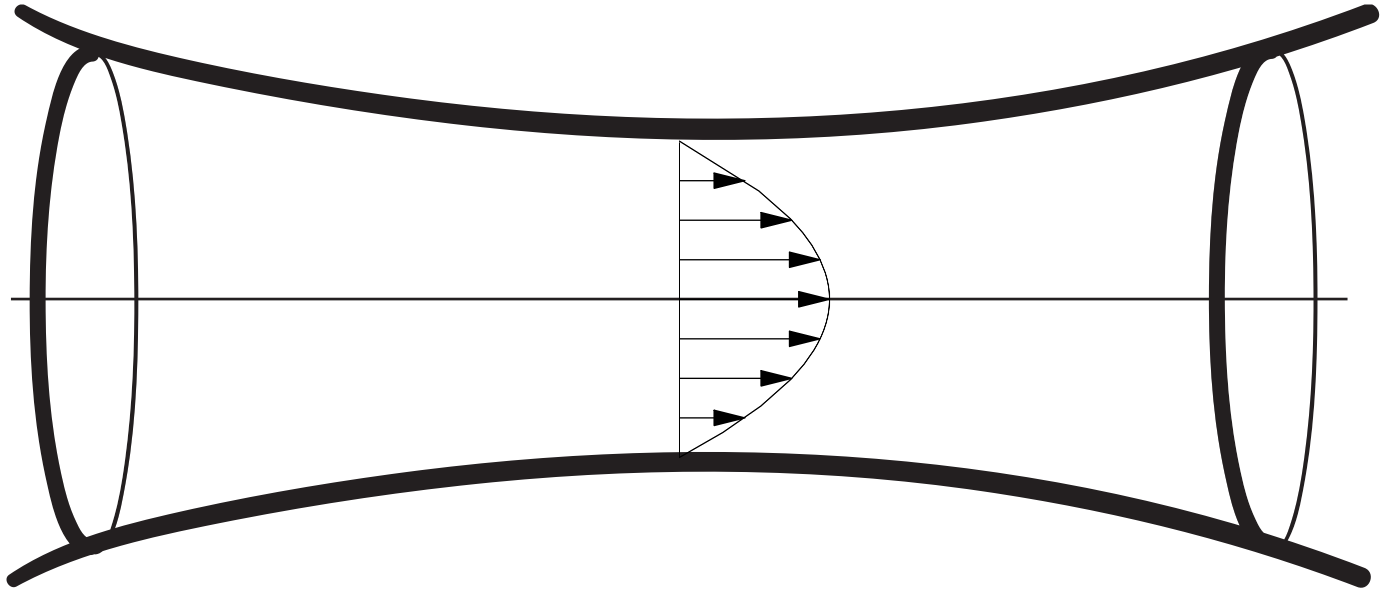
$$0 = -\frac{\partial p}{\rho \partial r}$$

Boundary conditions



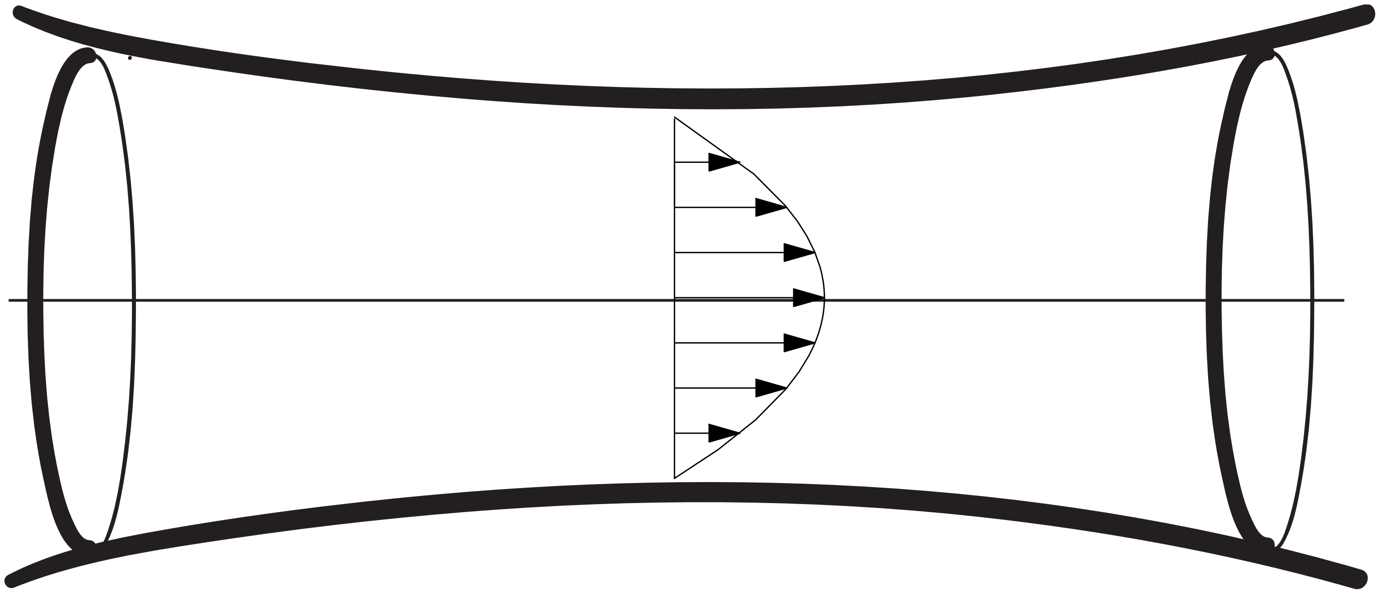
Rigid wall: $u = v = 0$

Boundary conditions



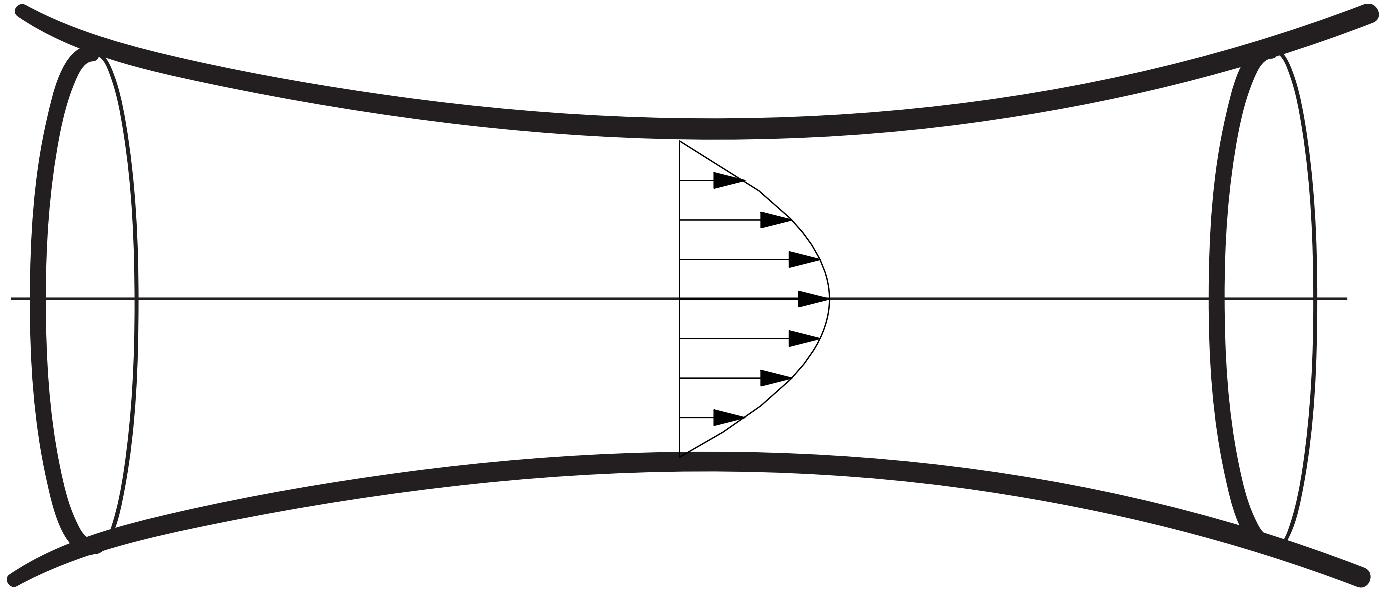
moving wall

Boundary conditions



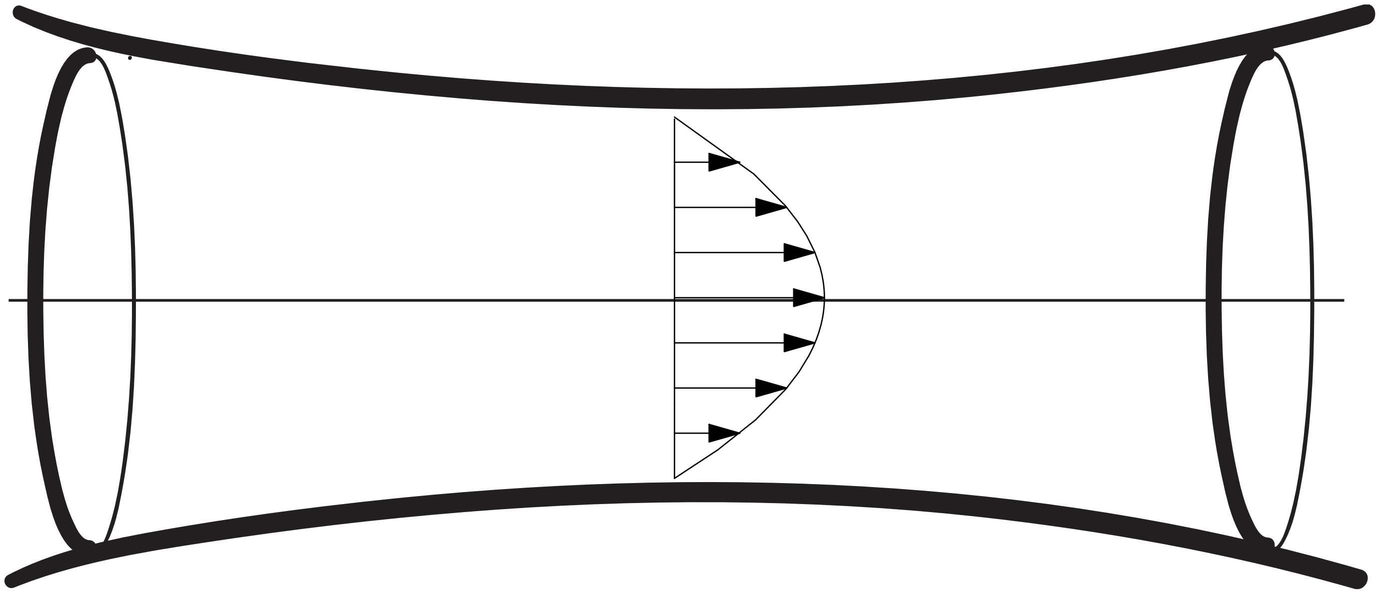
moving wall

Boundary conditions



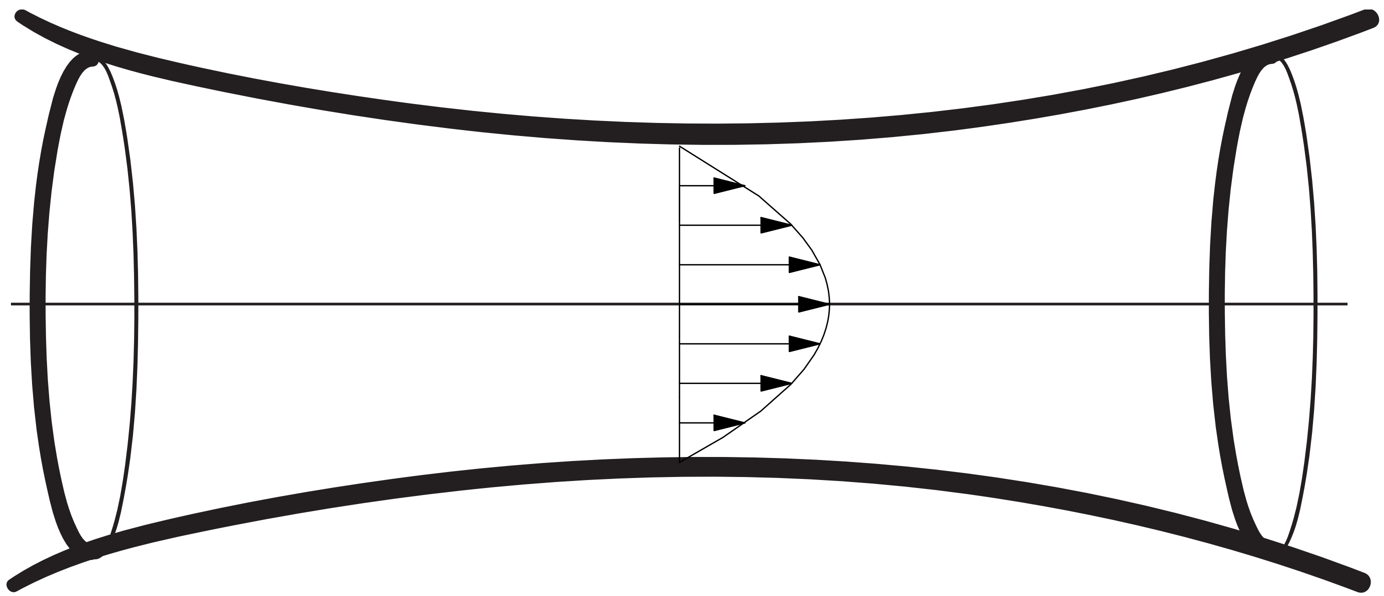
moving wall

Boundary conditions



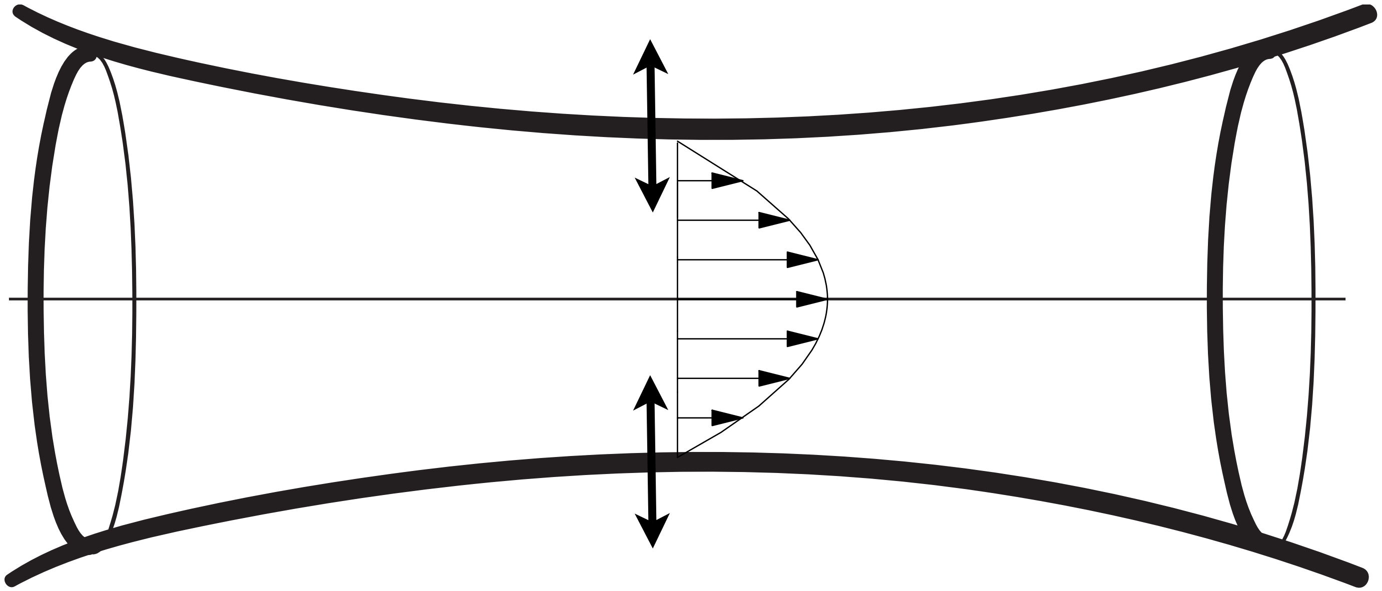
moving wall

Boundary conditions



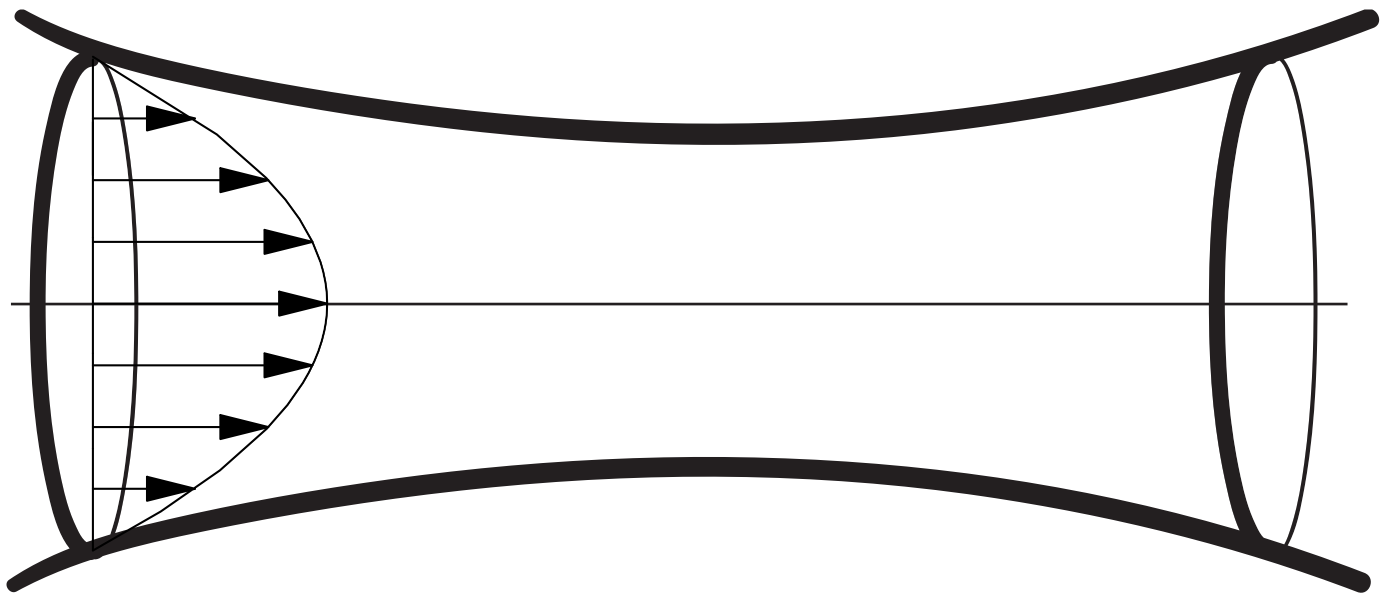
moving wall

Boundary conditions



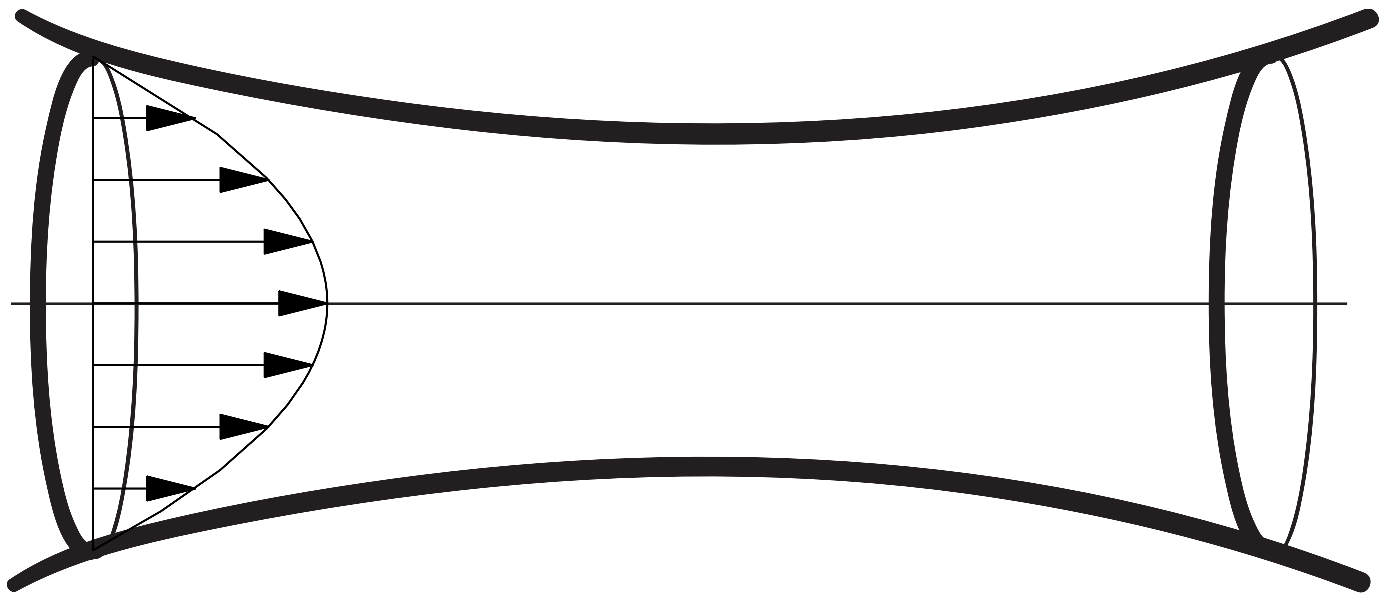
moving wall $$v = \frac{\partial R}{\partial t}$$

Boundary conditions



First given profile:

Boundary conditions

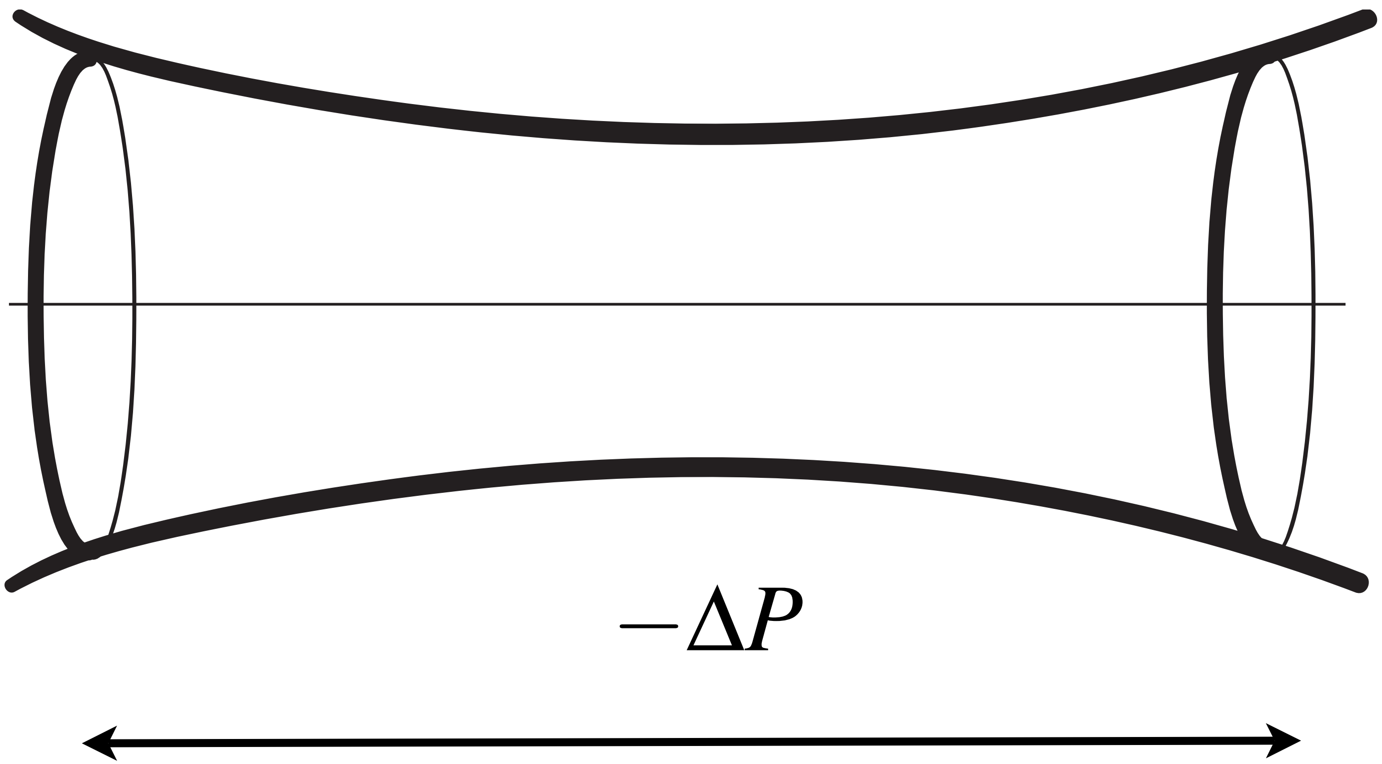


First given profile:

marching procedure

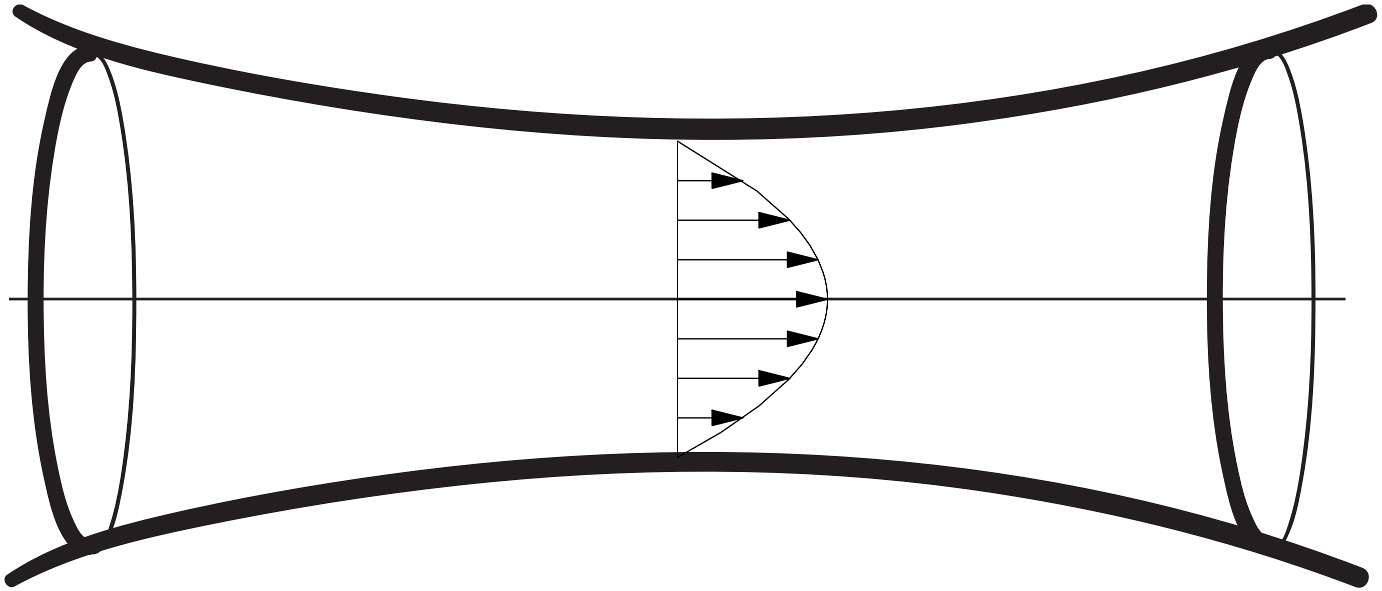
distribution of pressure is a result

Boundary conditions

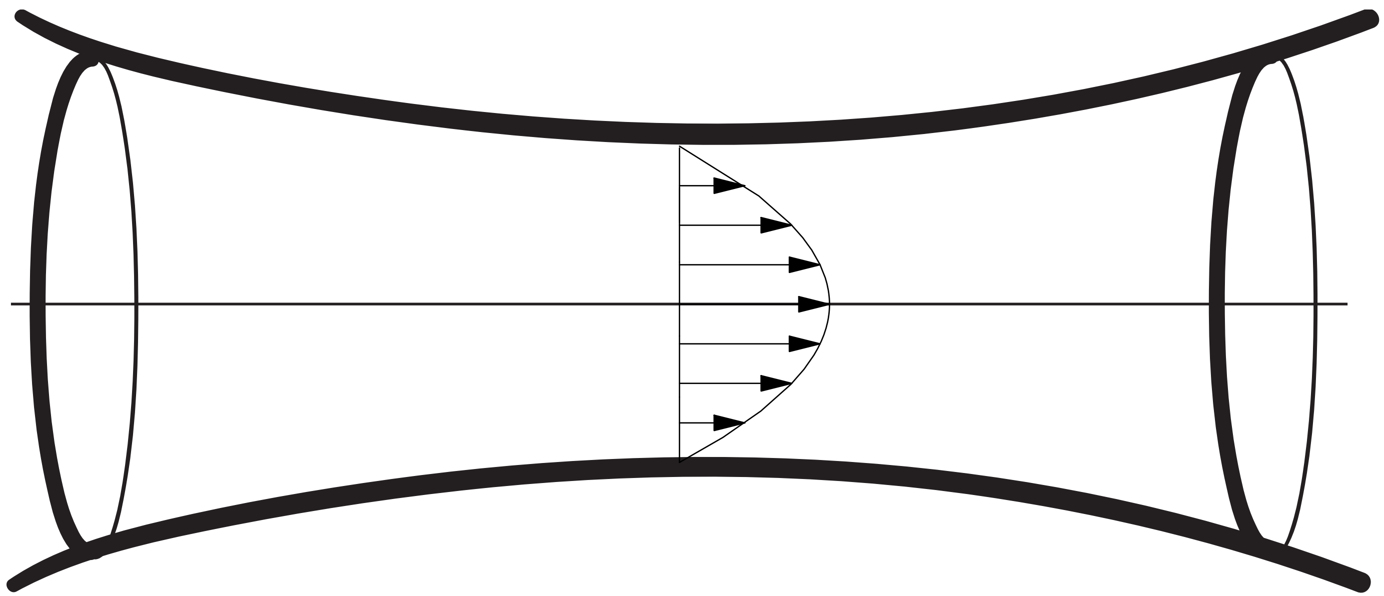


or given pressure drop
by Newton iteration on the entrance flux

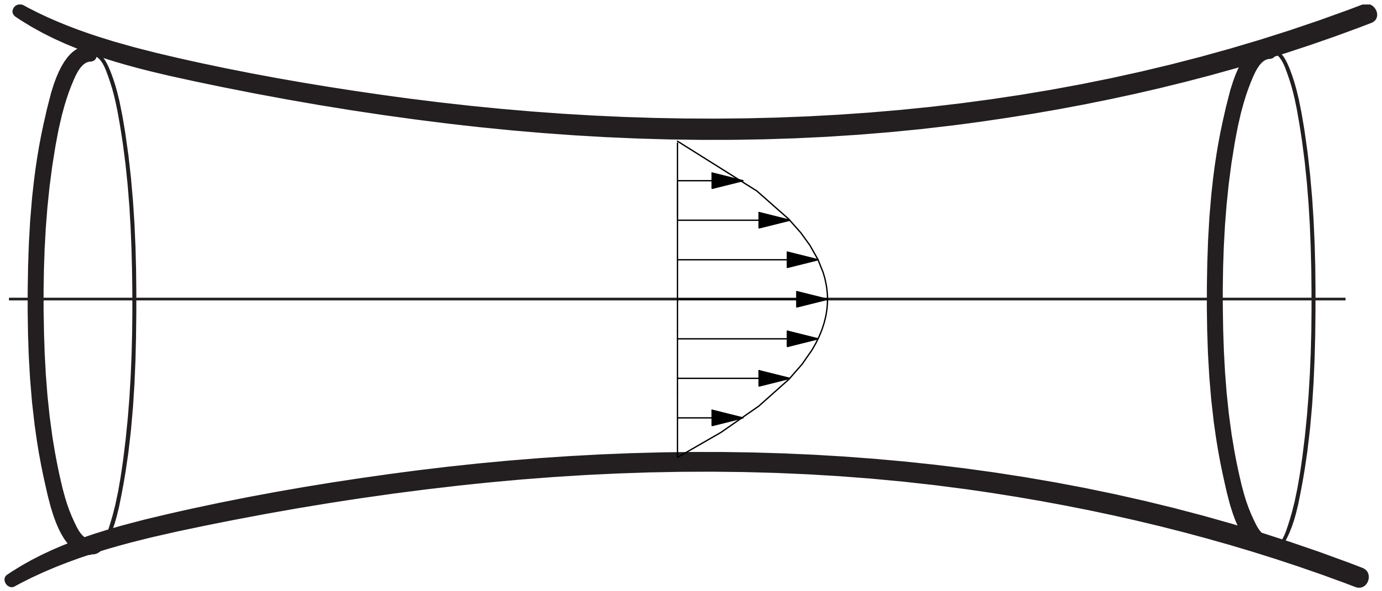
Numerical resolution



finite differences,
implicit in time

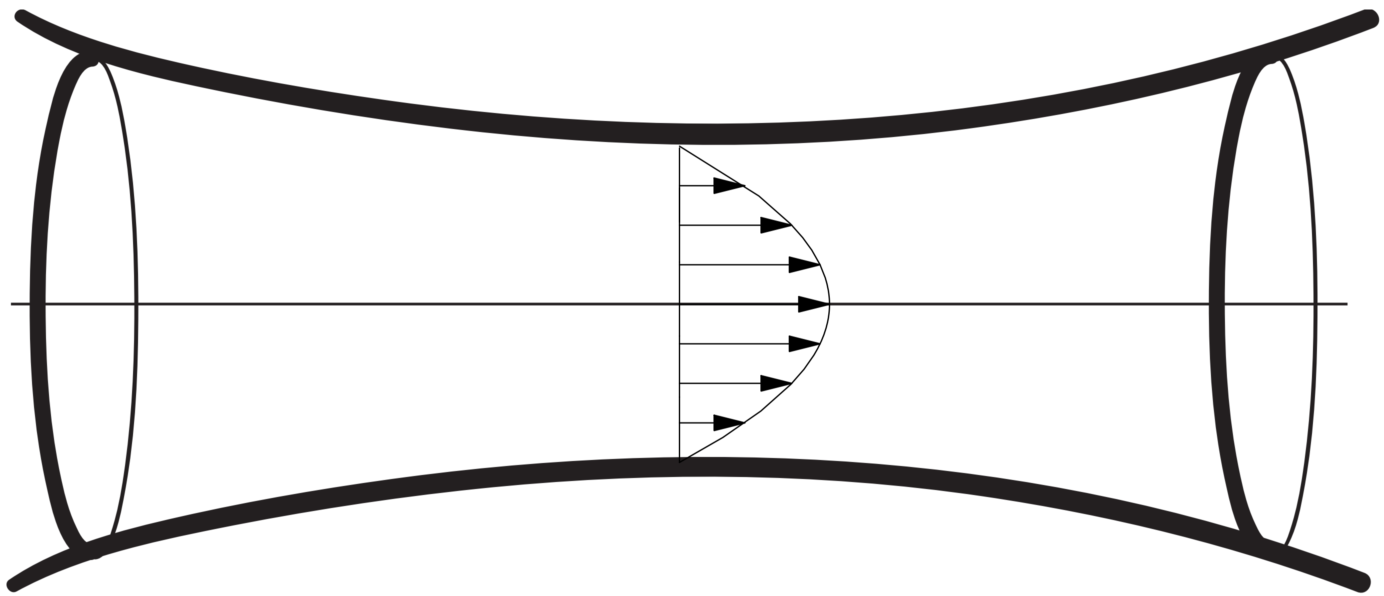


$$\frac{u^* - u^n}{\Delta t} + NL(u^n) = -\frac{\partial p^{given}}{\partial x} + \frac{\partial}{r\partial r} r \frac{\partial u^*}{\partial r}$$



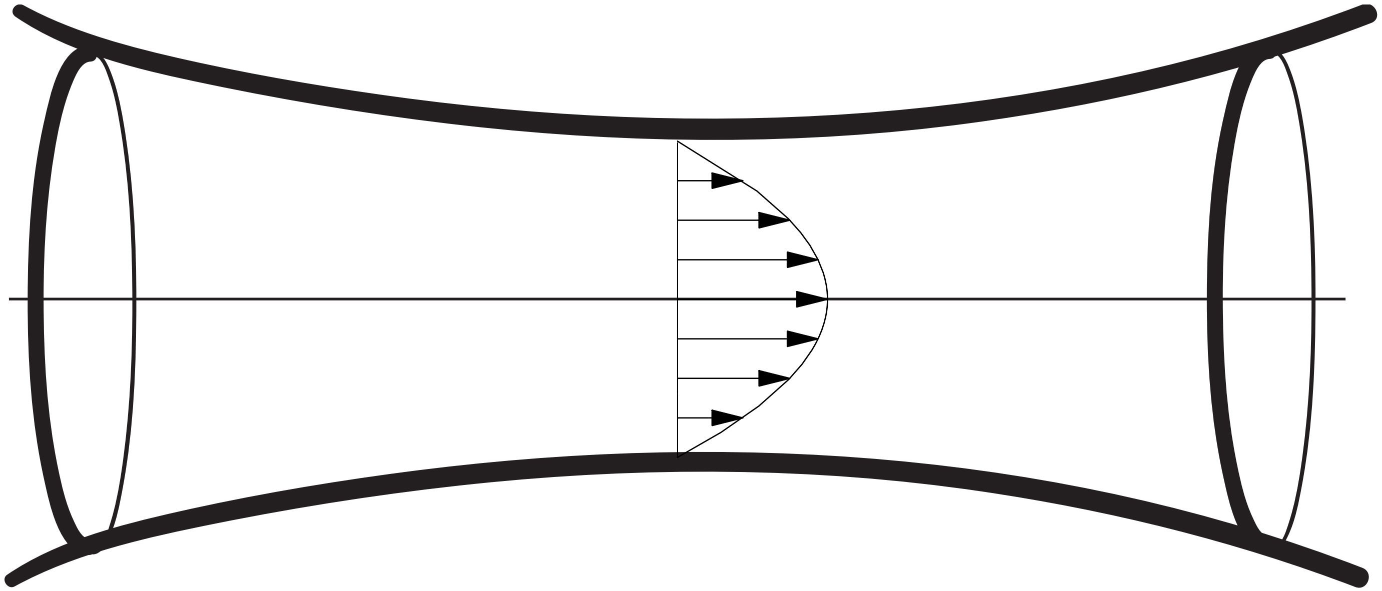
$$\frac{u^* - u^n}{\Delta t} + NL(u^n) = -\frac{\partial p^{given}}{\partial x} + \frac{\partial}{r\partial r} r \frac{\partial u^*}{\partial r}$$

$$p^{given} \rightarrow u^*$$



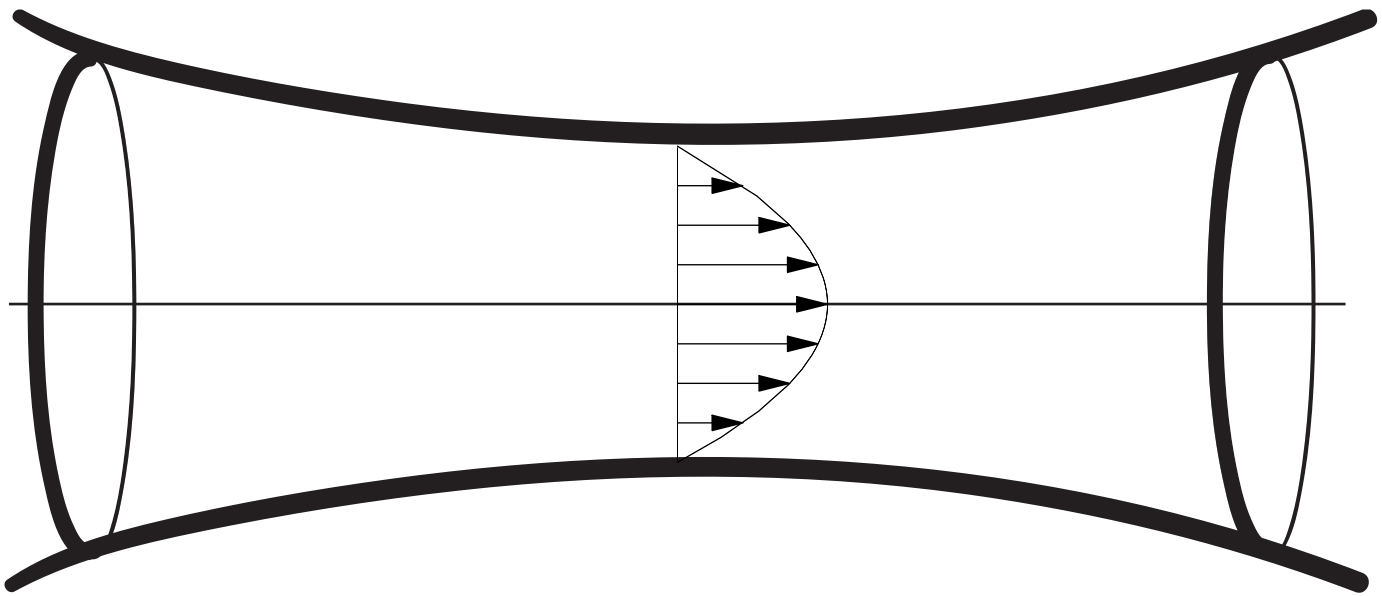
$$\frac{u^* - u^n}{\Delta t} + NL(u^n) = -\frac{\partial p^{given}}{\partial x} + \frac{\partial}{r\partial r} r \frac{\partial u^*}{\partial r}$$

$$p^{given} \rightarrow u^* \qquad\qquad rv^*(R) = -\int_0^R r \frac{\partial u^*}{\partial x} dr$$



$$\frac{u^* - u^n}{\Delta t} + NL(u^n) = -\frac{\partial p^{given}}{\partial x} + \frac{\partial}{r\partial r} r \frac{\partial u^*}{\partial r}$$

$$p^{given} \rightarrow u^* \qquad\qquad rv^*(R) = -\int_0^R r\frac{\partial u^*}{\partial x} dr \;\Big|\; \frac{\partial R}{\partial t} \; ? \quad 0?$$

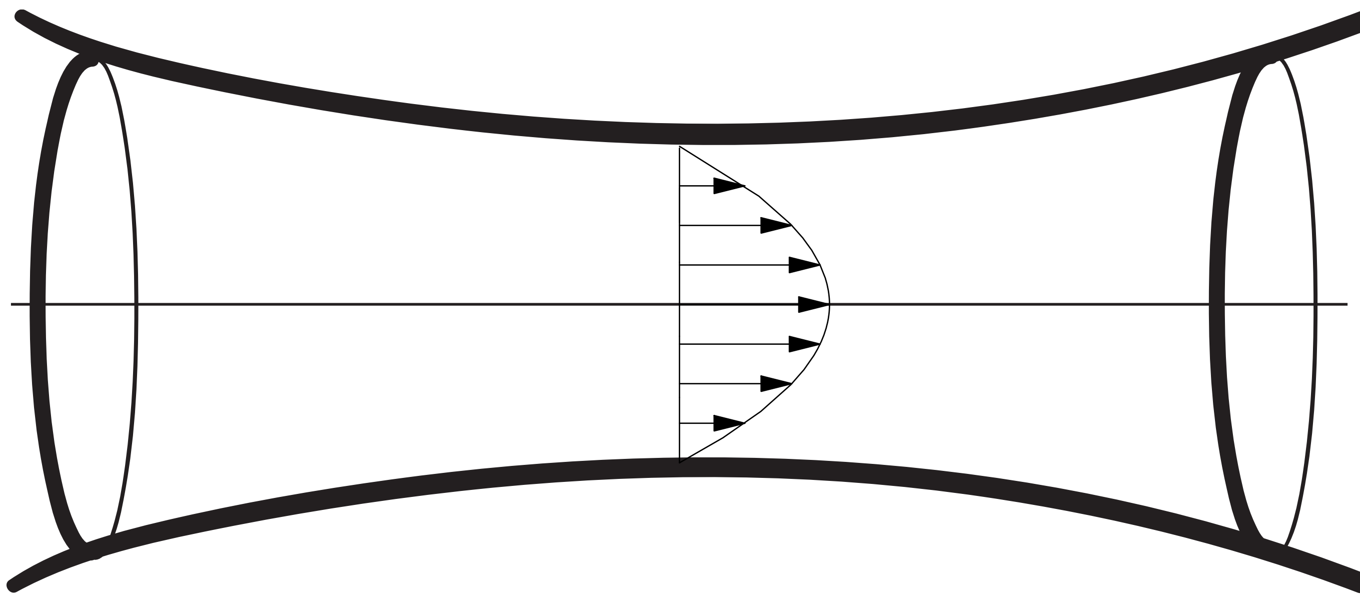


Newton on the pressure to obtain the boundary condition

$$\frac{u^* - u^n}{\Delta t} + NL(u^n) = -\frac{\partial p^{given}}{\partial x} + \frac{\partial}{r\partial r} r \frac{\partial u^*}{\partial r}$$

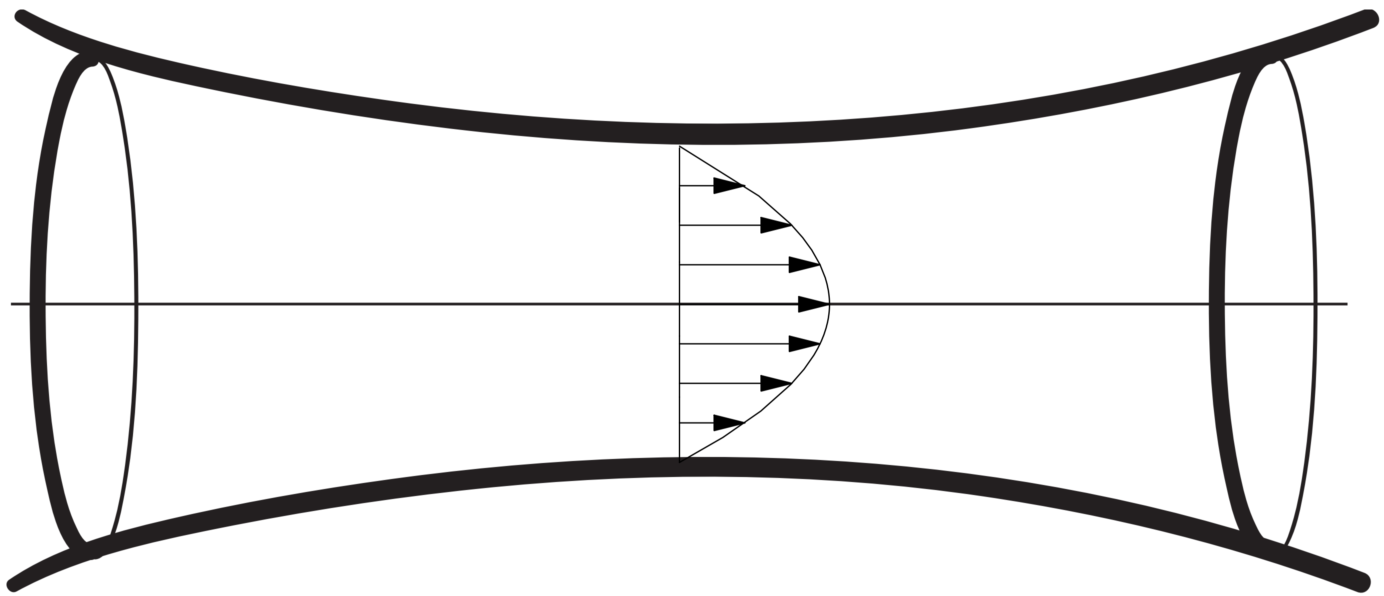
$$p^{given} \rightarrow u^* \longrightarrow rv^*(R) = -\int_0^R r \frac{\partial u^*}{\partial x} dr \Big| \frac{\partial R}{\partial t} ?$$

0?

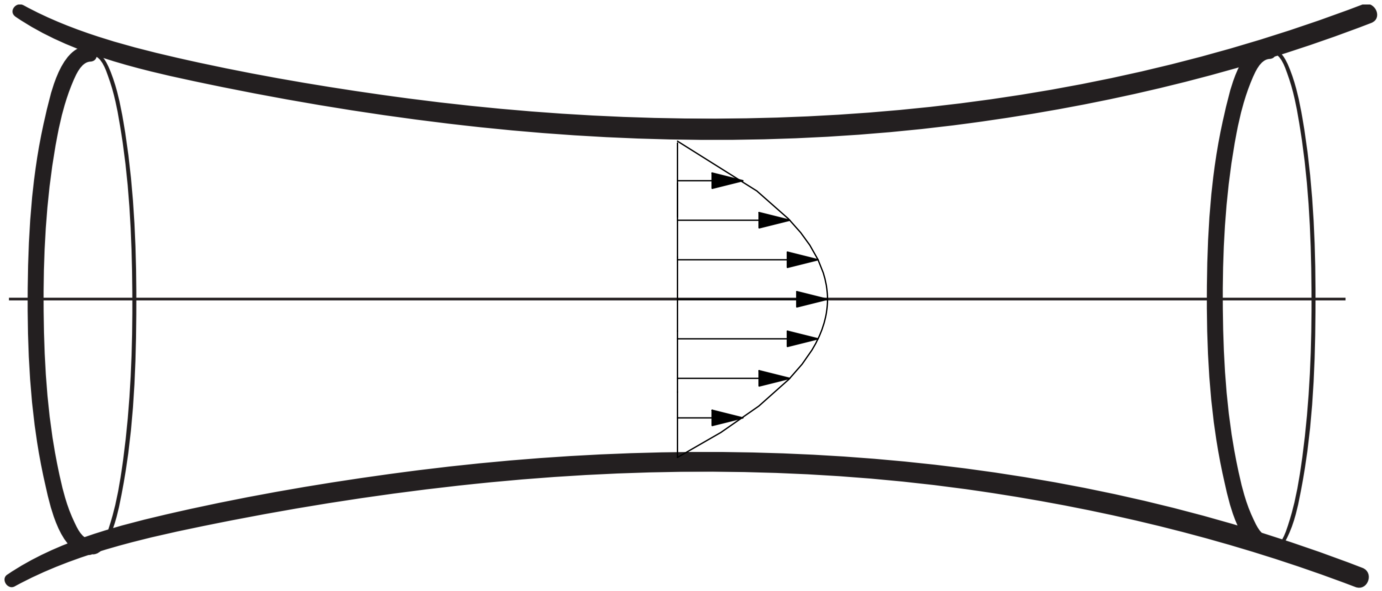


Pressure is a result of the computation

Integral resolution

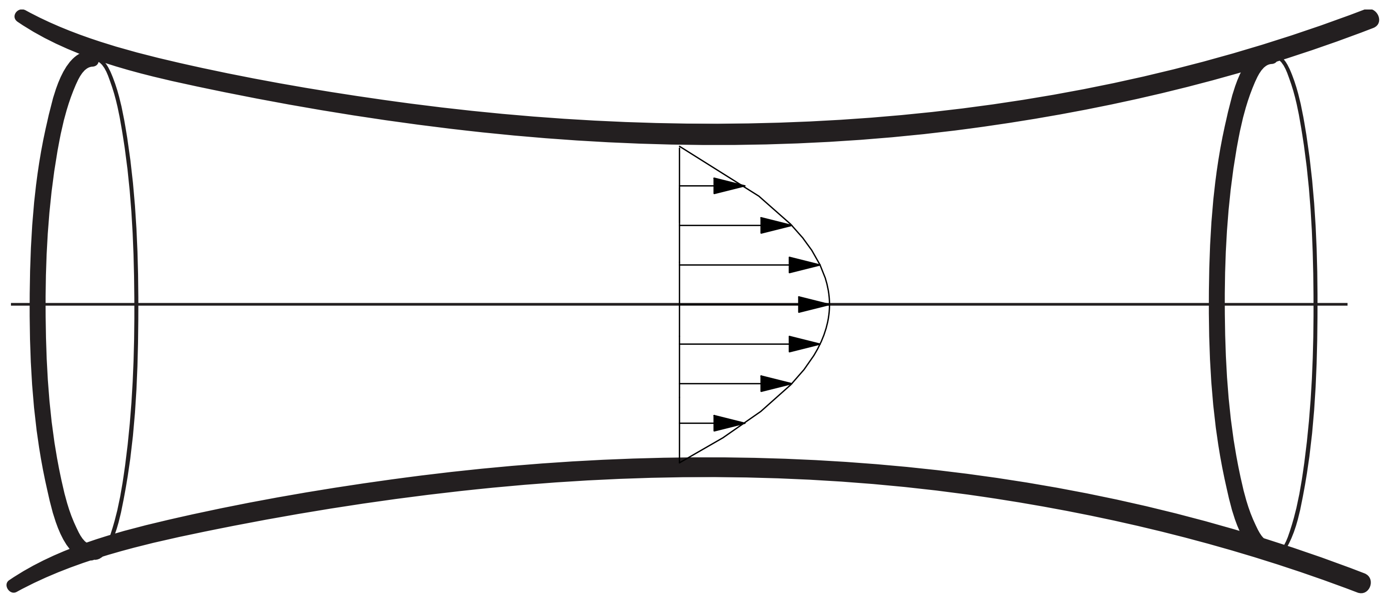


Integral resolution

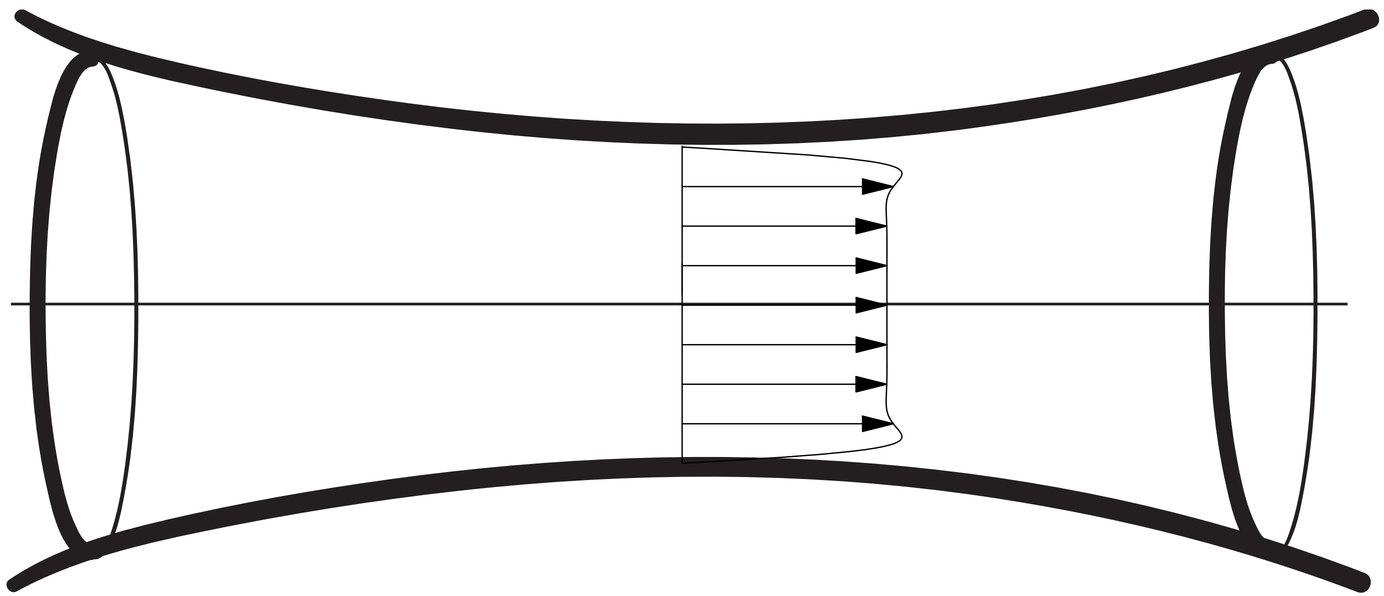


- integral system (1D) is included in RNSP
- we compute a more real profile

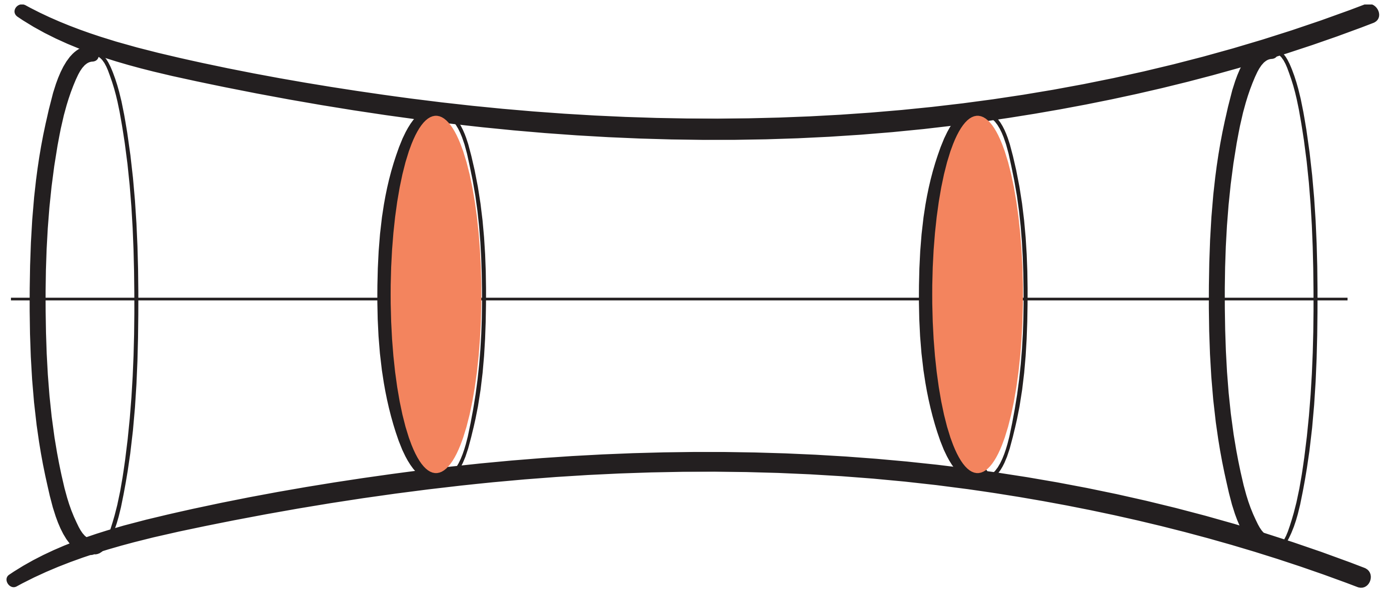
Integral resolution



Integral resolution

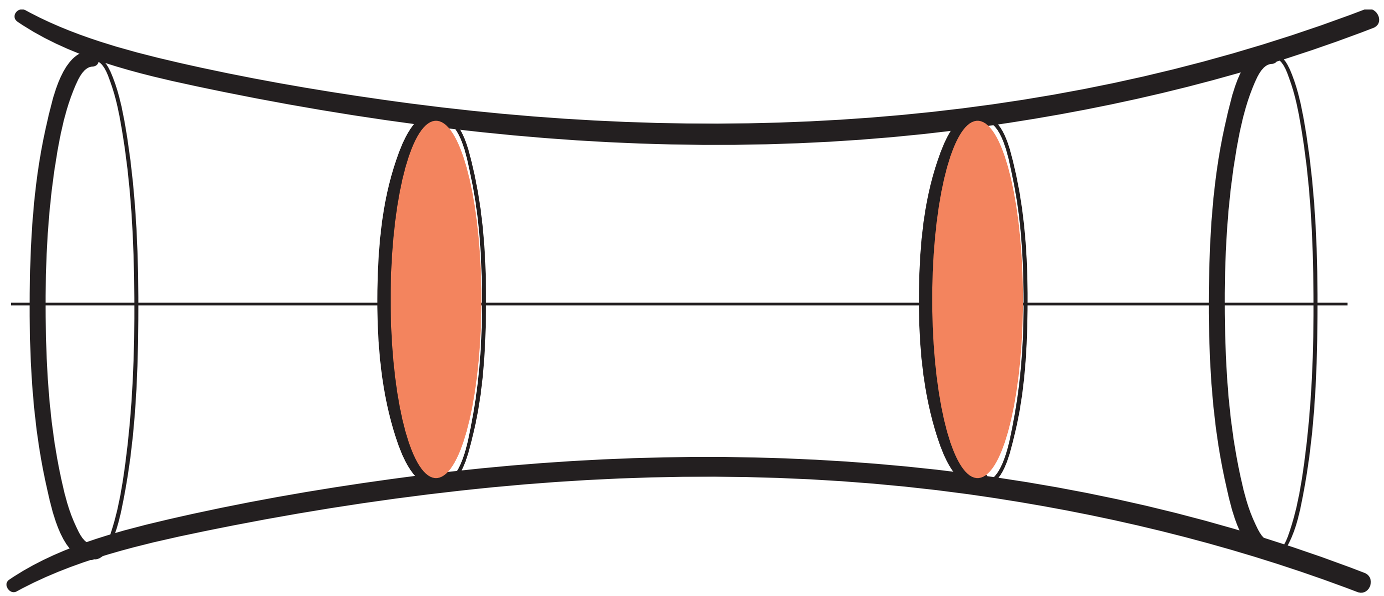


Integral resolution



$$Q = \int_0^R 2\pi r u dr$$

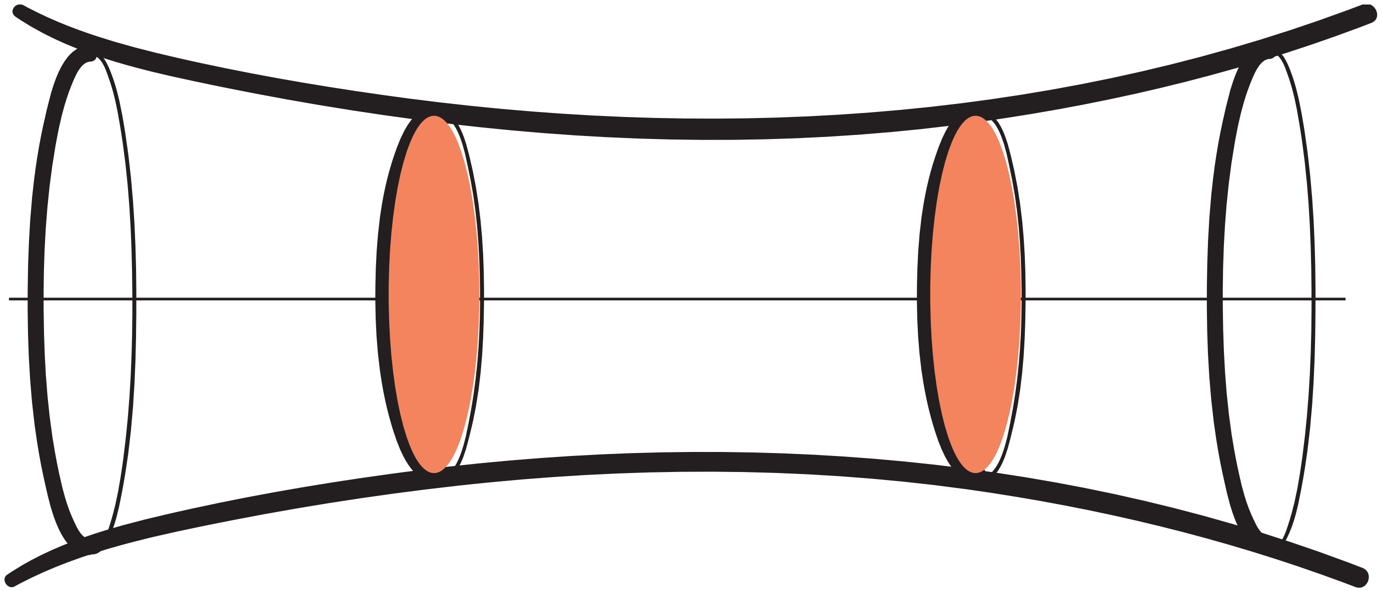
Integral resolution



$$Q = \int_0^R 2\pi r u dr$$

$$\frac{\partial u}{\partial x} + \frac{\partial r v}{r \partial r} = 0$$

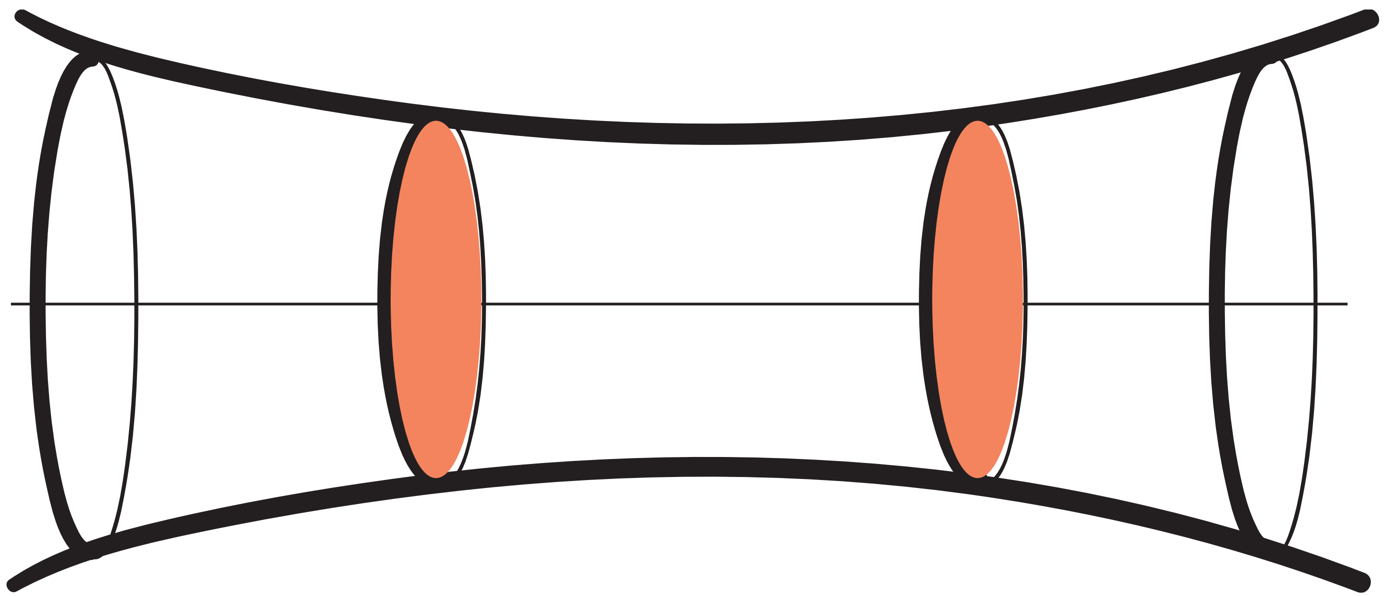
Integral resolution



$$Q = \int_0^R 2\pi r u dr$$

$$\int_0^R 2\pi r dr \cdot \left(\frac{\partial u}{\partial x} + \frac{\partial r v}{r \partial r} \right) = 0$$

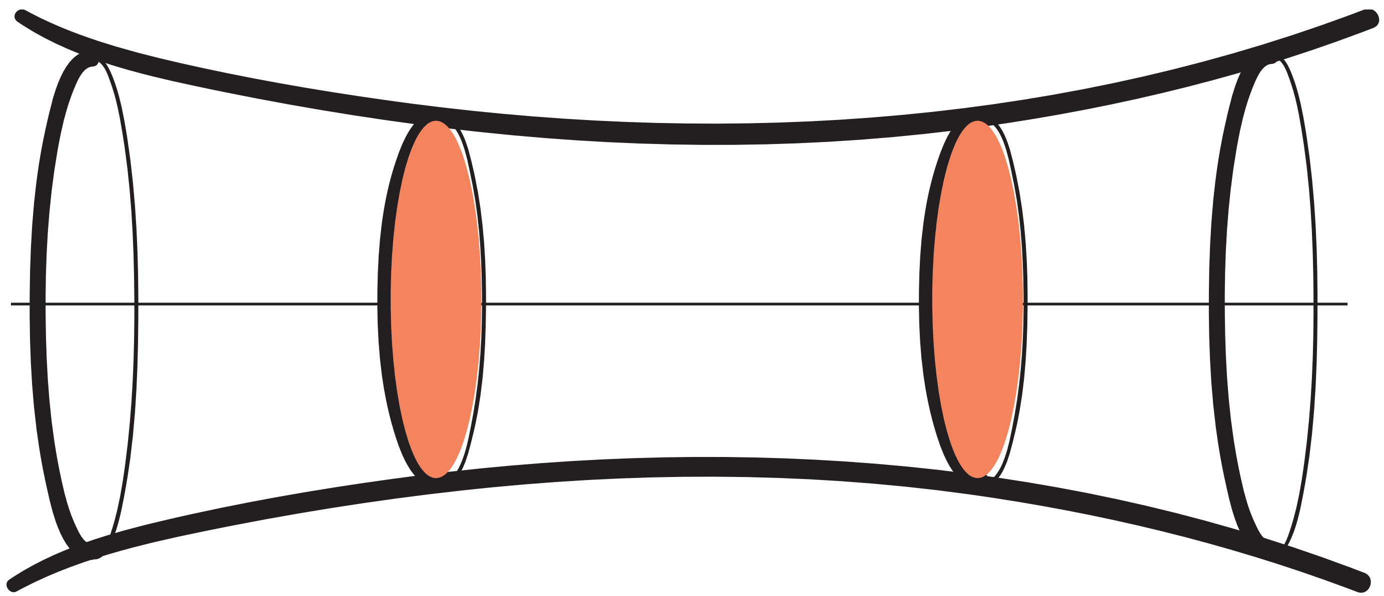
Integral resolution



$$Q = \int_0^R 2\pi r u dr$$

$$\int_0^R 2\pi r dr \cdot \left(\frac{\partial u}{\partial x} + \frac{\partial r v}{r \partial r} \right) = 0 \rightarrow \frac{\partial (2\pi R^2)}{\partial t} + \frac{\partial Q}{\partial x} = 0$$

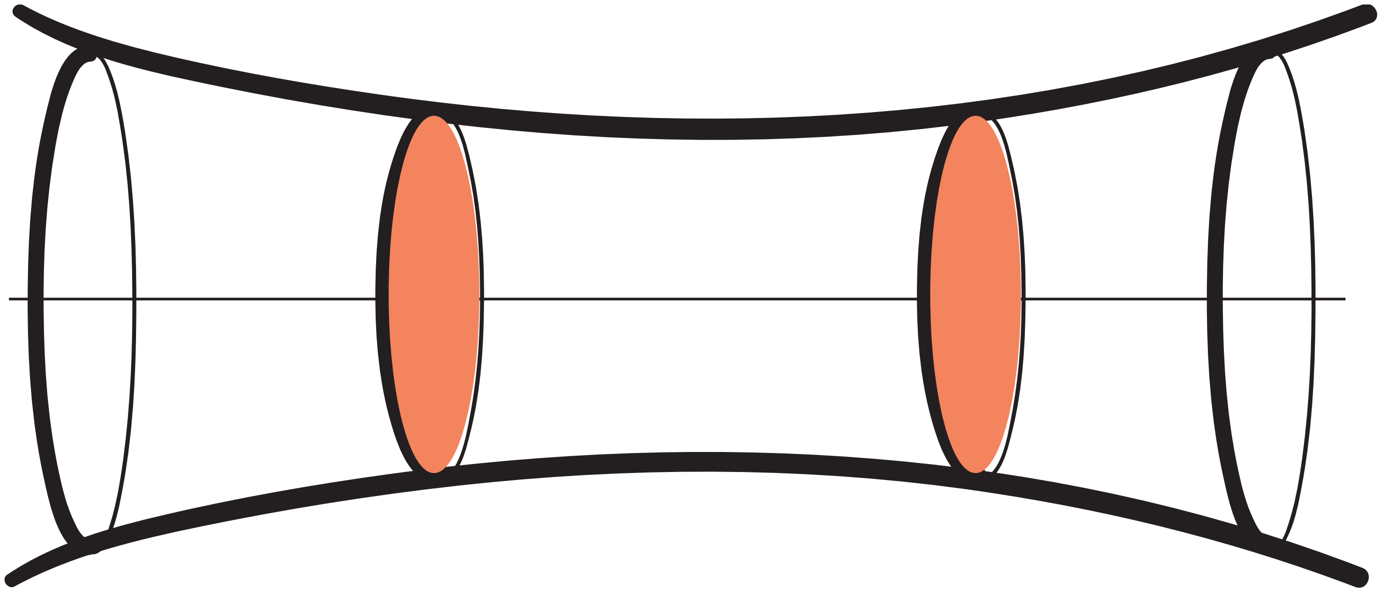
Integral resolution



$$Q = \int_0^R 2\pi r u dr$$

$$\frac{\partial(2\pi R^2)}{\partial t} + \frac{\partial Q}{\partial x} = 0$$

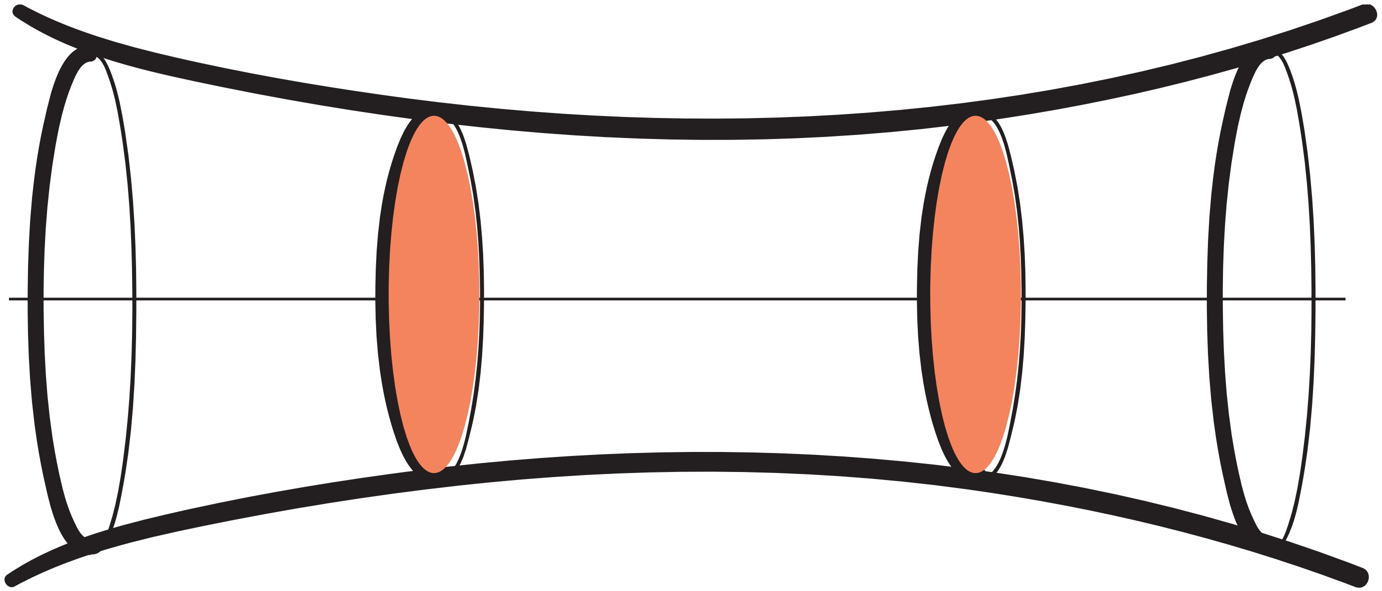
Integral resolution



$$Q = \int_0^R 2\pi r u \, dr$$

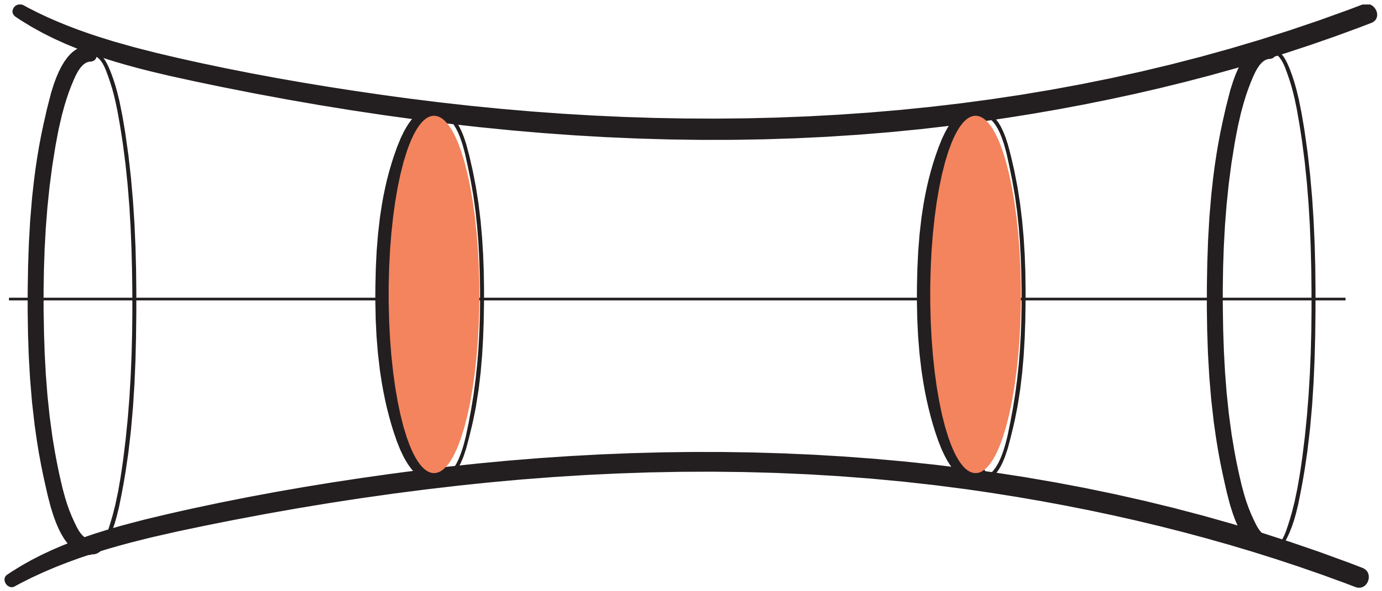
$$\tau = \frac{\partial u}{\partial r}$$

Integral resolution



$$Q = \int_0^R 2\pi r u \, dr \qquad Q_2 = \int_0^R 2\pi r u^2 \, dr \qquad \tau = \frac{\partial u}{\partial r}$$

Integral resolution

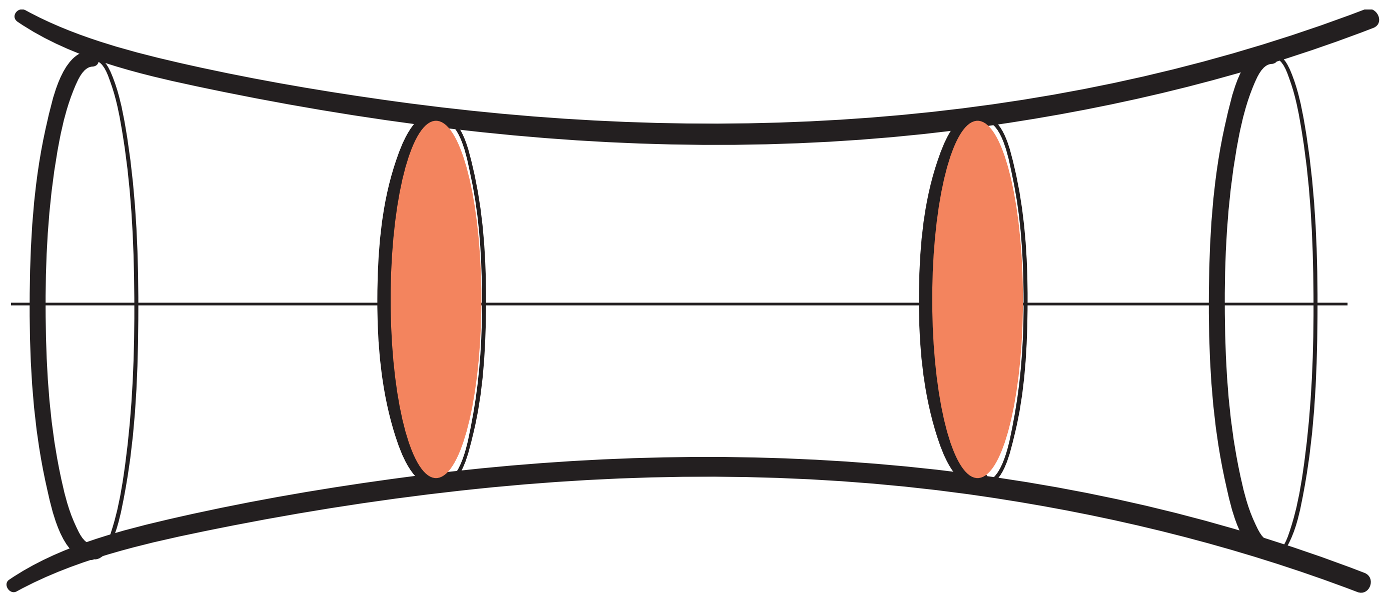


$$Q = \int_0^R 2\pi r u \, dr \qquad Q_2 = \int_0^R 2\pi r u^2 dr \qquad \tau = \frac{\partial u}{\partial r}$$

$$\int \left(\frac{\partial u}{\partial t} + u\frac{\partial}{\partial x}u + v\frac{\partial}{\partial r}u = -\frac{\partial p}{\rho \partial x} + \nu \frac{\partial}{r\partial r} r \frac{\partial u}{\partial r} \right)$$

$$0 = -\frac{\partial p}{\rho \partial r}$$

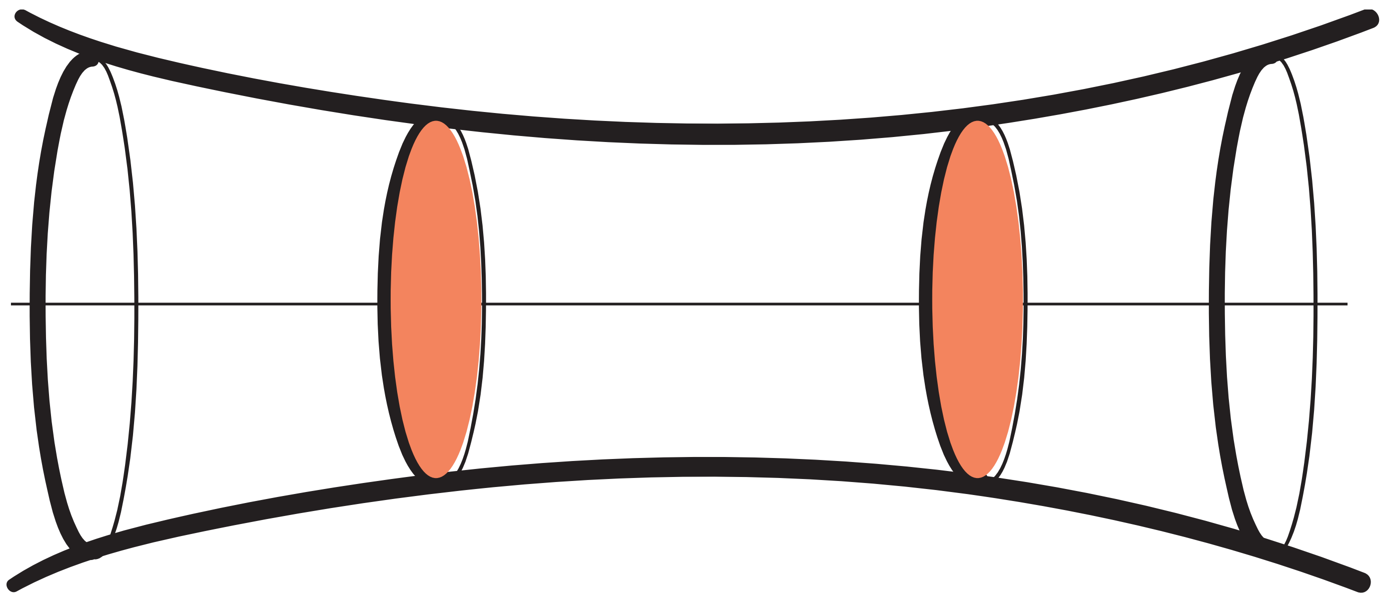
Integral resolution



$$Q = \int_0^R 2\pi r u\, dr \qquad Q_2 = \int_0^R 2\pi r u^2 dr \qquad \tau = \frac{\partial u}{\partial r}$$

$$\frac{\partial Q}{\partial t} + \frac{\partial Q_2}{\partial t} = -(2\pi R^2)\frac{\partial p}{\partial x} - \tau$$

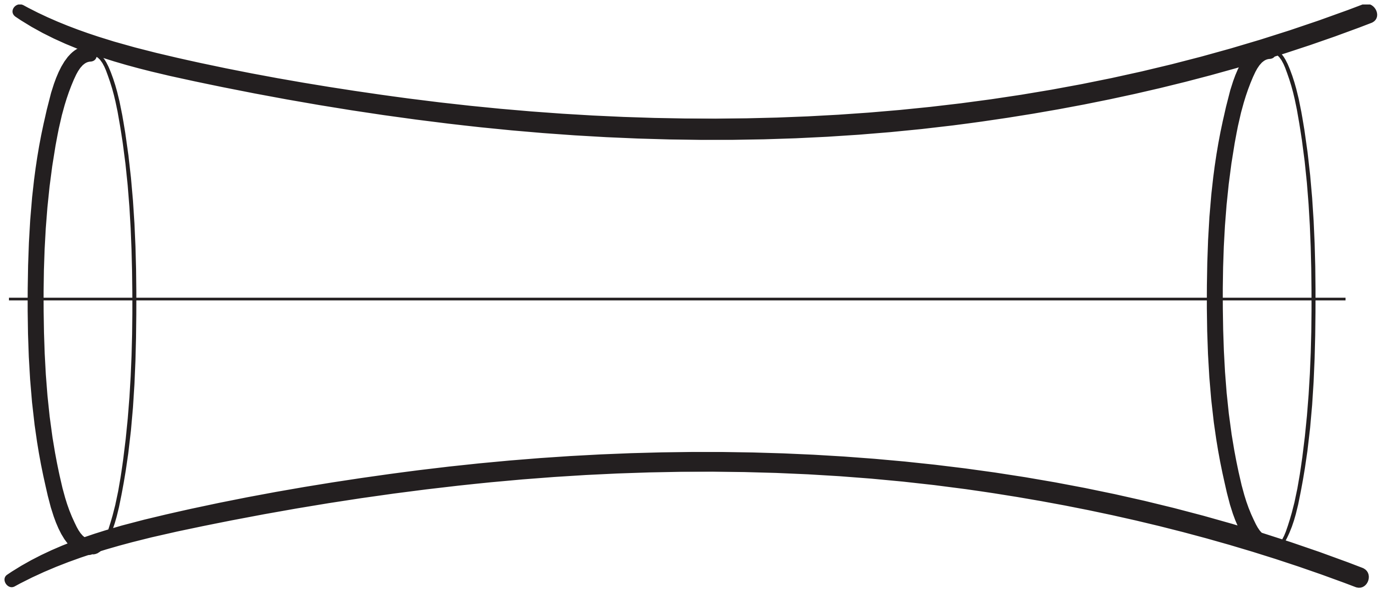
Integral resolution 1D equations



$$Q = \int_0^R 2\pi r u\, dr \qquad Q_2 = \int_0^R 2\pi r u^2 dr \qquad \tau = \frac{\partial u}{\partial r}$$

$$\frac{\partial(2\pi R^2)}{\partial t} + \frac{\partial Q}{\partial x} = 0 \qquad \frac{\partial Q}{\partial t} + \frac{\partial Q_2}{\partial t} = -(2\pi R^2)\frac{\partial p}{\partial x} - \tau$$

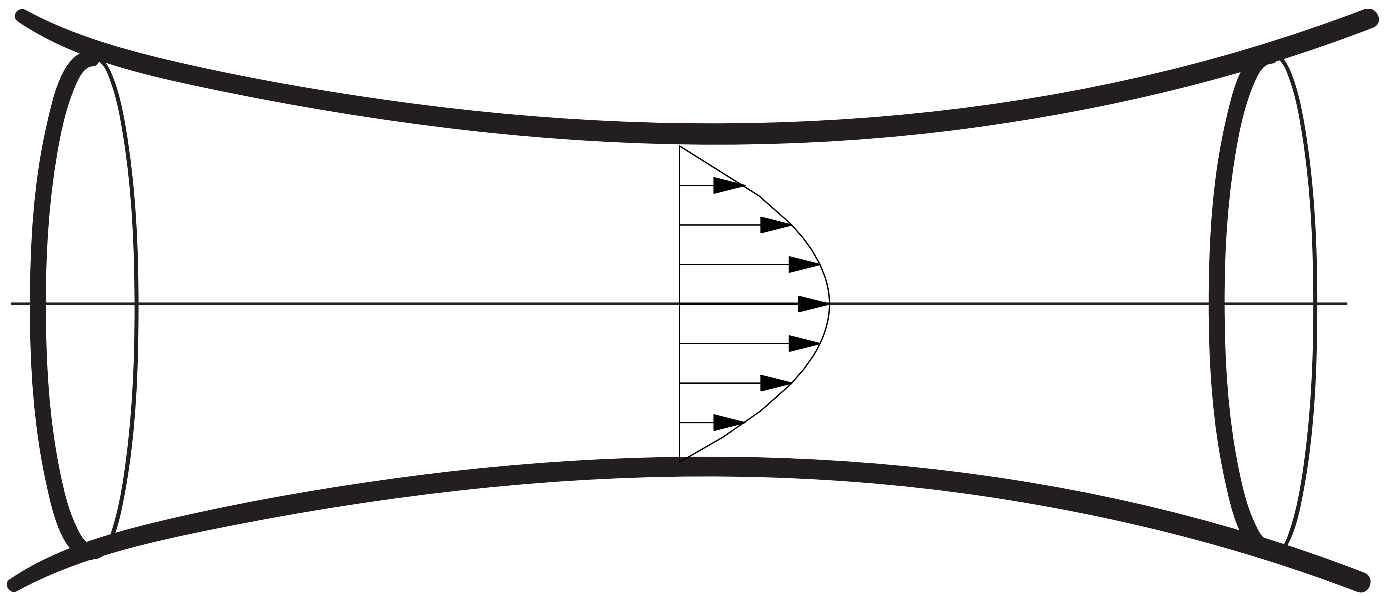
Integral resolution 1D equations



$$Q = \int_0^R 2\pi r u dr \qquad Q_2 = \int_0^R 2\pi r u^2 dr \qquad \tau = \frac{\partial u}{\partial r}$$

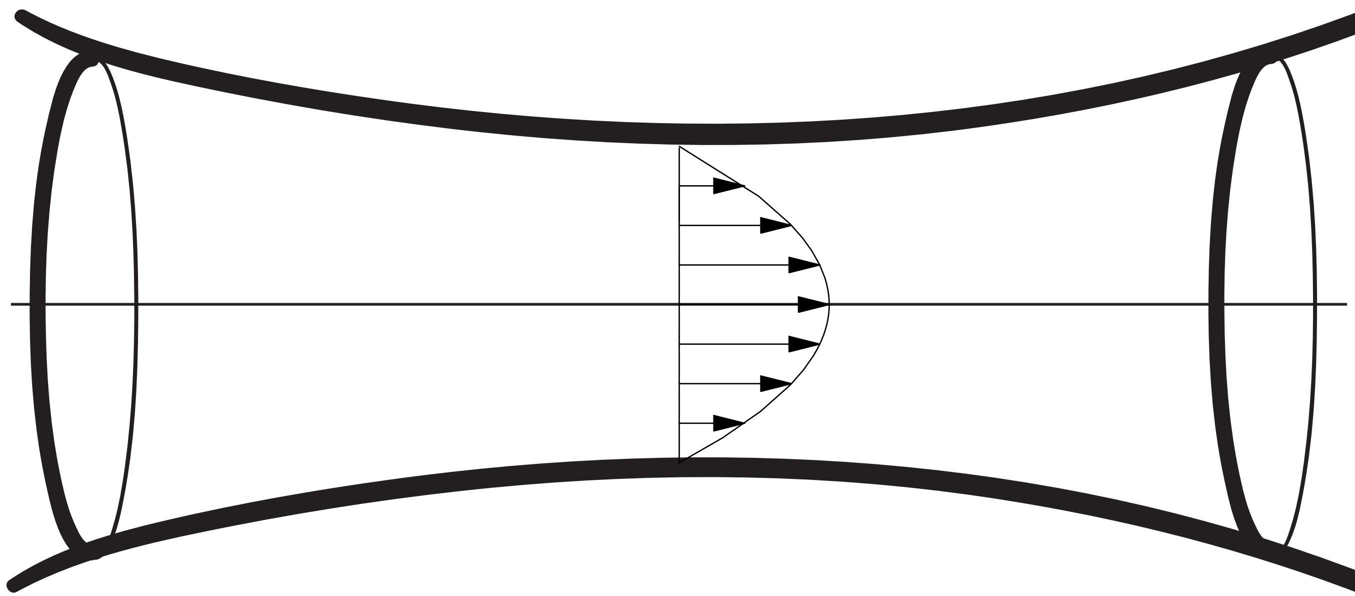
gives Q_2 as function of Q an τ as function Q

Integral resolution 1D equations



$$Q_2 = \int_0^R 2\pi r u^2 dr \qquad \tau = \frac{\partial u}{\partial r}$$

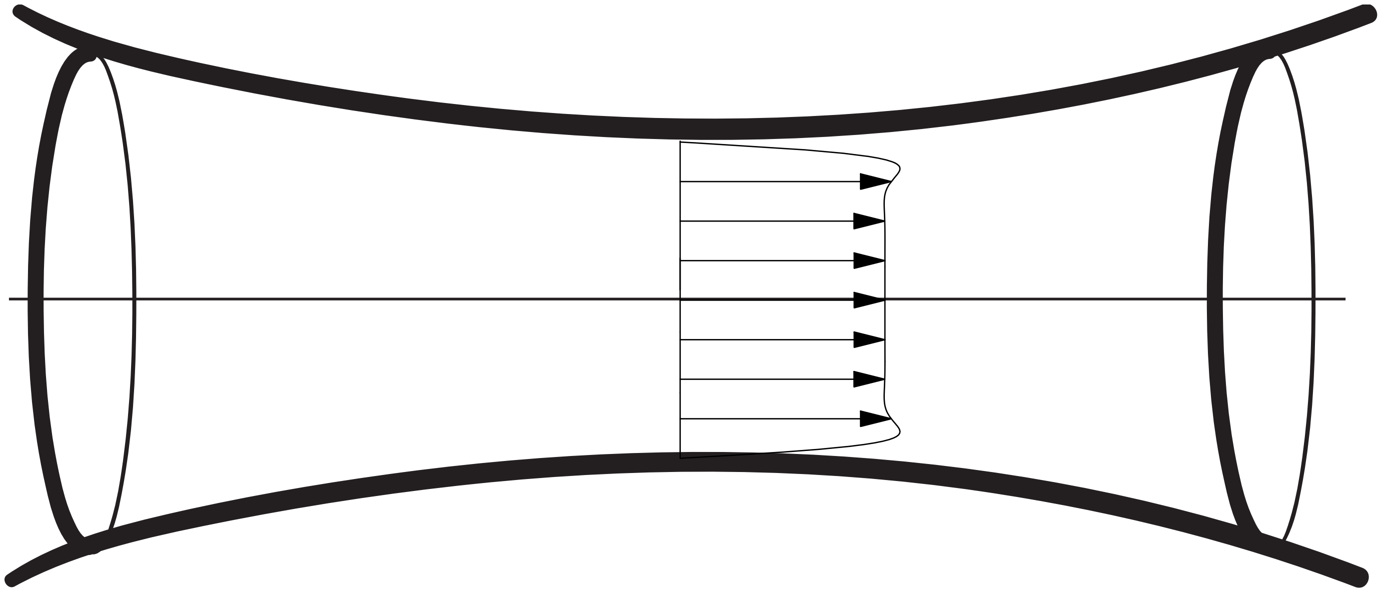
Integral resolution 1D equations



$$Q_2 = \int_0^R 2\pi r u^2 dr \qquad \tau = \frac{\partial u}{\partial r}$$

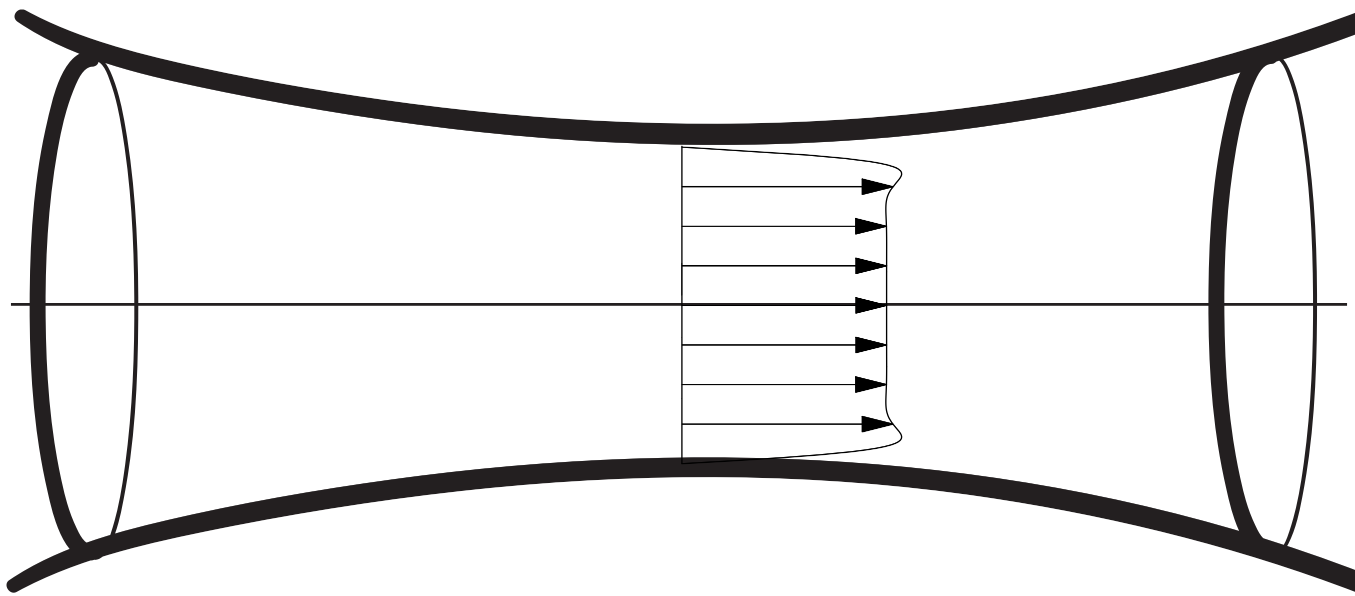
$$Q_2 = (\frac{4}{3})\frac{Q^2}{\pi R^2} \qquad \tau = (8\pi)\frac{Q}{\pi R^2}$$

Integral resolution 1D equations



$$Q_2 = \int_0^R 2\pi r u^2 dr \qquad \tau = \frac{\partial u}{\partial r}$$

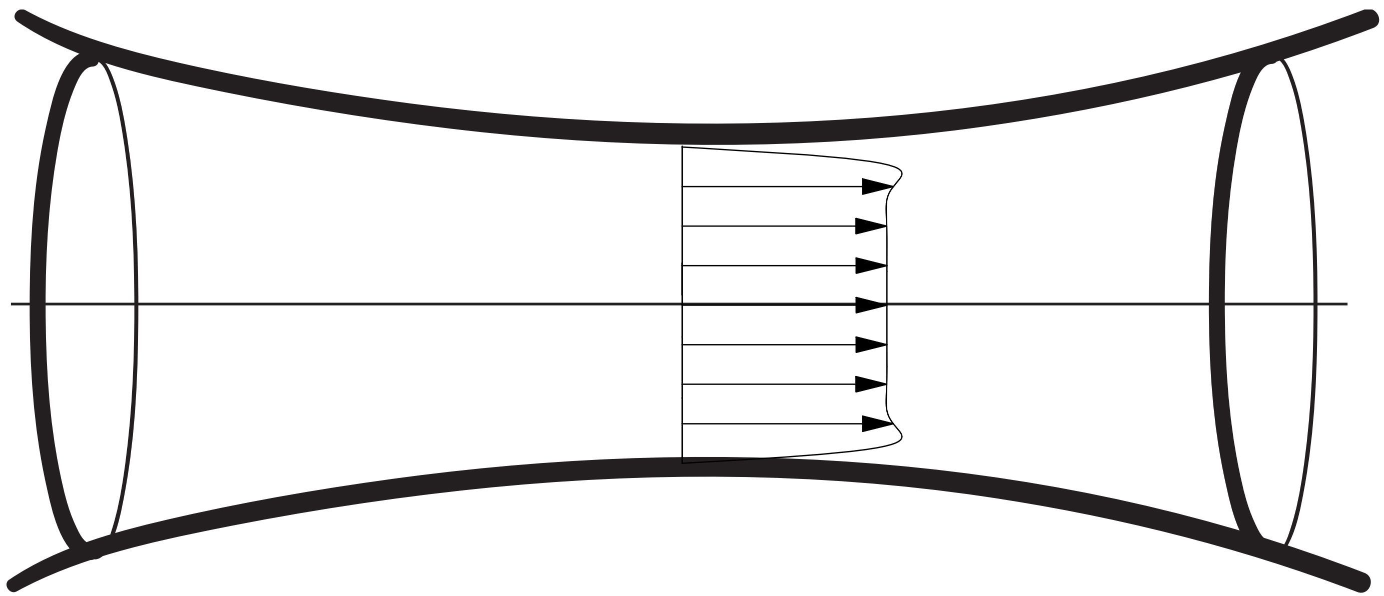
Integral resolution 1D equations



$$Q_2 = \int_0^R 2\pi r u^2 dr \qquad \tau = \frac{\partial u}{\partial r}$$

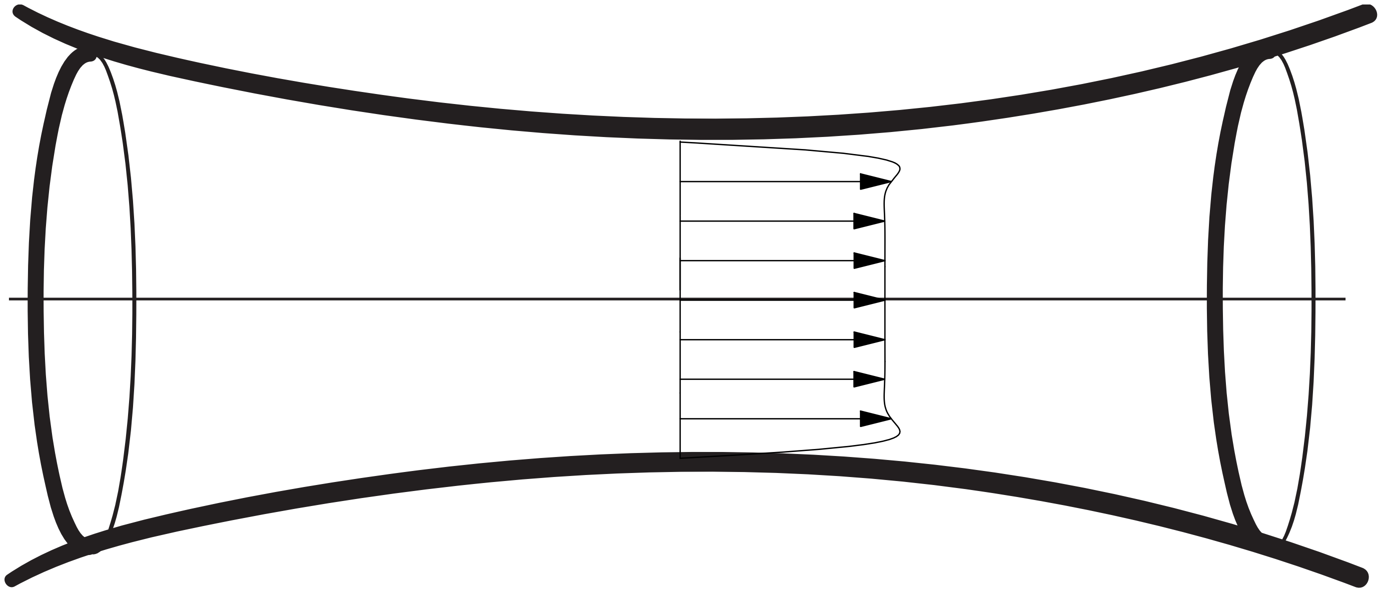
$$Q_2 = \frac{Q^2}{\pi R^2} \qquad \tau = F(Q)$$

Integral resolution 1D equations



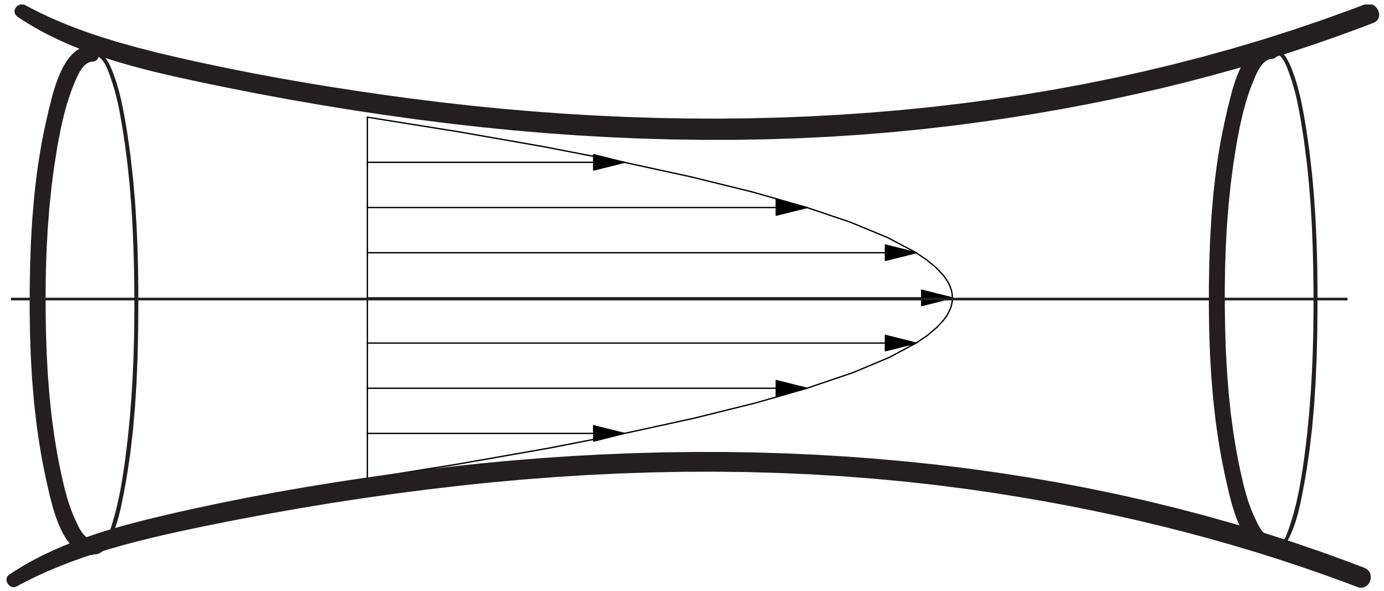
need of profile

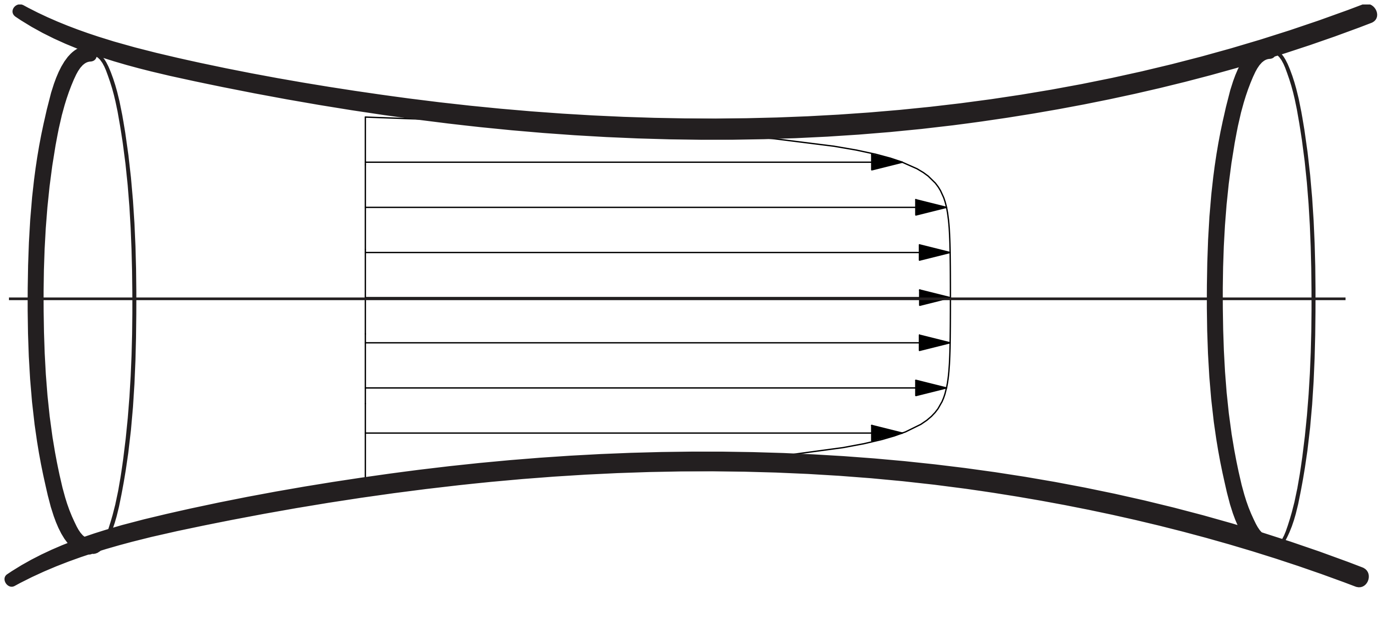
Integral resolution 1D equations

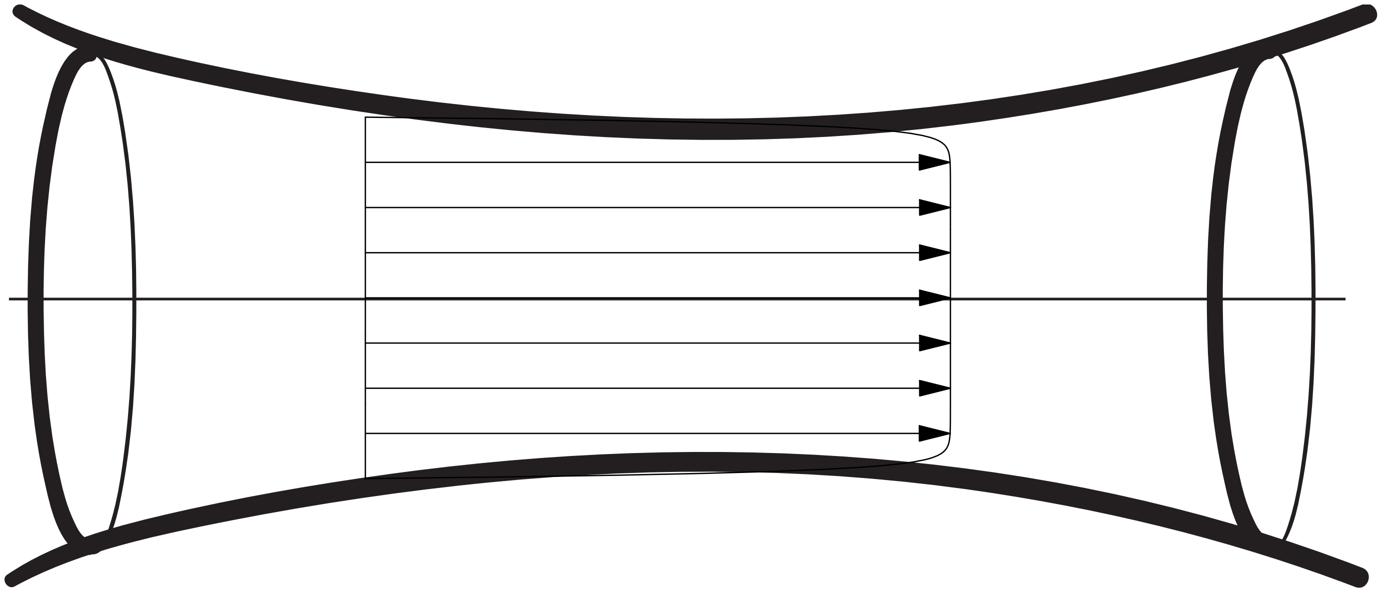


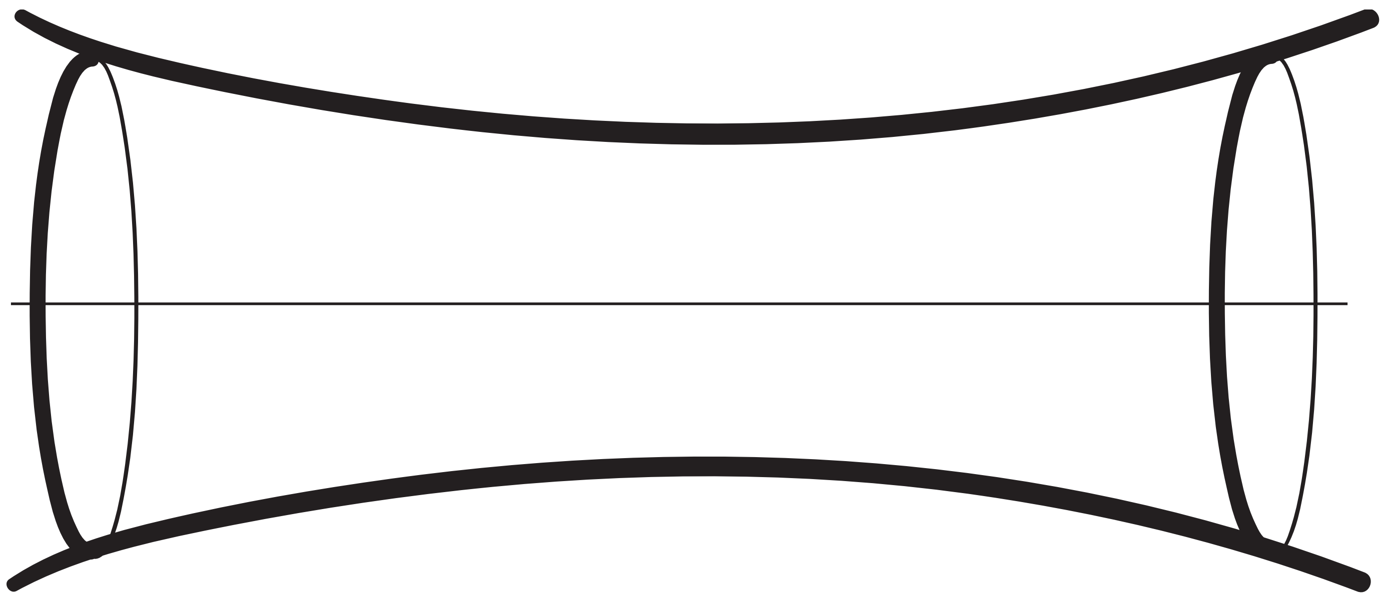
"usual" 1D equations are a simplification of RNSP

Choice of profiles

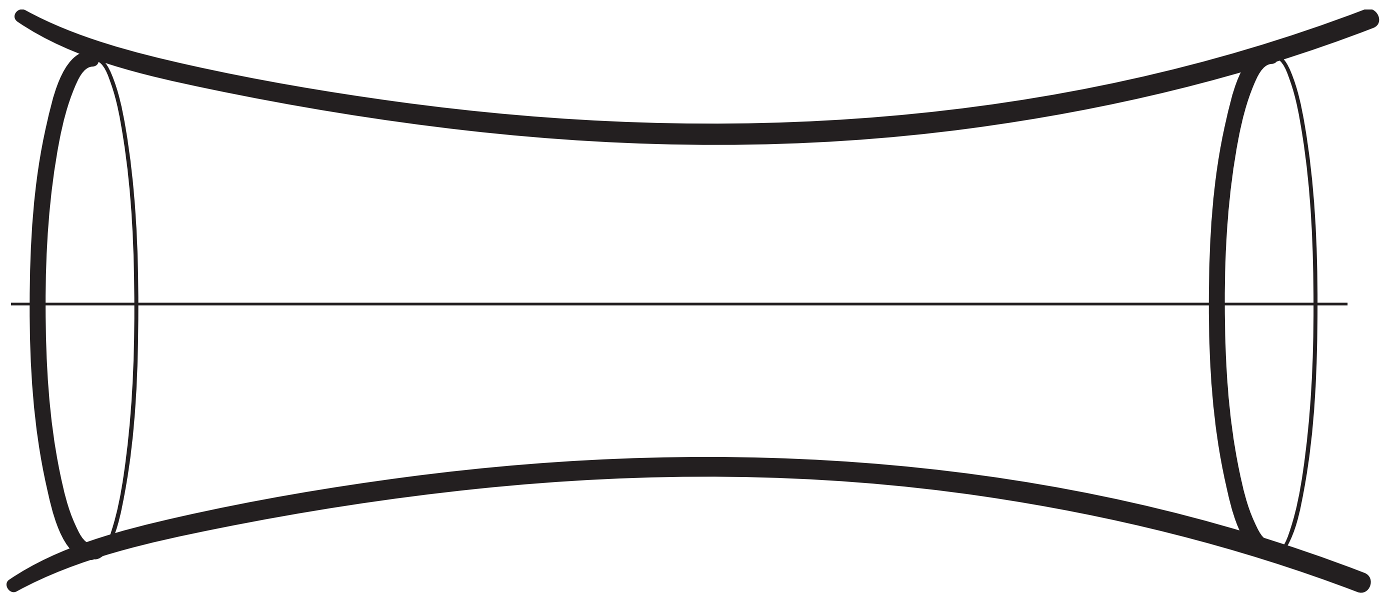








Choice of the family of simple profiles

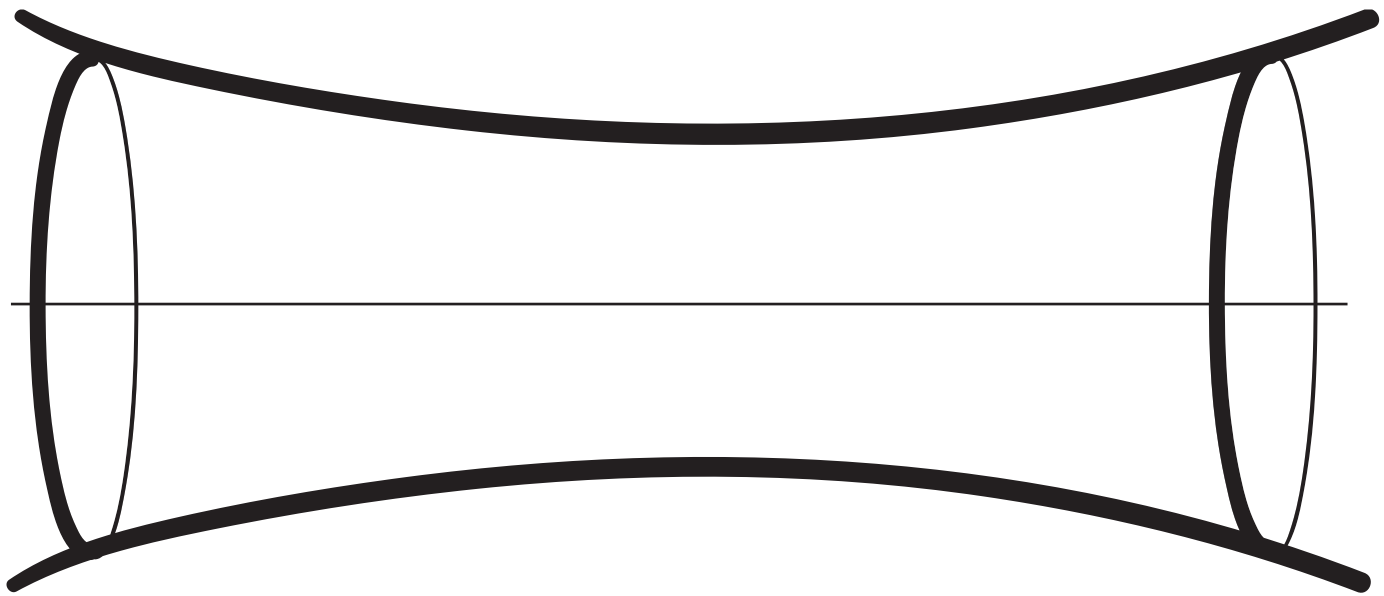


Choice of the family of simple profiles

In an unsteady flow it is natural to use Womersley

$$\frac{\partial u}{\partial t} + u\frac{\partial}{\partial x}u + v\frac{\partial}{\partial r}u = -\frac{\partial p}{\rho \partial x} + \nu \frac{\partial}{r \partial r} r \frac{\partial u}{\partial r}$$

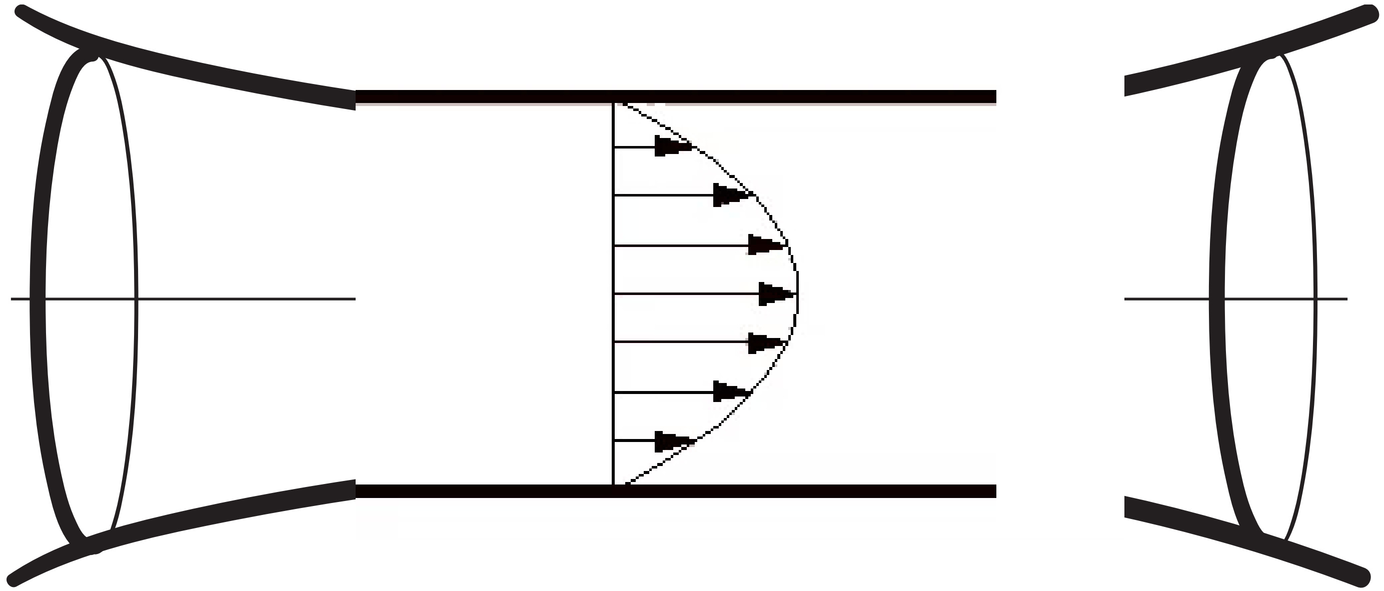
$$0 = -\frac{\partial p}{\rho \partial r}$$



Choice of the family of simple profiles

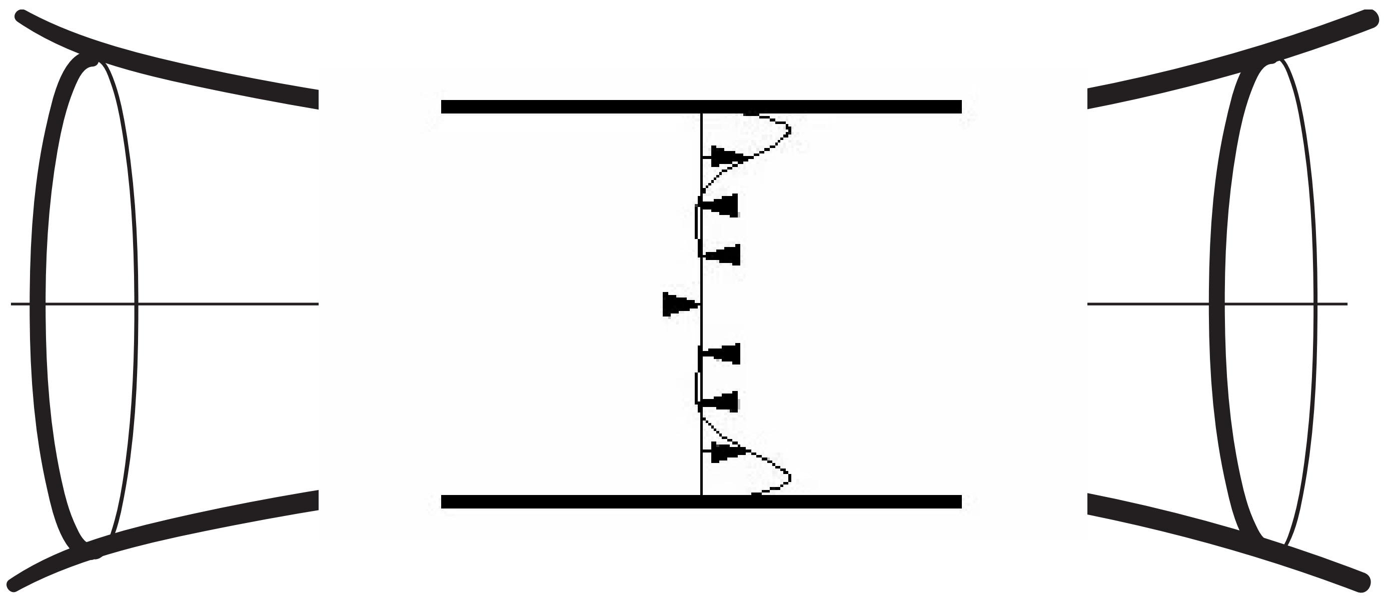
In an unsteady flow it is natural to use Womersley

Womersley profiles are solution of RNSP



Choice of the family of simple profiles

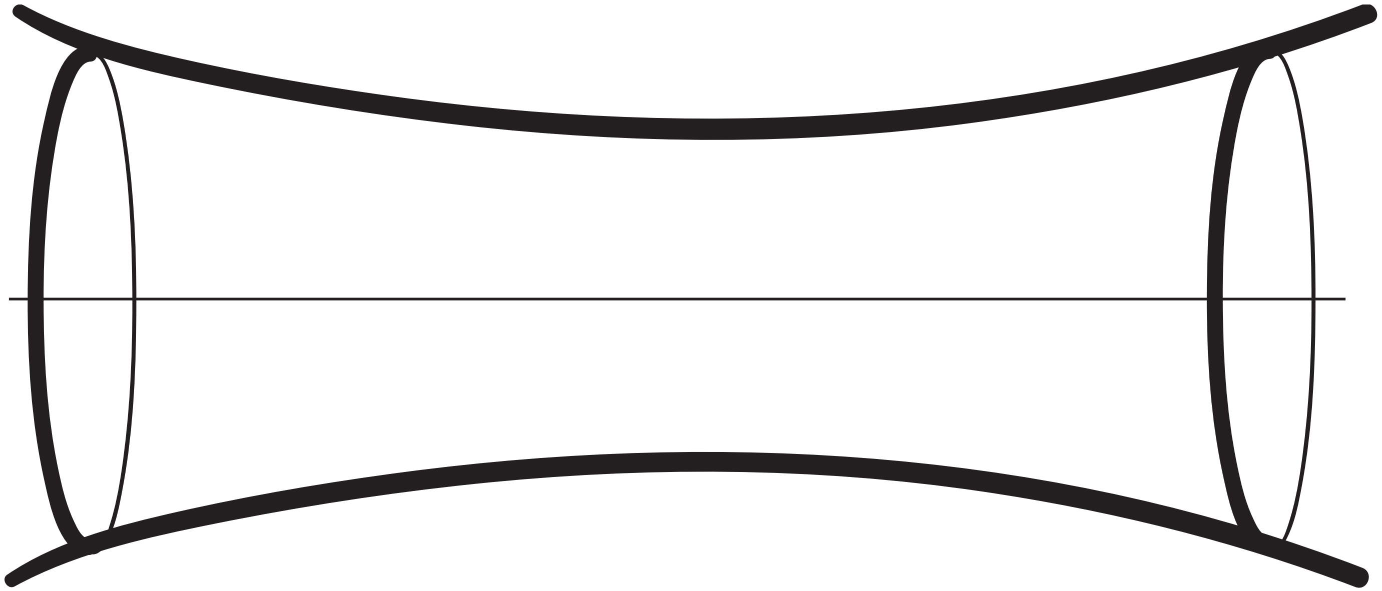
In an unsteady flow it is natural to use Womersley



Choice of the family of simple profiles

In an unsteady flow it is natural to use Womersley

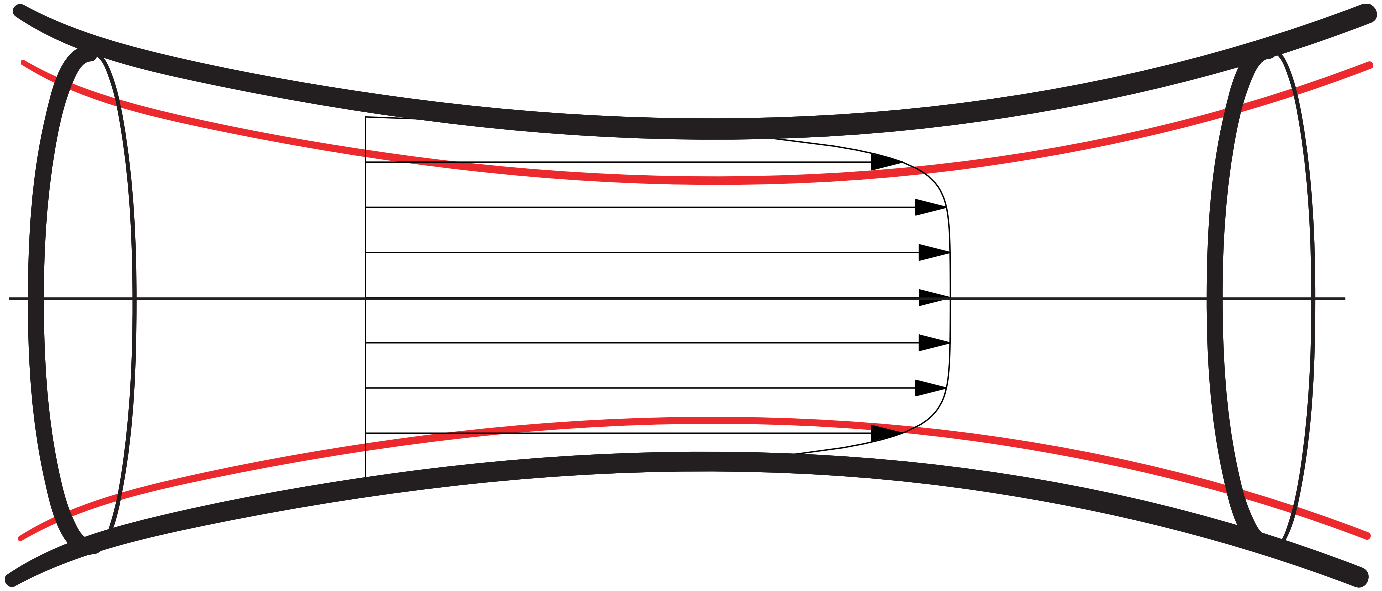
Integral resolution



$$Q = \int_0^R 2\pi r u dr \qquad Q_2 = \int_0^R 2\pi r u^2 dr \qquad \tau = \frac{\partial u}{\partial r}$$

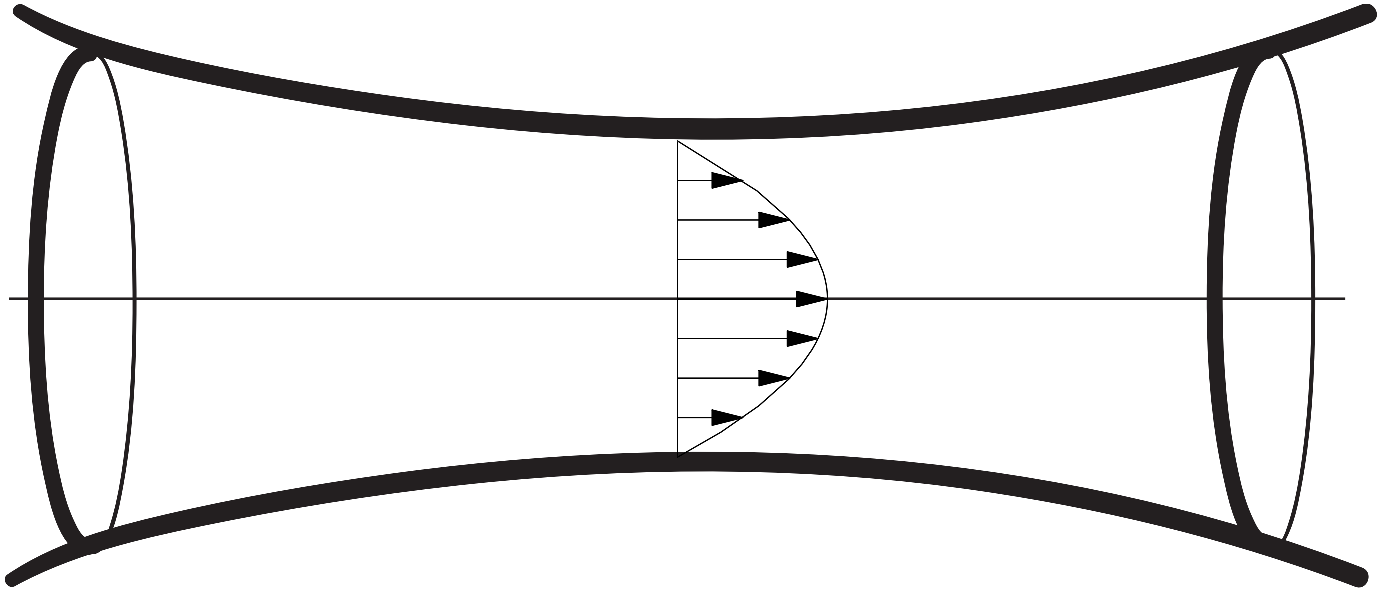
gives Q_2 as function of Q an τ as function Q

Integral resolution

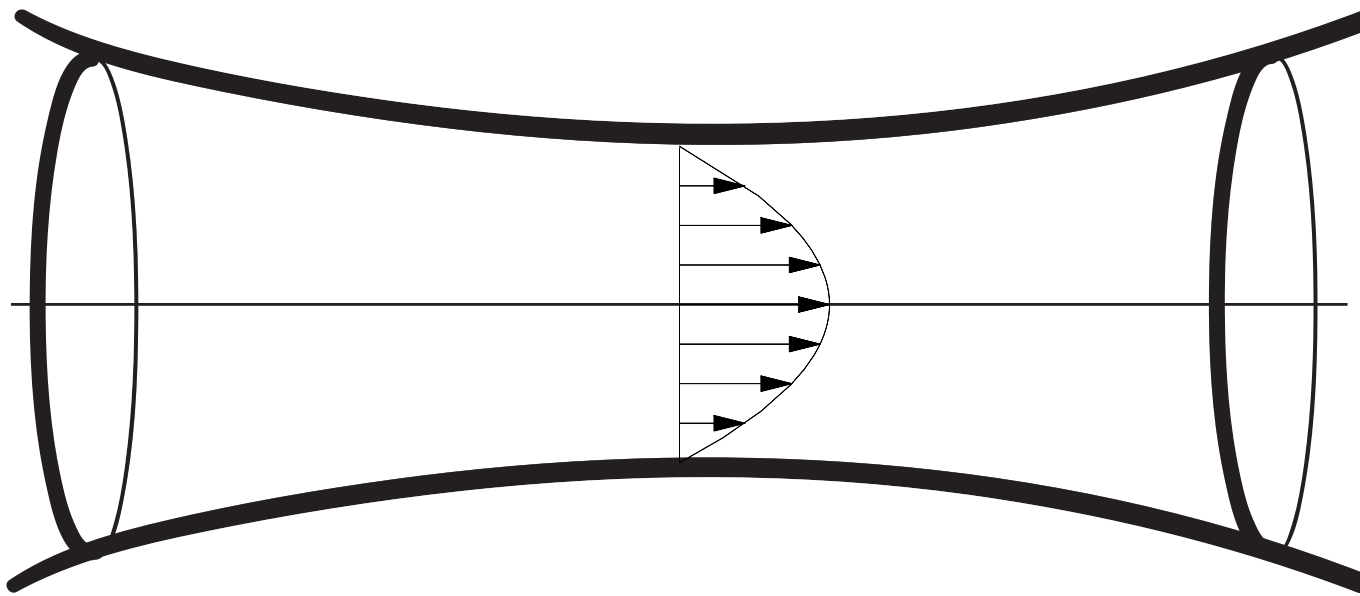


Numerical resolution:
finite differences

Interactive Boundary Layer

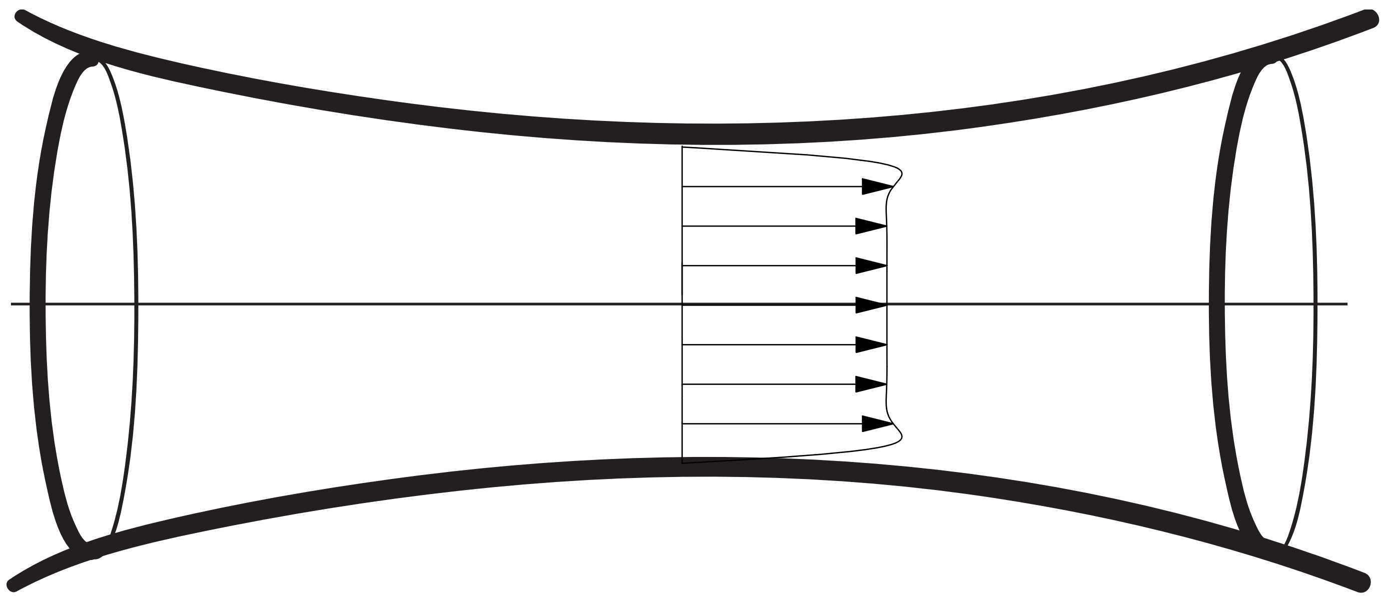


Interactive Boundary Layer

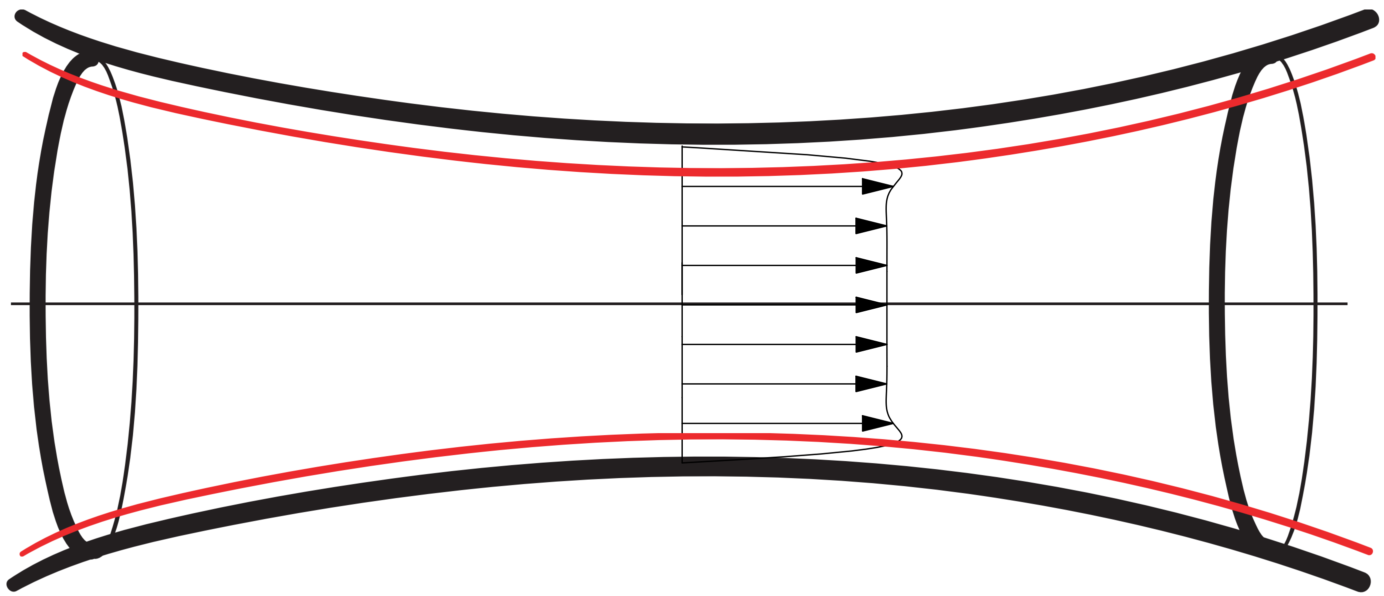


IBL is included in RNSP

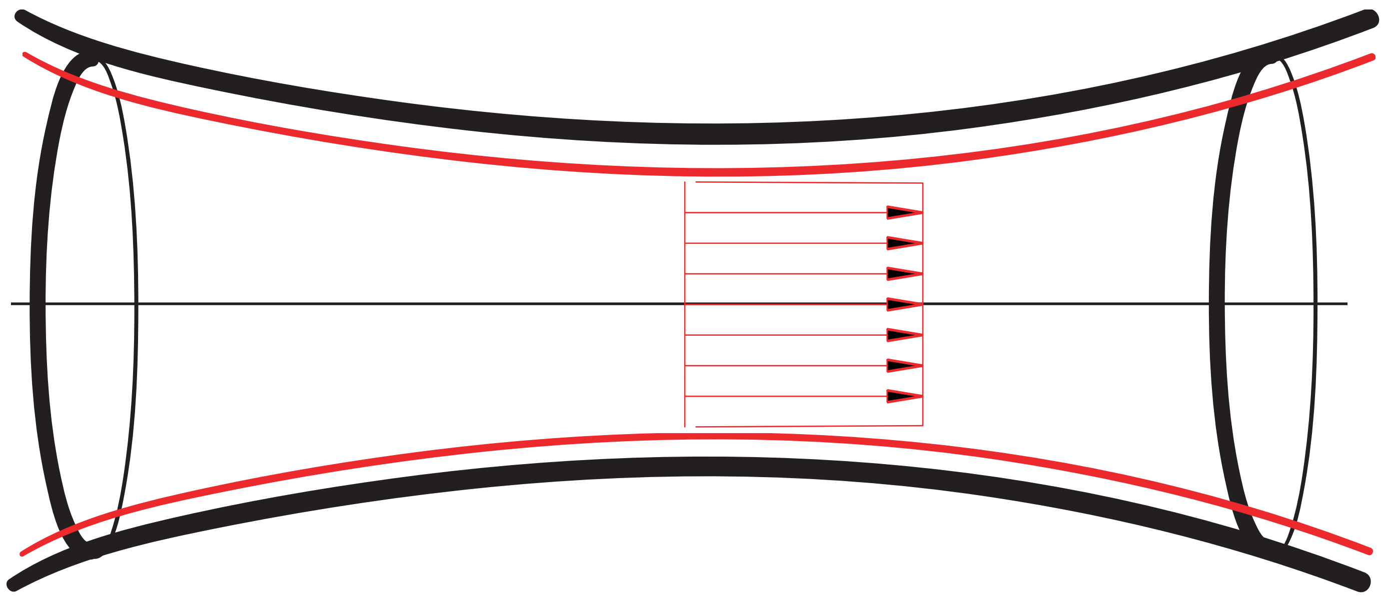
Interactive Boundary Layer



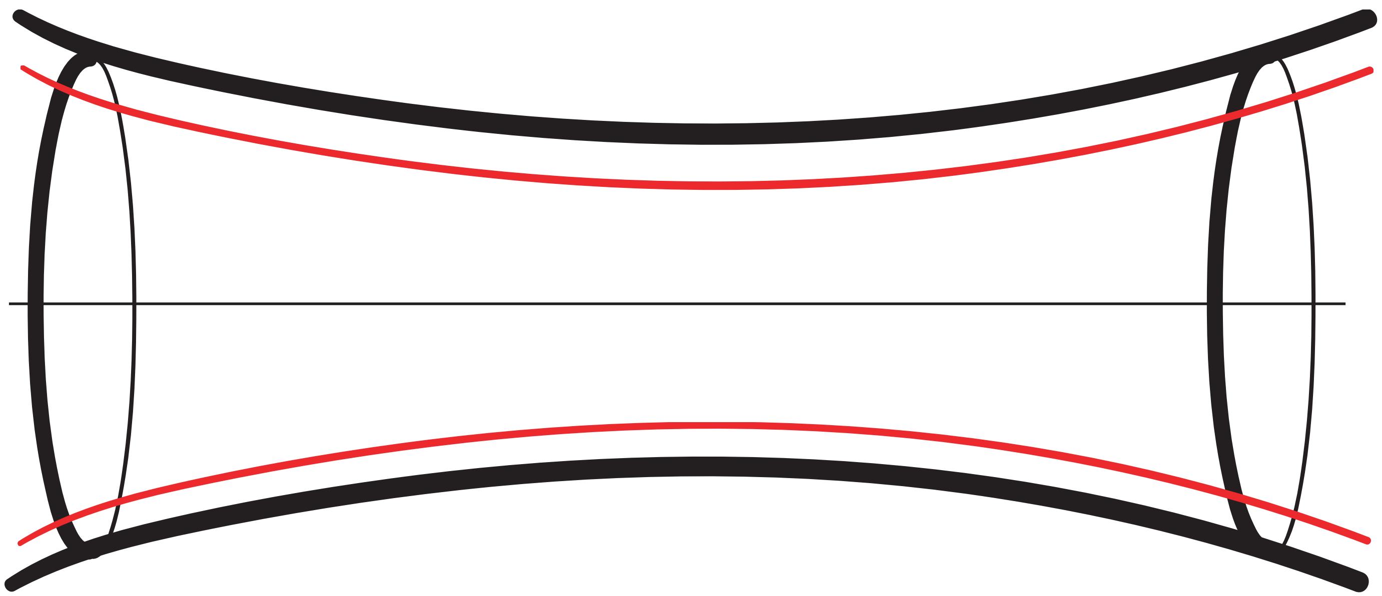
Interactive Boundary Layer



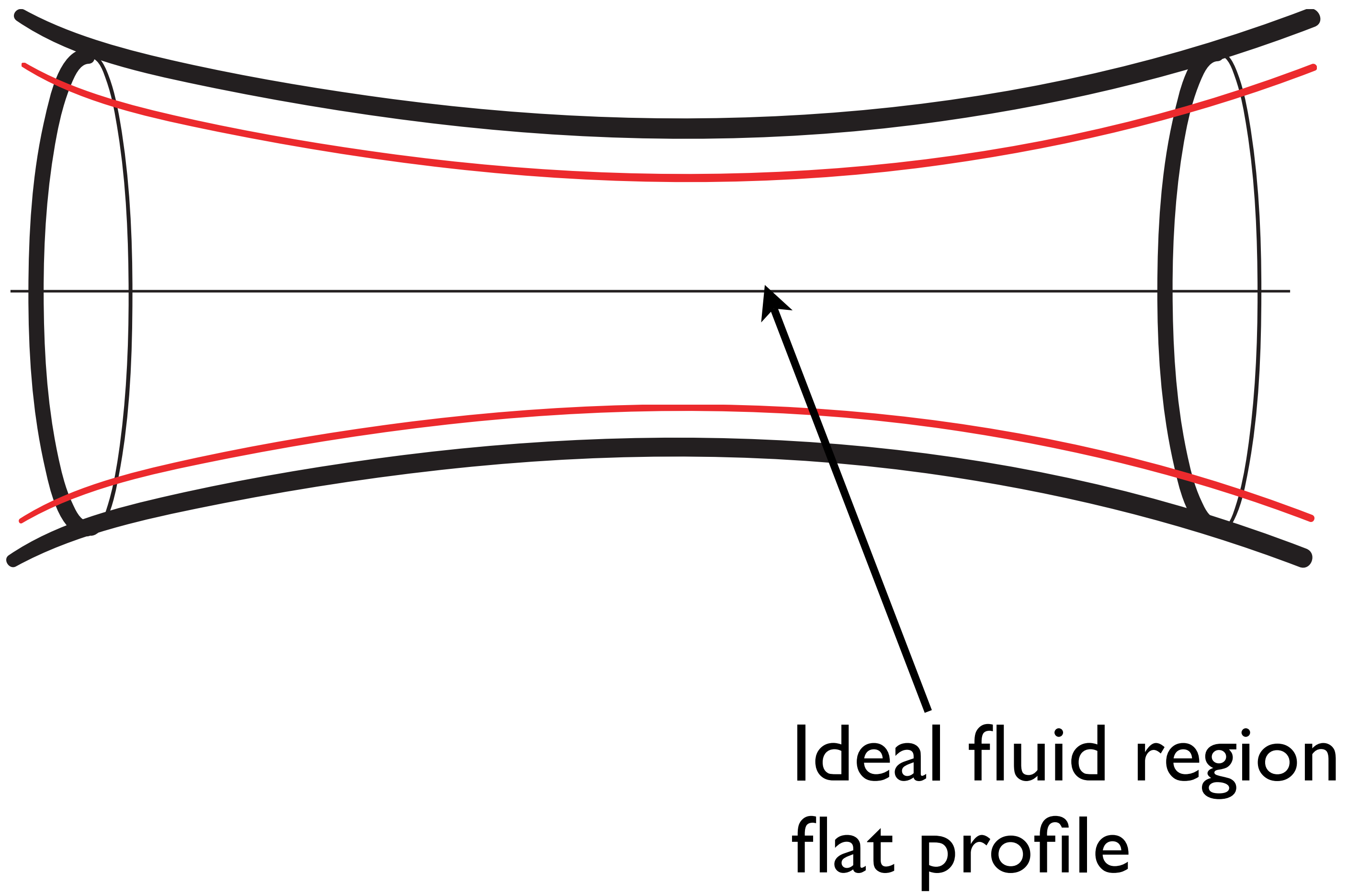
Interactive Boundary Layer



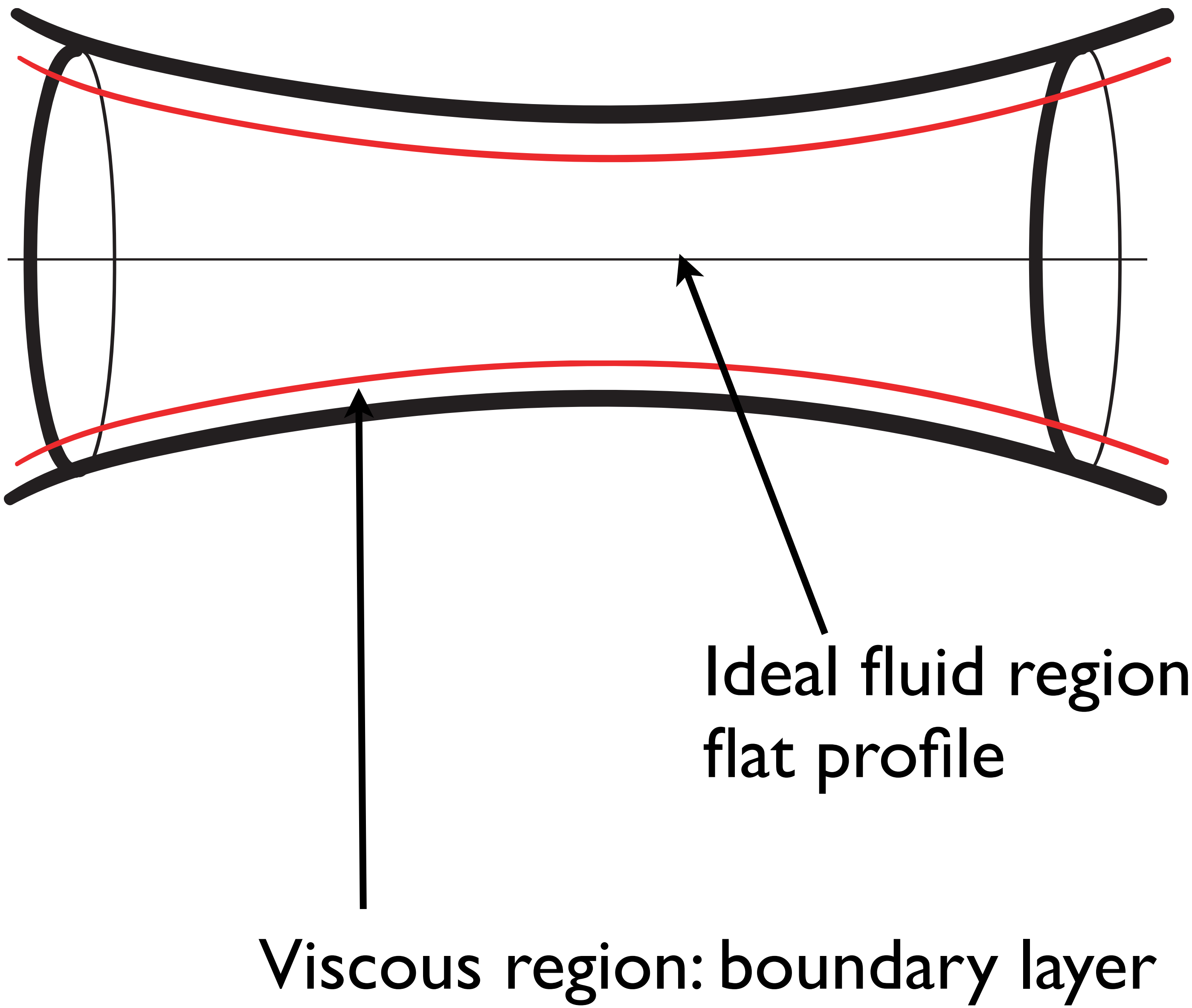
Interactive Boundary Layer



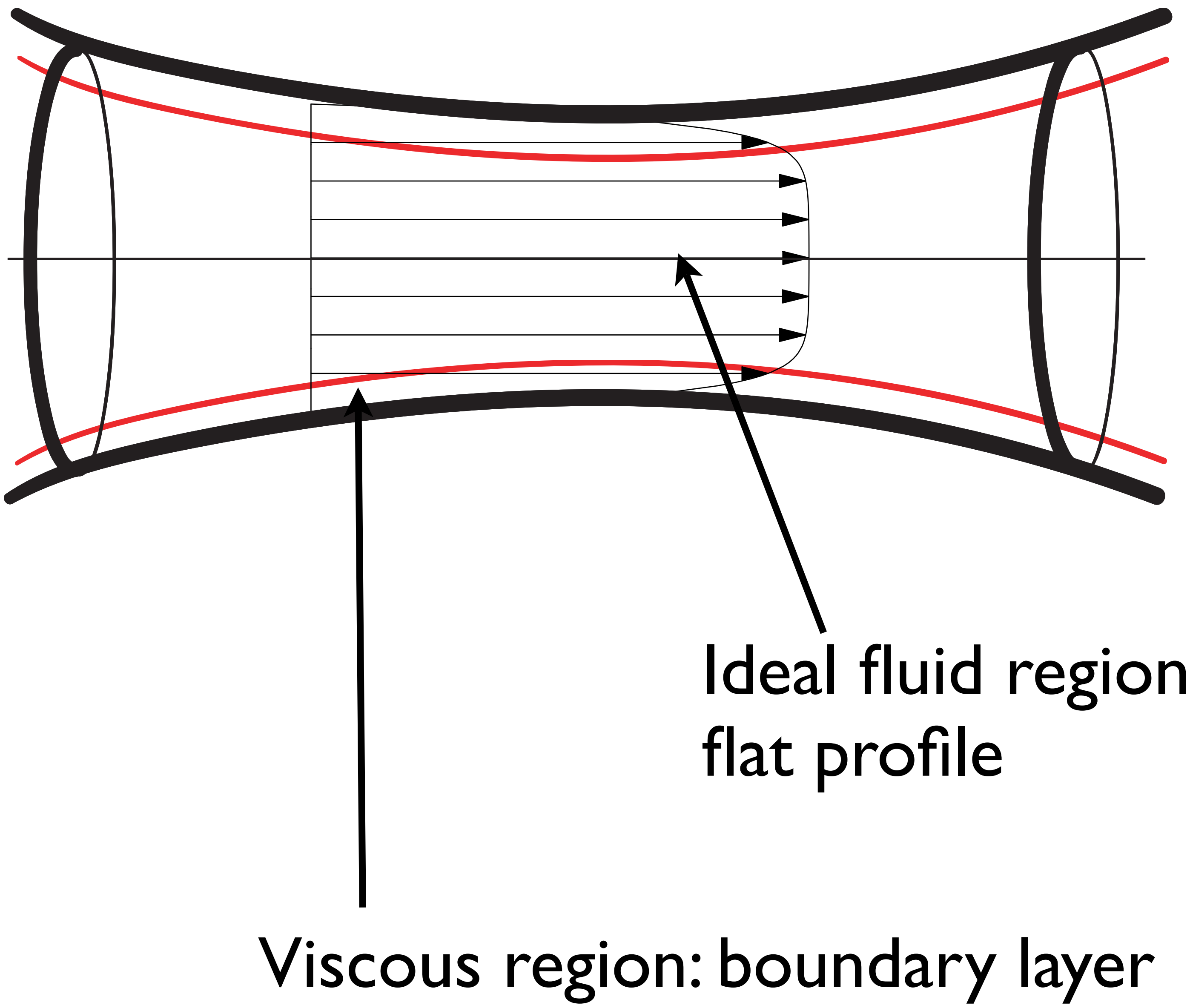
Interactive Boundary Layer



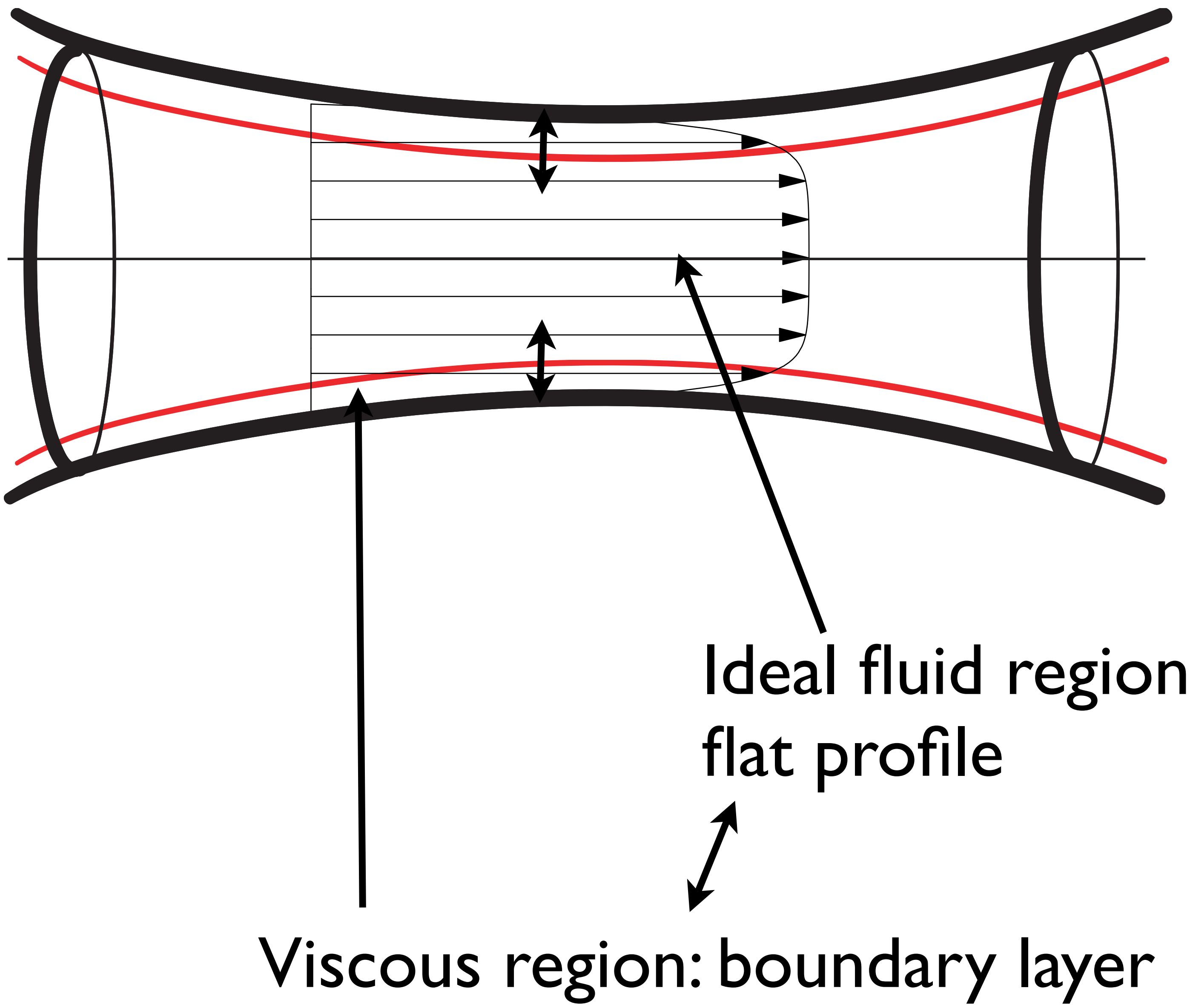
Interactive Boundary Layer



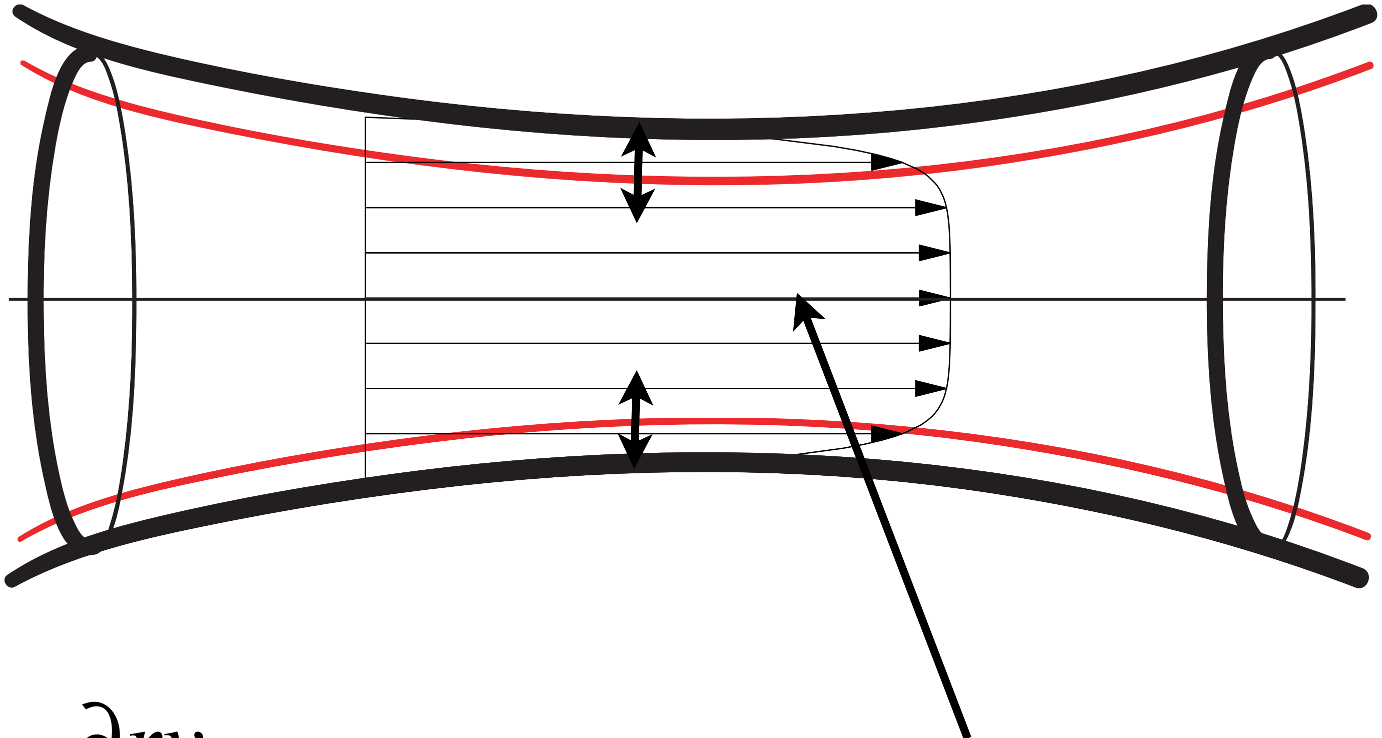
Interactive Boundary Layer



Integral resolution



Interactive Boundary Layer

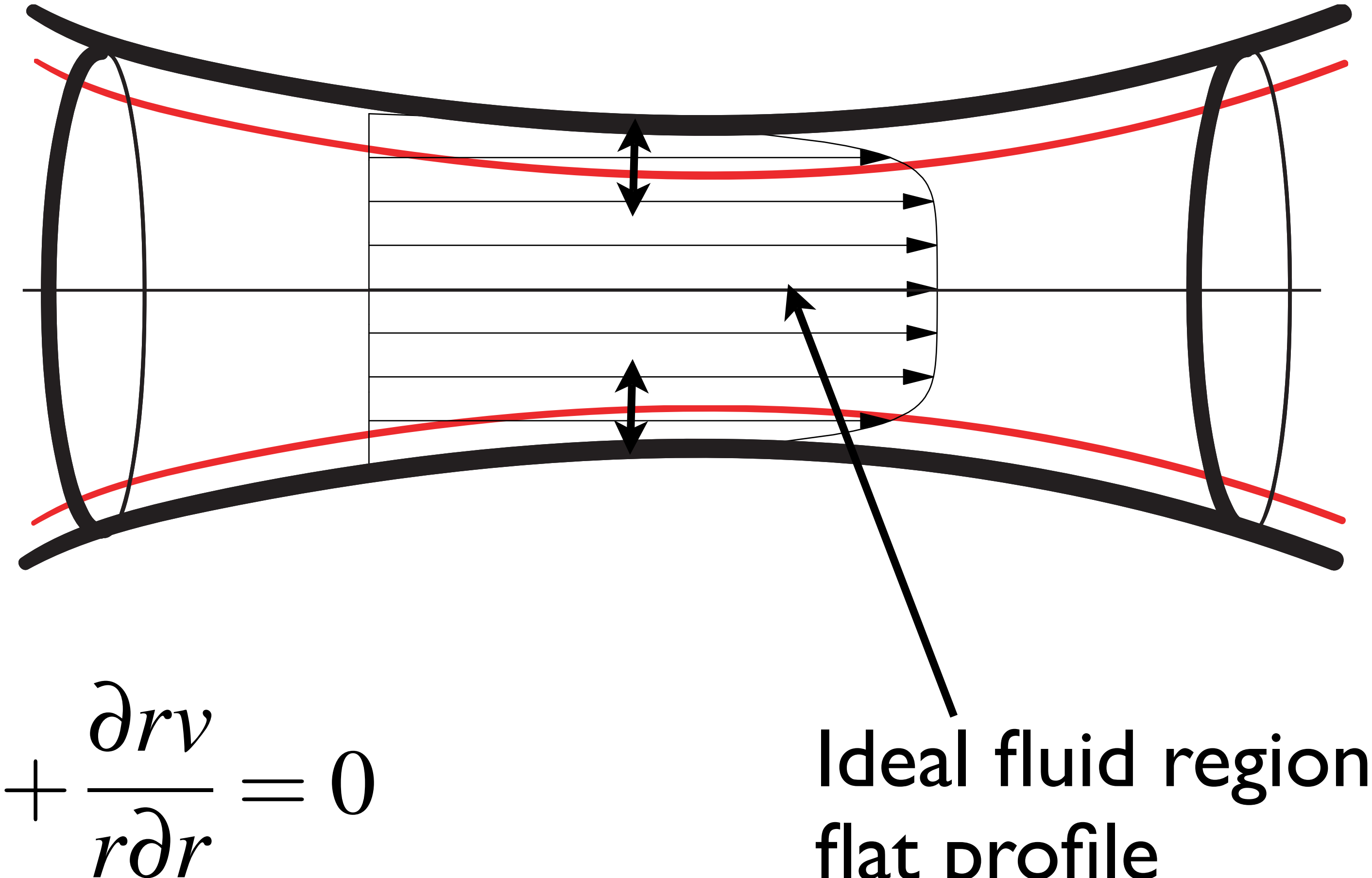


Ideal fluid region
flat profile

$$\frac{\partial u}{\partial x} + \frac{\partial r v}{r \partial r} = 0$$

$$\frac{\partial u}{\partial t} + u \frac{\partial}{\partial x} u + v \frac{\partial}{\partial r} u = -\frac{\partial p}{\rho \partial x} + \nu \frac{\partial}{r \partial r} r \frac{\partial u}{\partial r} \qquad 0 = -\frac{\partial p}{\rho \partial r}$$

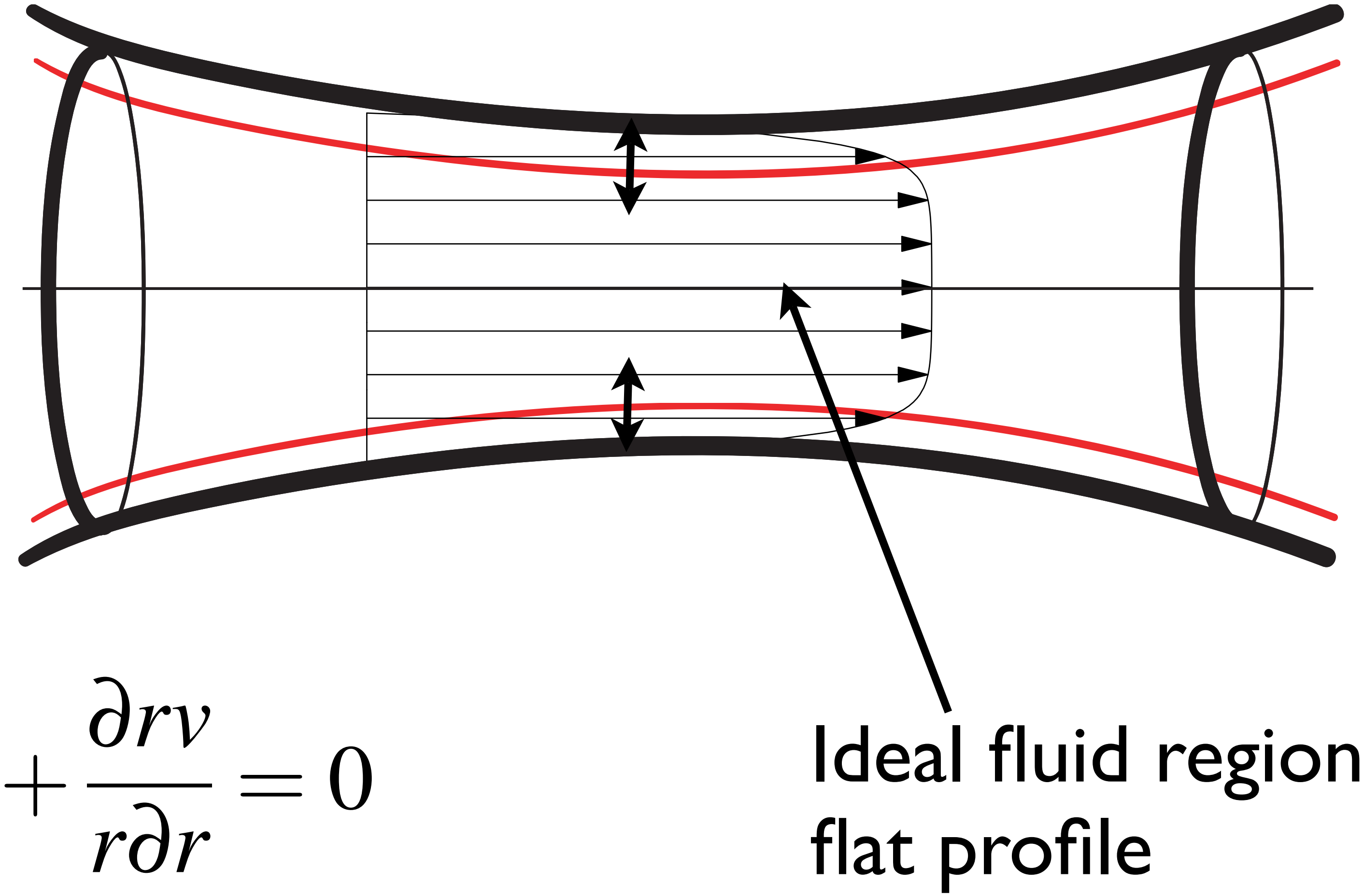
Interactive Boundary Layer



$$\frac{\partial u}{\partial x} + \frac{\partial r v}{r \partial r} = 0$$

$$\frac{\partial u}{\partial t} + u\frac{\partial}{\partial x}u + v\frac{\partial}{\partial r}u = -\frac{\partial p}{\rho \partial x} + \nu \frac{\partial}{r\partial r} r \frac{\partial u}{\partial r} \qquad 0 = -\frac{\partial p}{\rho \partial r}$$

Interactive Boundary Layer

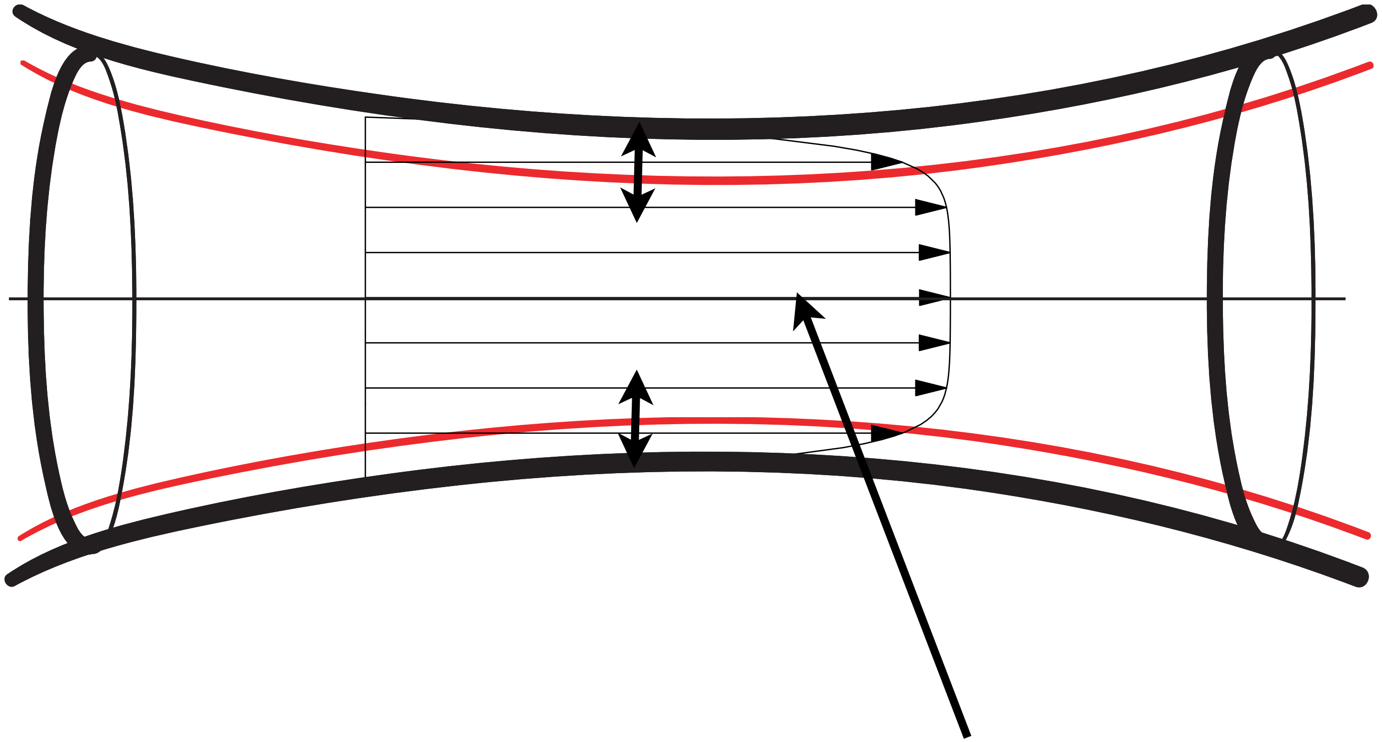


$$\frac{\partial u}{\partial x} + \frac{\partial rv}{r\partial r} = 0$$

$$\cancel{\frac{\partial u}{\partial t}} + u\frac{\partial}{\partial x}u + v\frac{\partial}{\partial r}u = -\frac{\partial p}{\rho\partial x} + \nu\cancel{\frac{\partial}{r\partial r}}r\frac{\partial u}{\partial r} \qquad 0 = -\frac{\partial p}{\rho\partial r}$$

steady/ or large convective acceleration

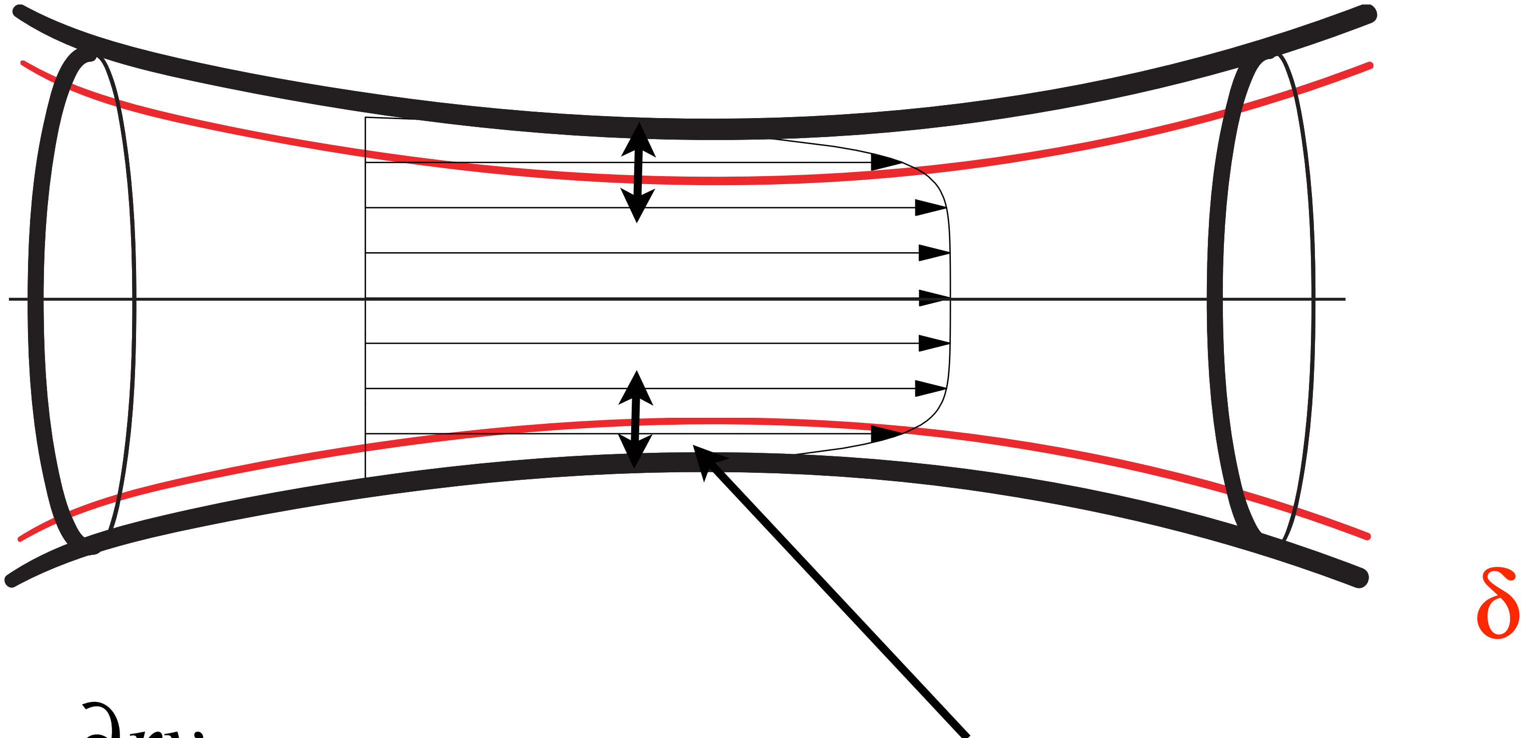
Interactive Boundary Layer



Ideal fluid region
flat profile

$$U_e S = cst$$

Interactive Boundary Layer



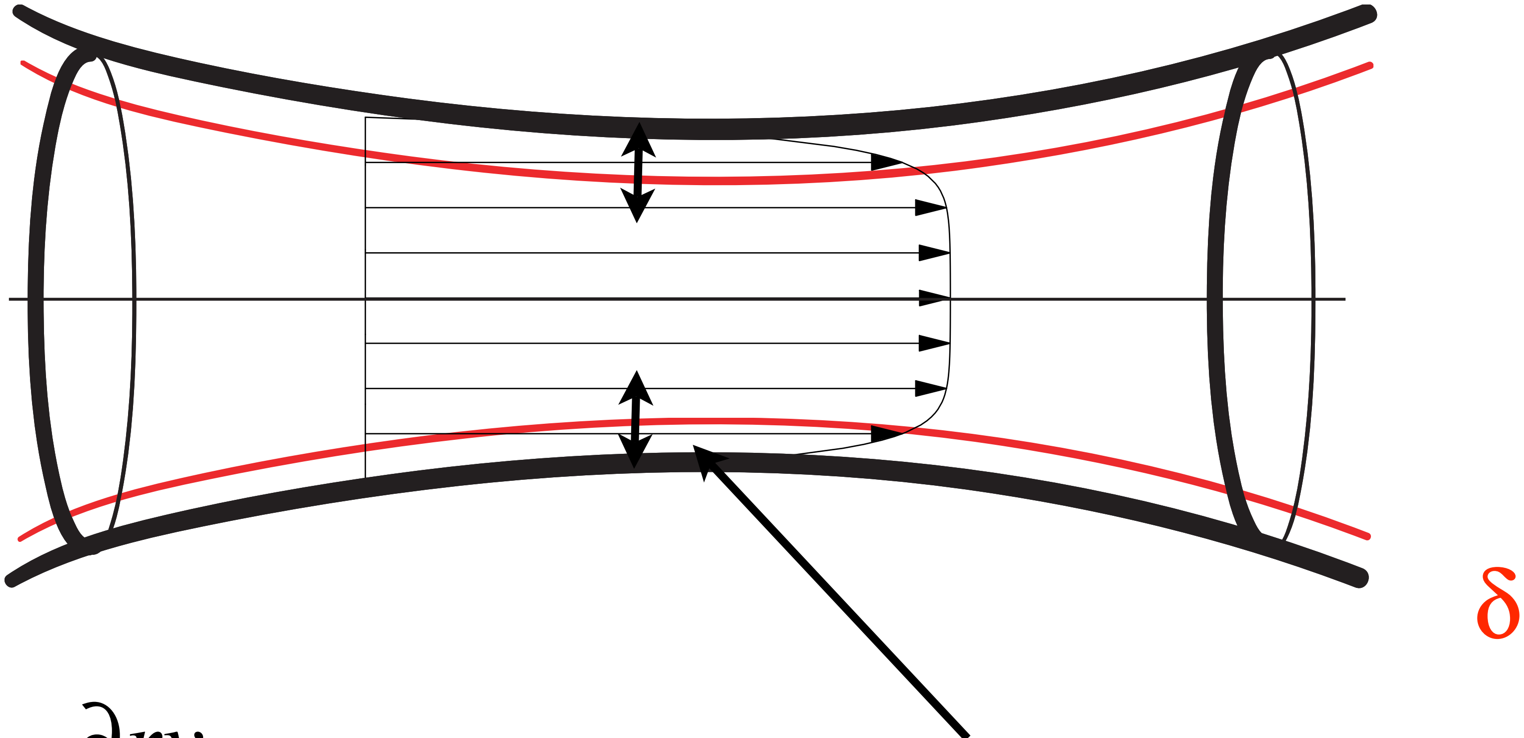
Viscous region: boundary layer

$$\frac{\partial u}{\partial x} + \frac{\partial r v}{r \partial r} = 0$$

$$\frac{\partial u}{\partial t} + u\frac{\partial}{\partial x}u + v\frac{\partial}{\partial r}u = -\frac{\partial p}{\rho \partial x} + \nu \frac{\partial}{r \partial r} r \frac{\partial u}{\partial r} \qquad 0 = -\frac{\partial p}{\rho \partial r}$$

steady/ or large convective acceleration

Interactive Boundary Layer



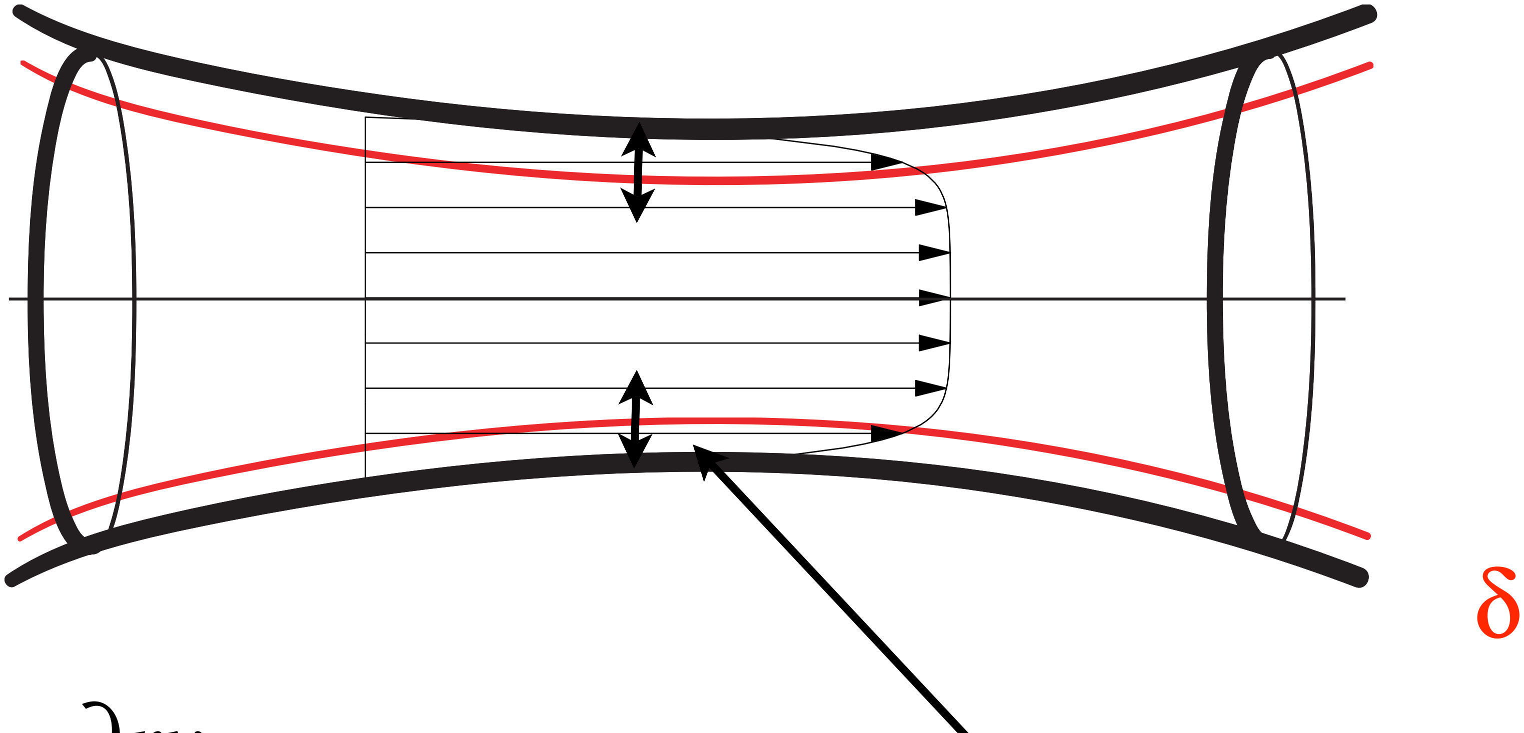
$$\frac{\partial u}{\partial x} + \frac{\partial r v}{r \partial r} = 0$$

Viscous region: boundary layer

$$\boxed{\frac{U_0^2}{\lambda}} = -\frac{\partial p}{\rho \partial x} + \boxed{\frac{\nu}{U_0 \lambda} \frac{\lambda^2}{\delta^2} \frac{U_0^2}{\lambda}} \qquad 0 = -\frac{\partial p}{\rho \partial r}$$

steady/ or large convective acceleration

Interactive Boundary Layer



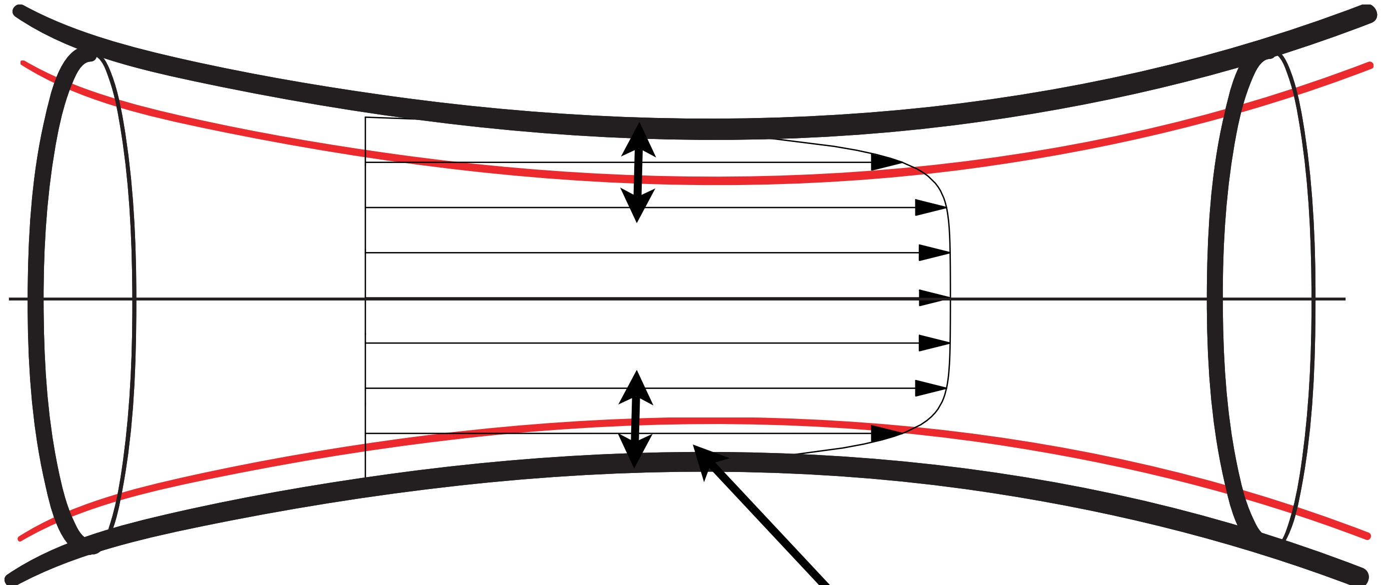
$$\frac{\partial u}{\partial x}+\frac{\partial rv}{r\partial r}=0$$

Viscous region: boundary layer

$$\boxed{\frac{U_0^2}{\lambda}} = -\frac{\partial p}{\rho\partial x}+\boxed{\frac{1}{Re}\frac{\lambda^2}{\delta^2}\frac{U_0^2}{\lambda}} \qquad 0=-\frac{\partial p}{\rho\partial r}$$

steady/ or large convective acceleration

Interactive Boundary Layer



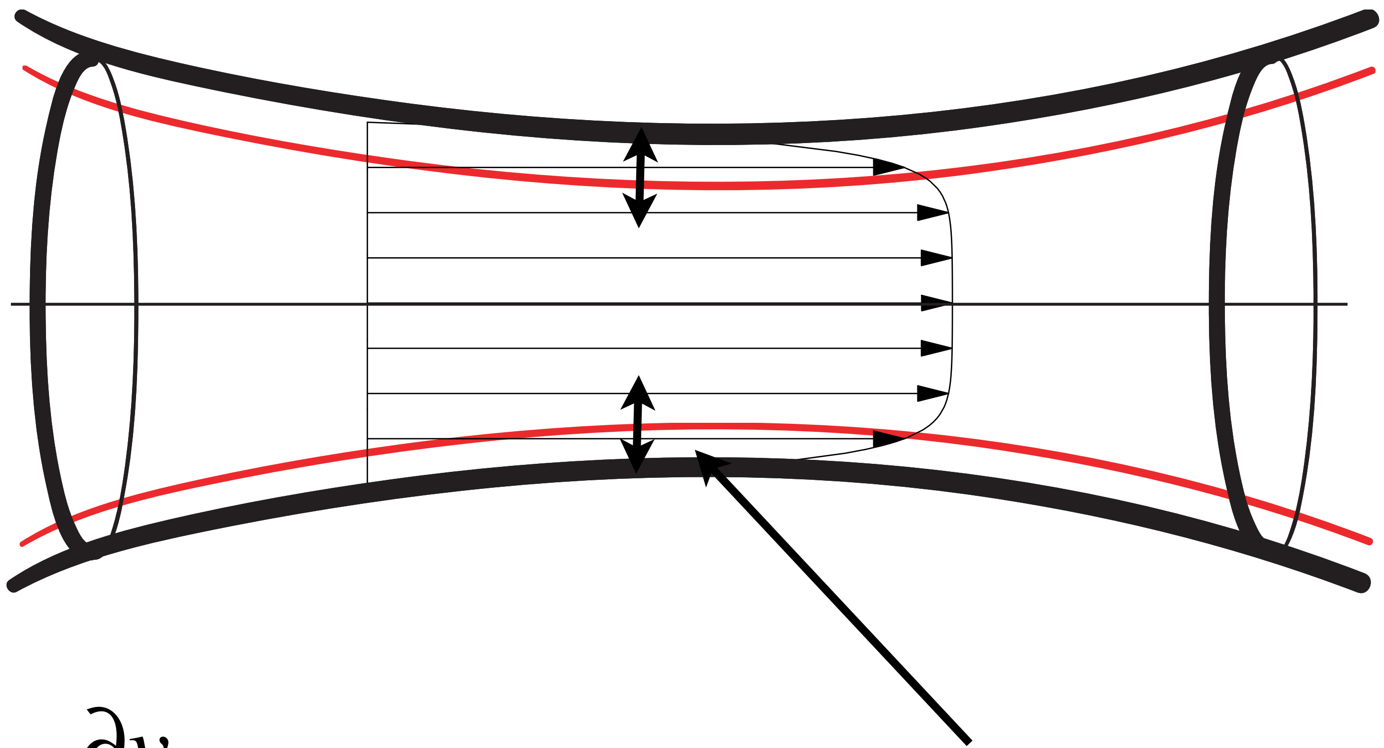
$$\delta \sim \frac{\lambda}{Re^{1/2}}$$

Viscous region: boundary layer

$$\frac{\partial u}{\partial x} + \frac{\partial rv}{r\partial r} = 0$$

$$\boxed{\frac{U_0^2}{\lambda}} = -\frac{\partial p}{\rho \partial x} + \boxed{\frac{1}{Re}\frac{\lambda^2}{\delta^2}\frac{U_0^2}{\lambda}} \quad 0 = -\frac{\partial p}{\rho \partial r}$$

Interactive Boundary Layer

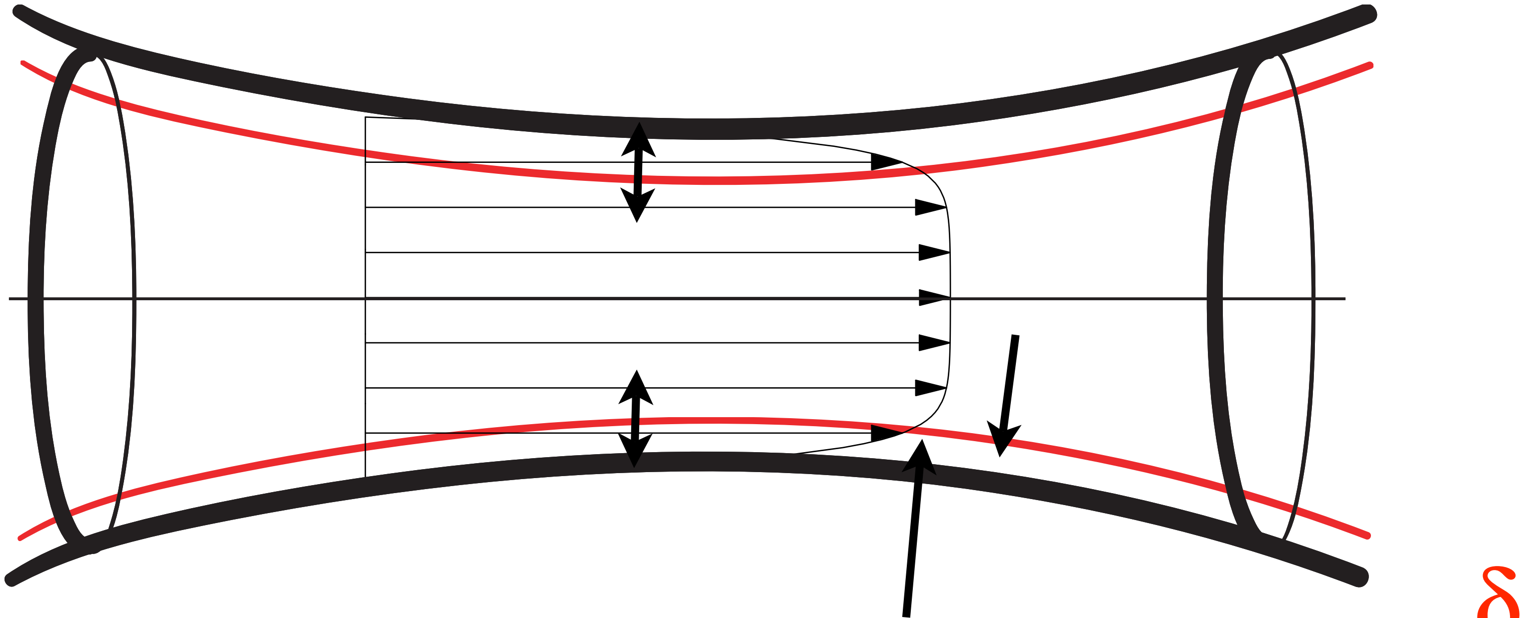


$$\delta \sim \frac{\lambda}{Re^{1/2}}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial n} = 0$$

$$u\frac{\partial}{\partial x}u + v\frac{\partial}{\partial n}u = -\frac{\partial p}{\partial x} + \frac{\partial^2}{\partial n^2}u \qquad 0 = -\frac{\partial p}{\partial n}$$

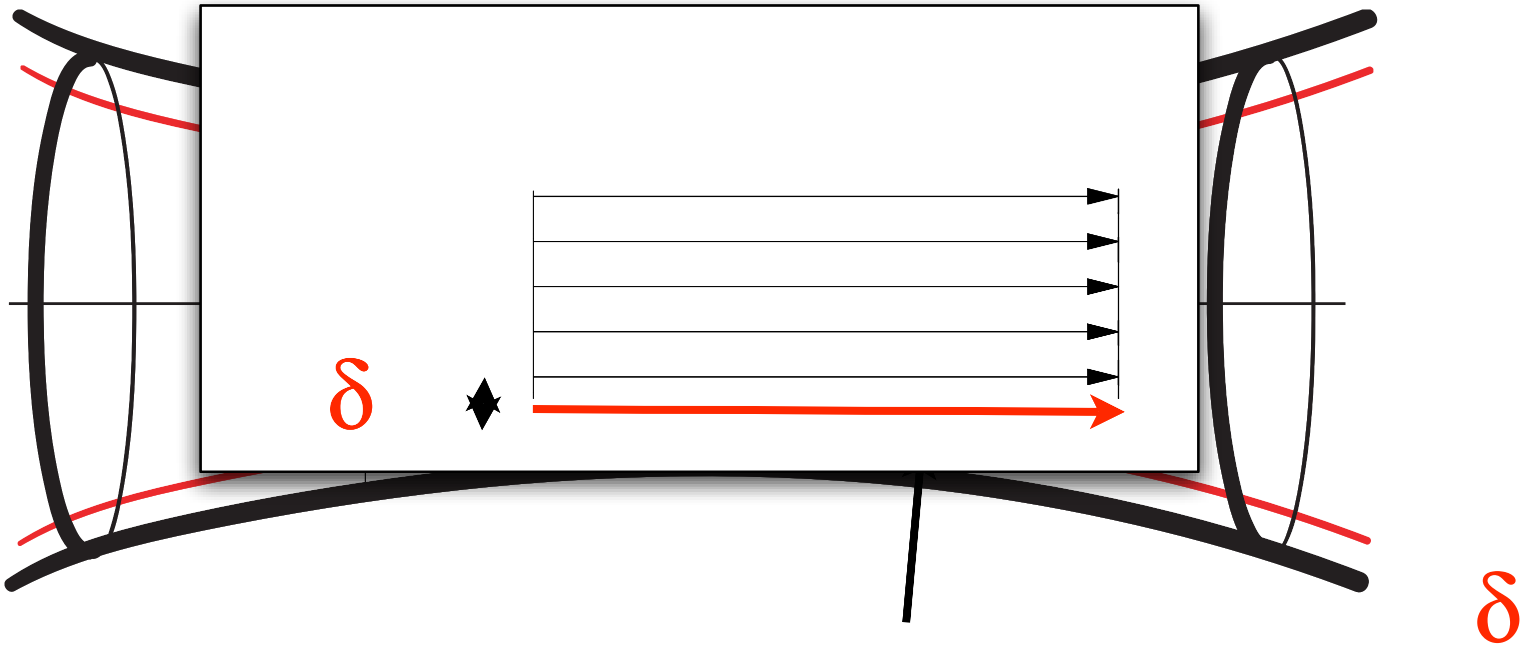
Interactive Boundary Layer



$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial n} = 0$$

$$u\frac{\partial}{\partial x}u + v\frac{\partial}{\partial n}u = -\frac{\partial p}{\partial x} + \frac{\partial^2}{\partial n^2}u \qquad 0 = -\frac{\partial p}{\partial n}$$

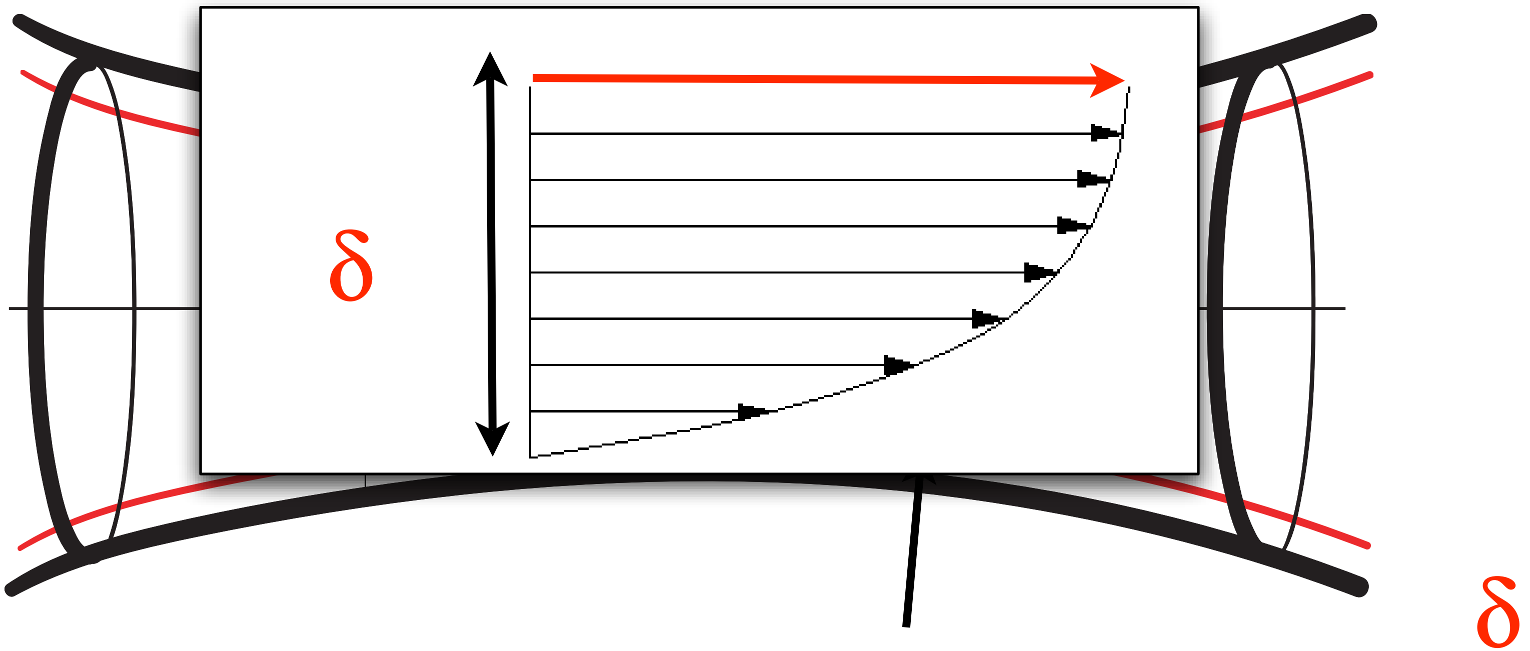
Interactive Boundary Layer



Matching of velocity from invicid/ viscous

U_e at the wall

Interactive Boundary Layer

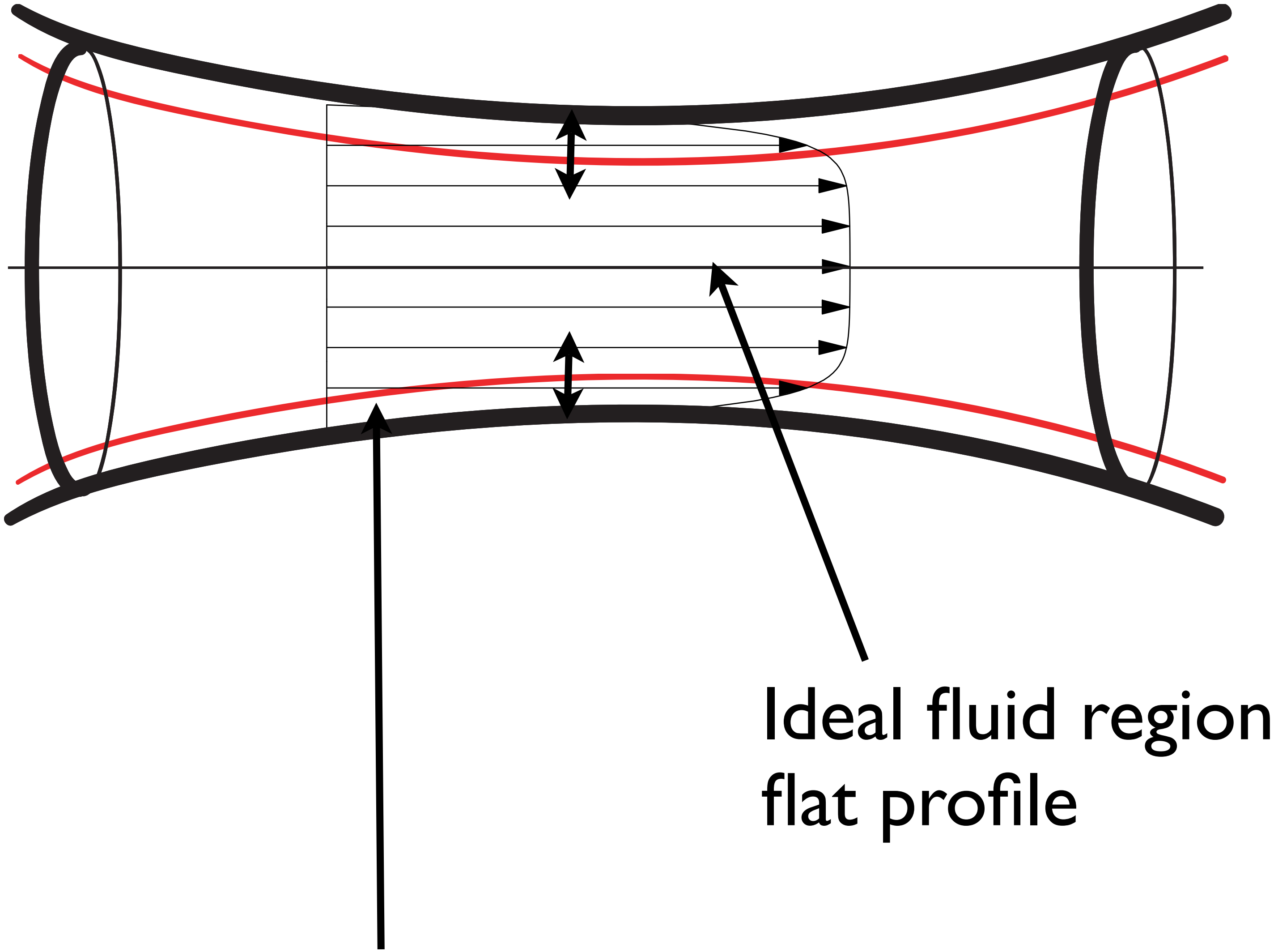


U_e at the wall

is the velocity at the edge of the boundary layer $u(x,\infty)$

at "infinity"

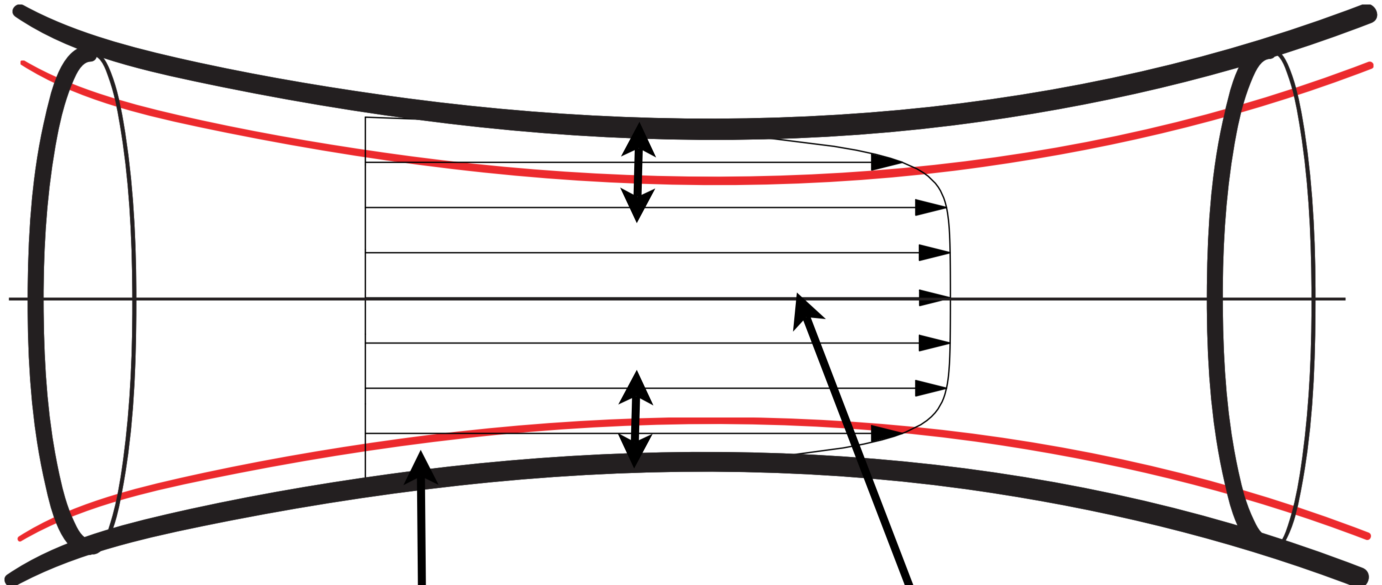
Interactive Boundary Layer



$$\delta_1 = \int_0^\infty (1 - \frac{u}{U_e}) dn$$

displacement of stream lines

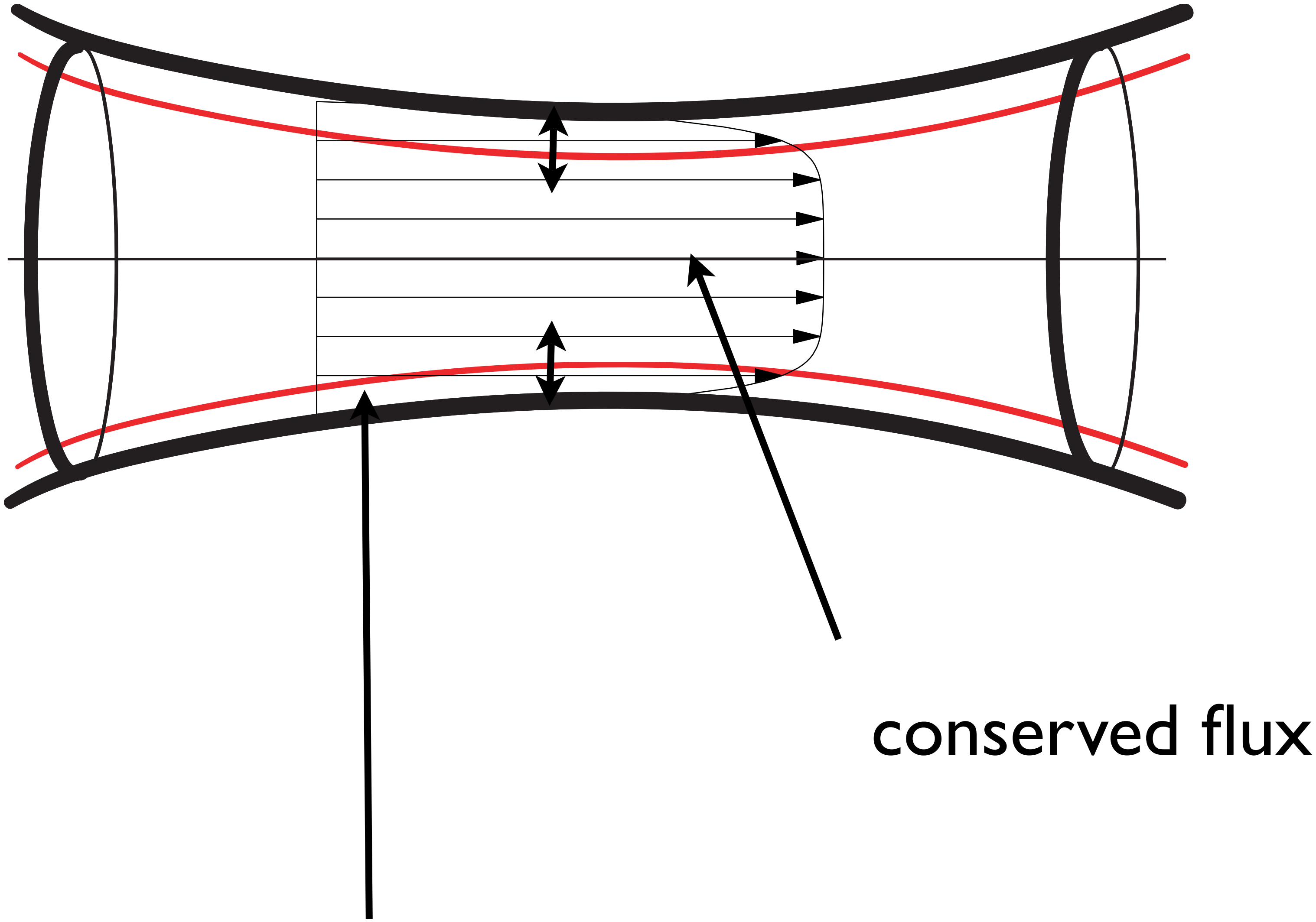
Interactive Boundary Layer



Ideal fluid region
flat profile perturbed by the
boundary layer thickness

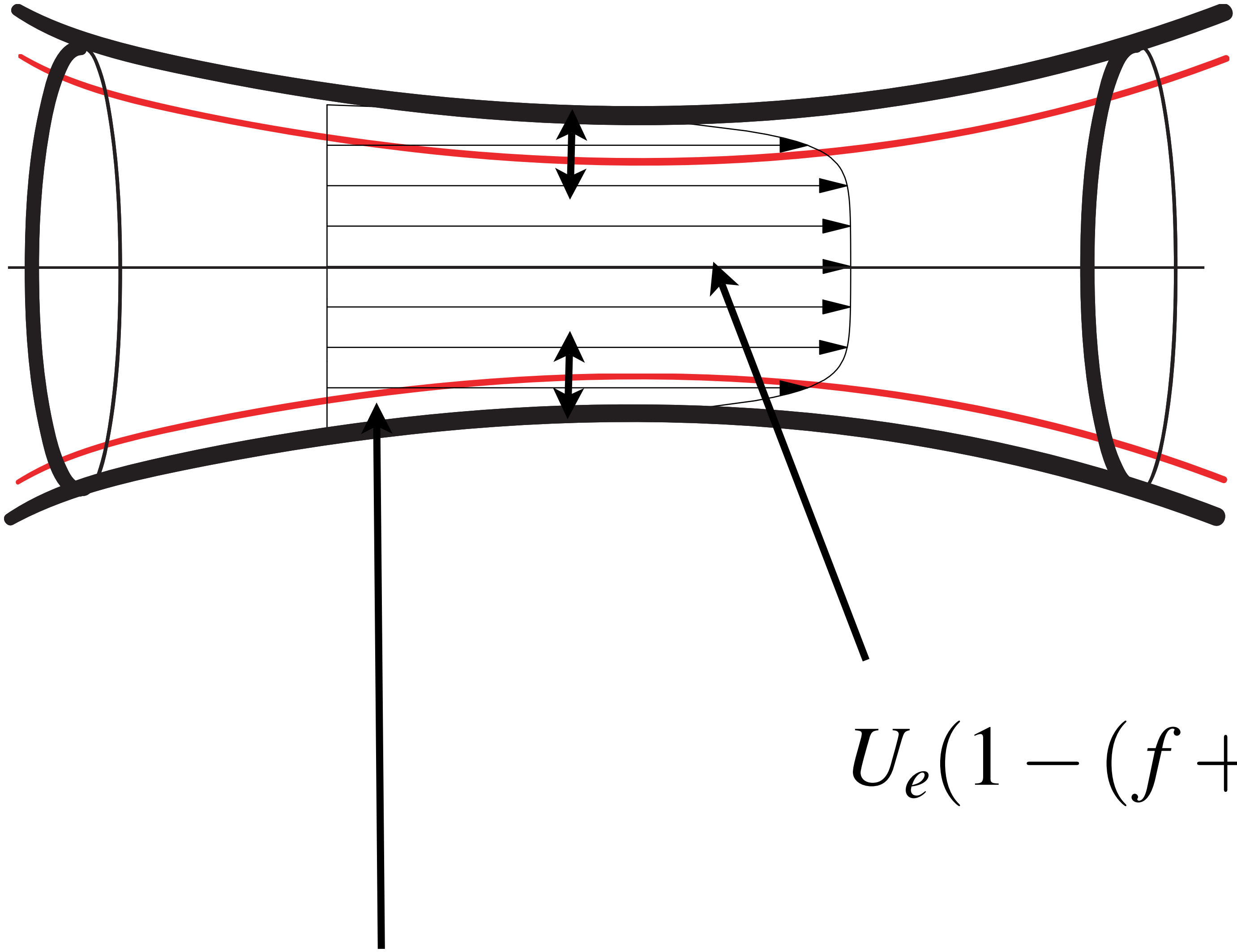
$$\delta_1 = \int_0^\infty (1 - \frac{u}{U_e}) dn$$

Interactive Boundary Layer



$$\delta_1 = \int_0^{\infty} (1 - \frac{u}{U_e}) dn$$

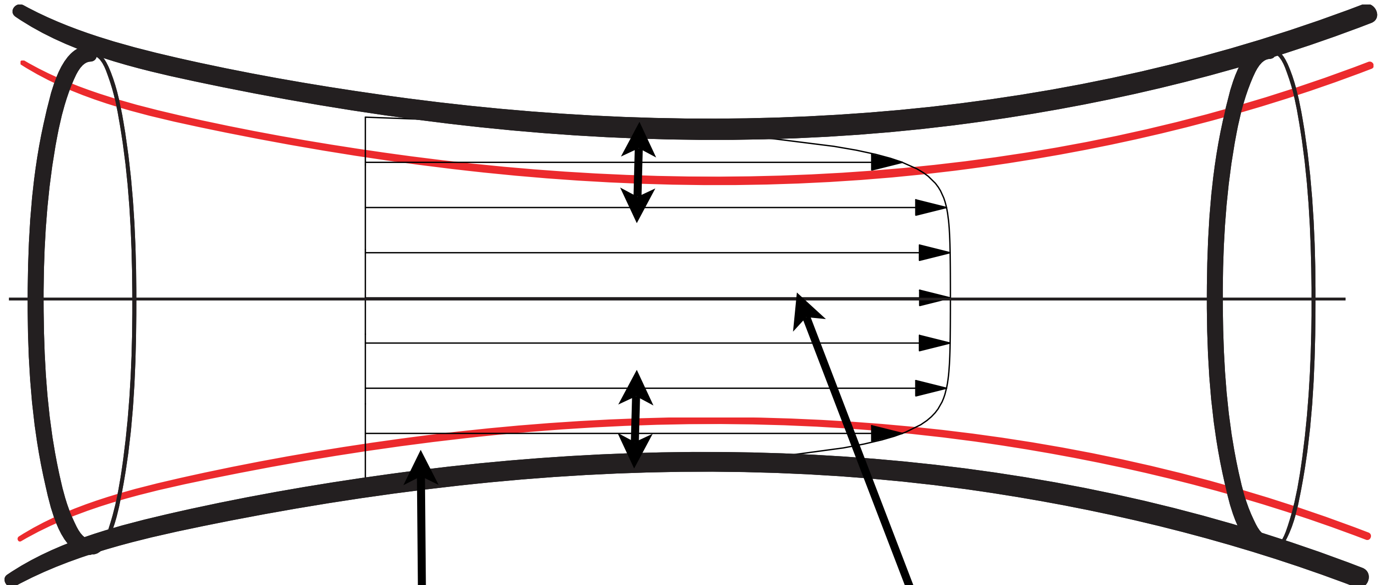
Interactive Boundary Layer



$$U_e(1-(f+\delta_1))^2 = 1$$

$$\delta_1 = \int_0^\infty (1 - \frac{u}{U_e}) dn$$

Interactive Boundary Layer



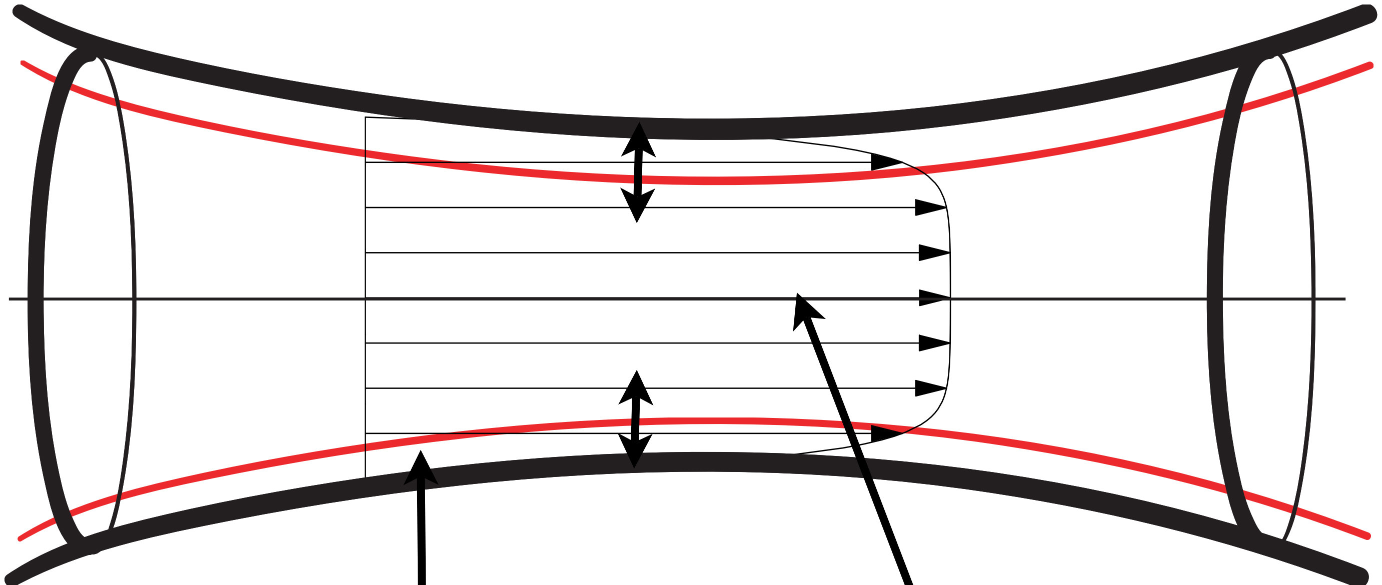
$$U_e(1-(f+\delta_1))^2 = 1$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial n} = 0$$

$$u\frac{\partial}{\partial x}u + v\frac{\partial}{\partial n}u = \frac{dU_e}{dx} + \frac{\partial^2}{\partial n^2}u$$

$$\delta_1 = \int_0^\infty (1 - \frac{u}{U_e})dn$$

Interactive Boundary Layer



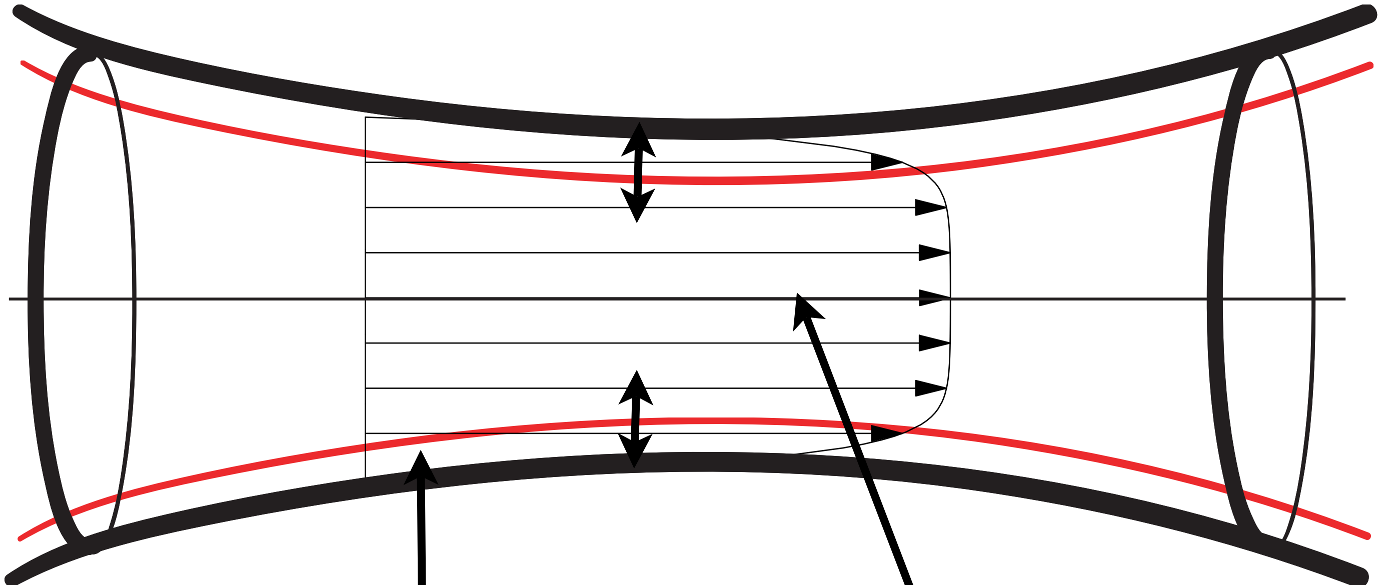
$$U_e(1-(f+\delta_1))^2 = 1$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial n} = 0 \qquad u(x,\infty) = U_e$$

$$u\frac{\partial}{\partial x}u + v\frac{\partial}{\partial n}u = \frac{dU_e}{dx} + \frac{\partial^2}{\partial n^2}u$$

$$\delta_1 = \int_0^\infty (1-\frac{u}{U_e})dn$$

Interactive Boundary Layer



$$U_e(1-(f+\delta_1))^2 = 1$$

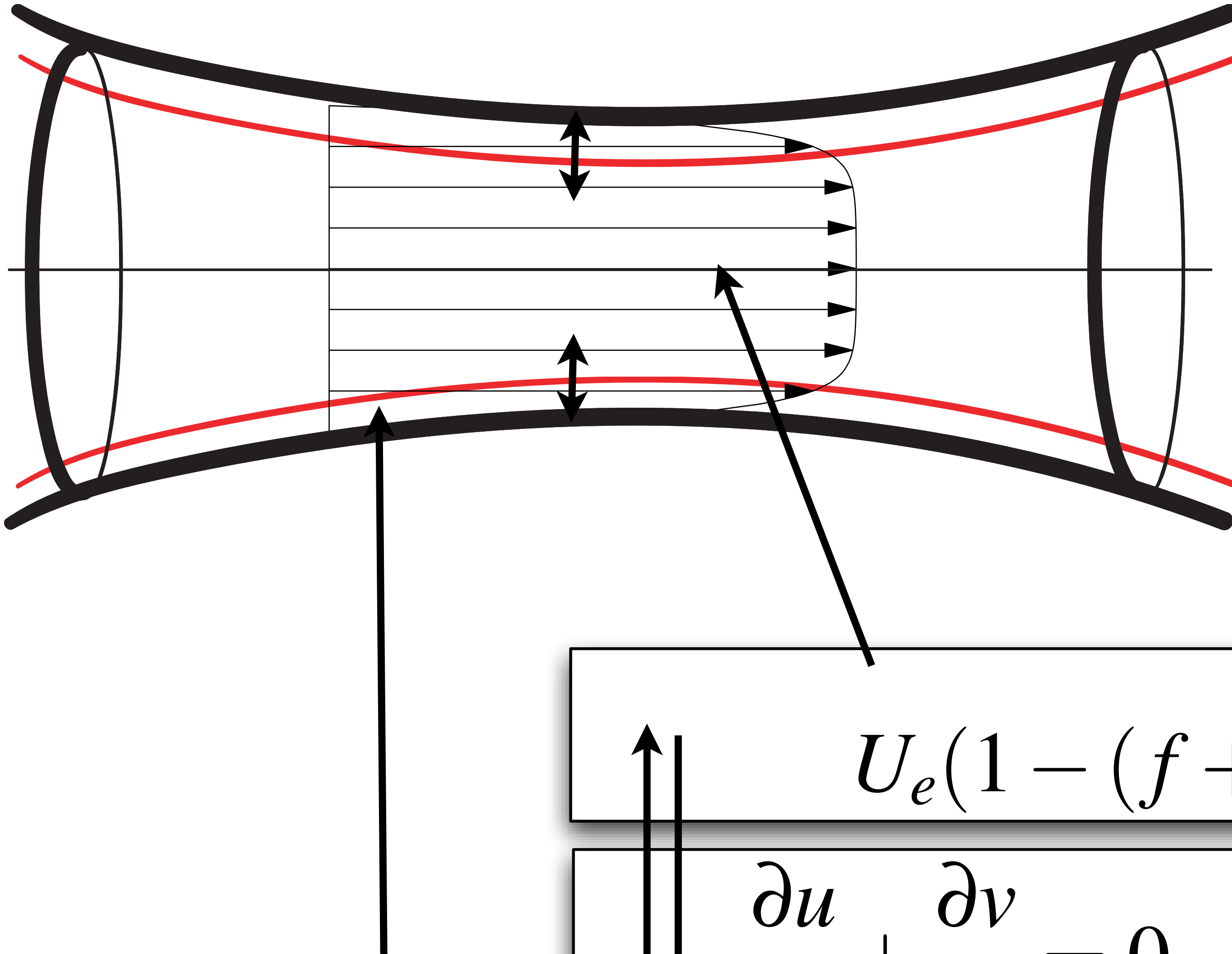
$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial n}=0$$

$$u(x,\infty)=U_e$$

$$u\frac{\partial}{\partial x}u+v\frac{\partial}{\partial n}u=\frac{dU_e}{dx}+\frac{\partial^2}{\partial n^2}u$$

$$\delta_1=\int_0^\infty\left(1-\frac{u}{U_e}\right)dn$$

Interactive Boundary Layer



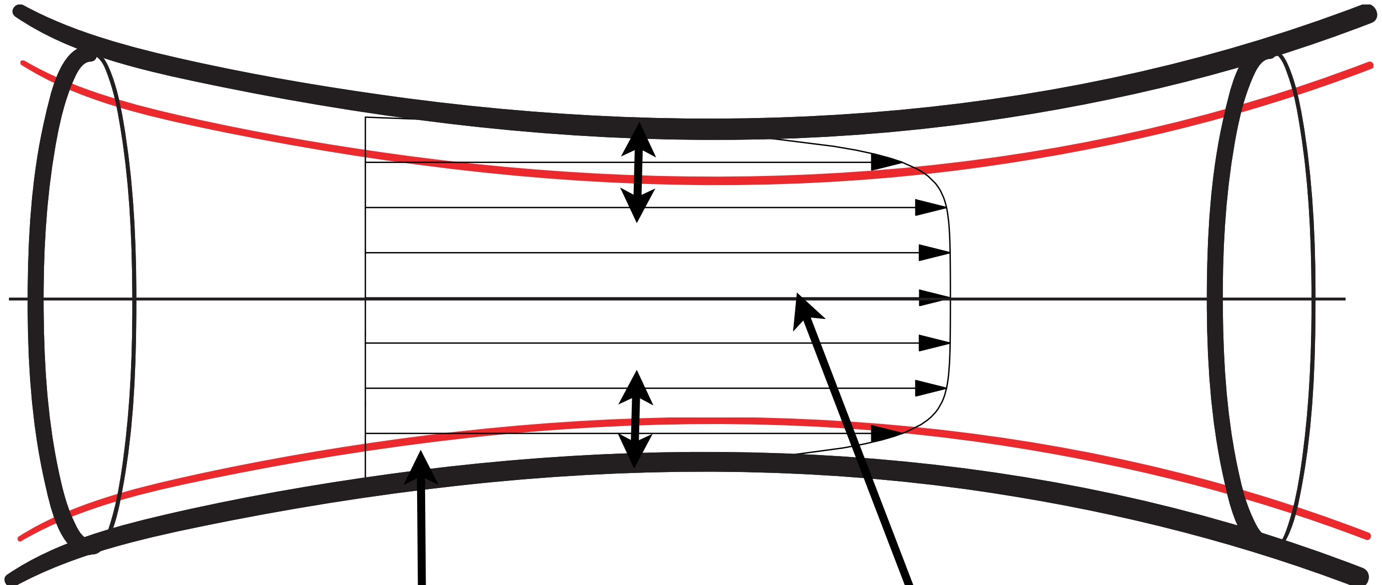
$$U_e(1-(f+\delta_1))^2 = 1$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial n}=0 \qquad u(x,\infty)=U_e$$

$$u\frac{\partial}{\partial x}u+v\frac{\partial}{\partial n}u=\frac{dU_e}{dx}+\frac{\partial^2}{\partial n^2}u$$

$$\delta_1=\int_0^\infty (1-\frac{u}{U_e})dn$$

Interactive Boundary Layer



Coupled System to solve

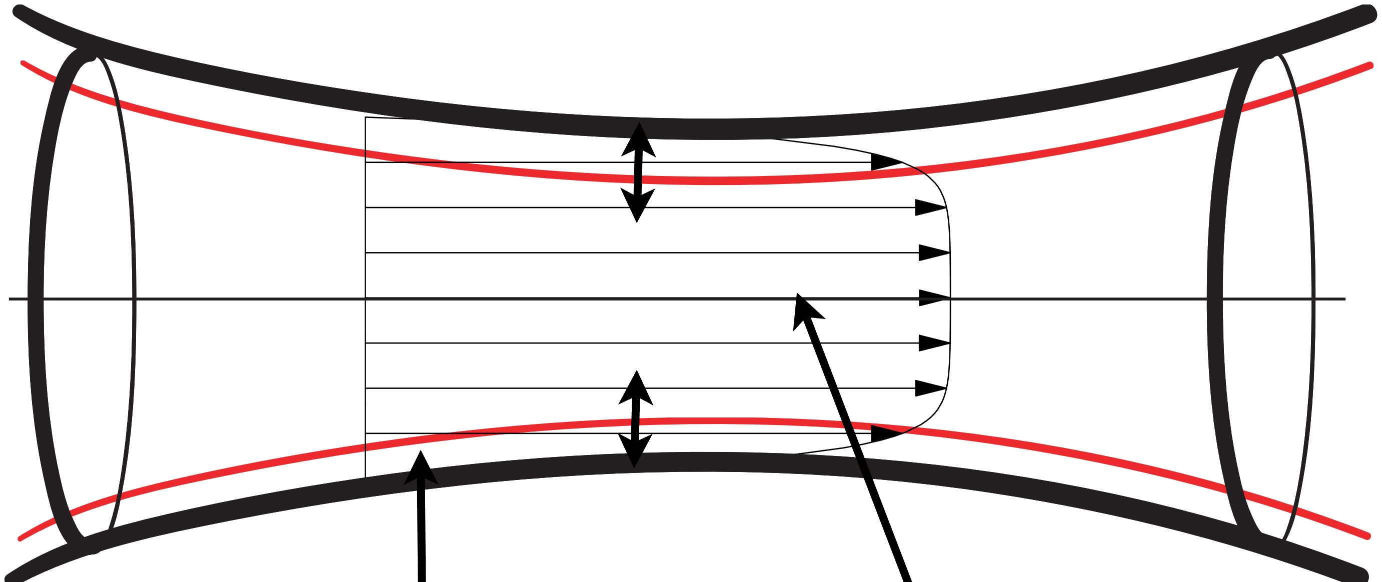
$$U_e(1-(f+\delta_1))^2 = 1$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial n} = 0 \qquad u(x,\infty) = U_e$$

$$u\frac{\partial}{\partial x}u + v\frac{\partial}{\partial n}u = \frac{dU_e}{dx} + \frac{\partial^2}{\partial n^2}u$$

$$\delta_1 = \int_0^\infty (1 - \frac{u}{U_e})dn$$

Interactive Boundary Layer



Coupled System to solve

$$U_e(1-(f+\delta_1))^2 = 1$$

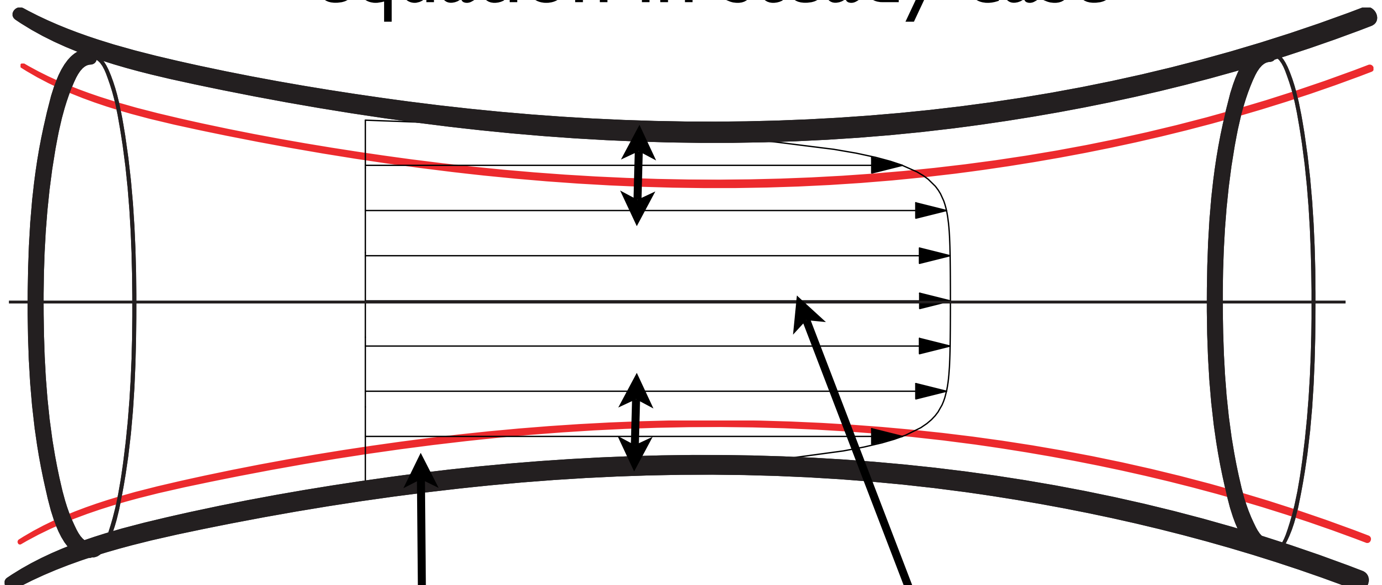
$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial n} = 0 \qquad u(x,\infty) = U_e$$

$$u\frac{\partial}{\partial x}u + v\frac{\partial}{\partial n}u = \frac{dU_e}{dx} + \frac{\partial^2}{\partial n^2}u$$

$$\int_0^\infty dn$$

$$\delta_1 = \int_0^\infty (1 - \frac{u}{U_e})dn$$

Integral resolution equation in steady case

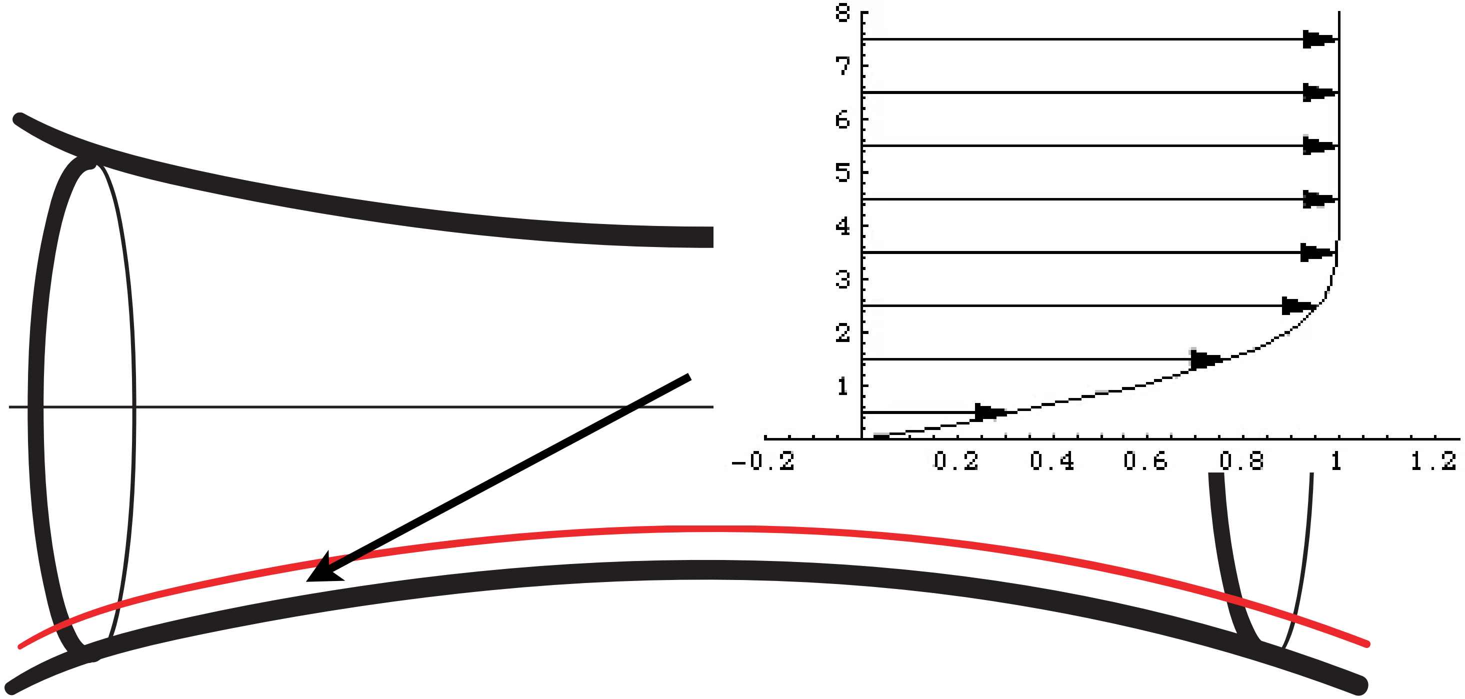


Coupled System to solve

$$U_e(1-(f+\delta_1))^2 = 1$$

$$\frac{d}{dx}\left(\frac{\delta_1}{H}\right) + \frac{\delta_1}{U_e}\left(1+\frac{2}{H}\right)\frac{dU_e}{dx} = \frac{f_2 H}{\delta_1 U_e}$$

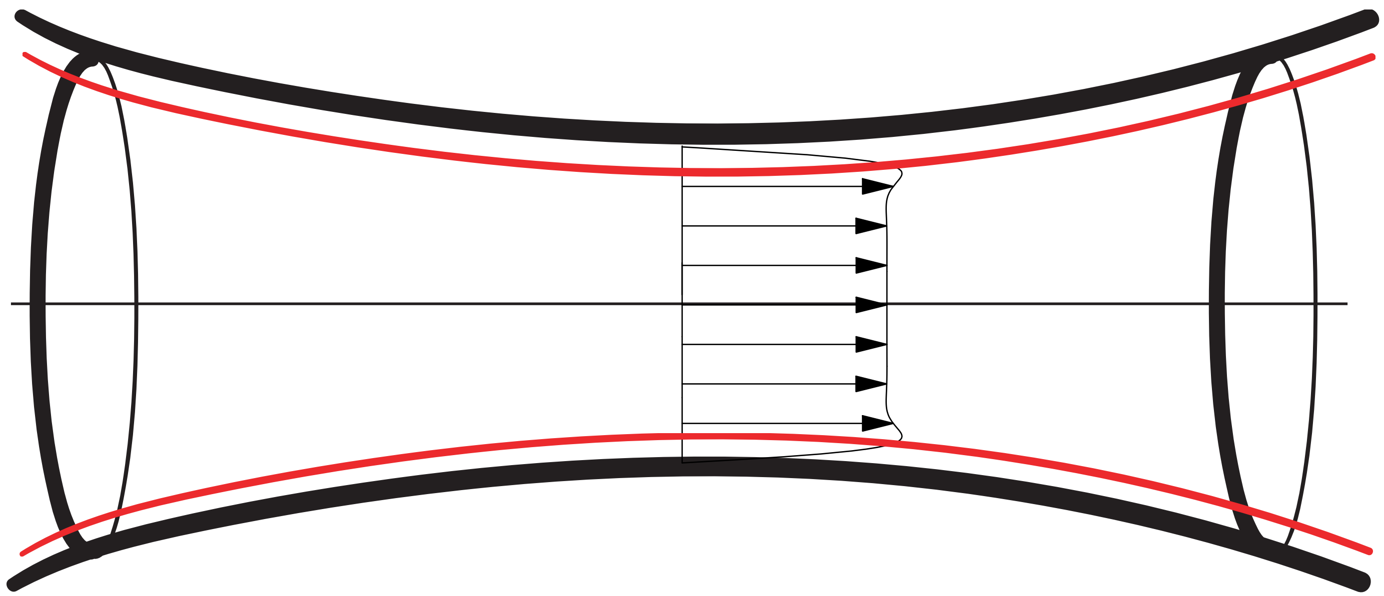
$$\delta_1 = \int_0^\infty \left(1-\frac{u}{U_e}\right)dn$$



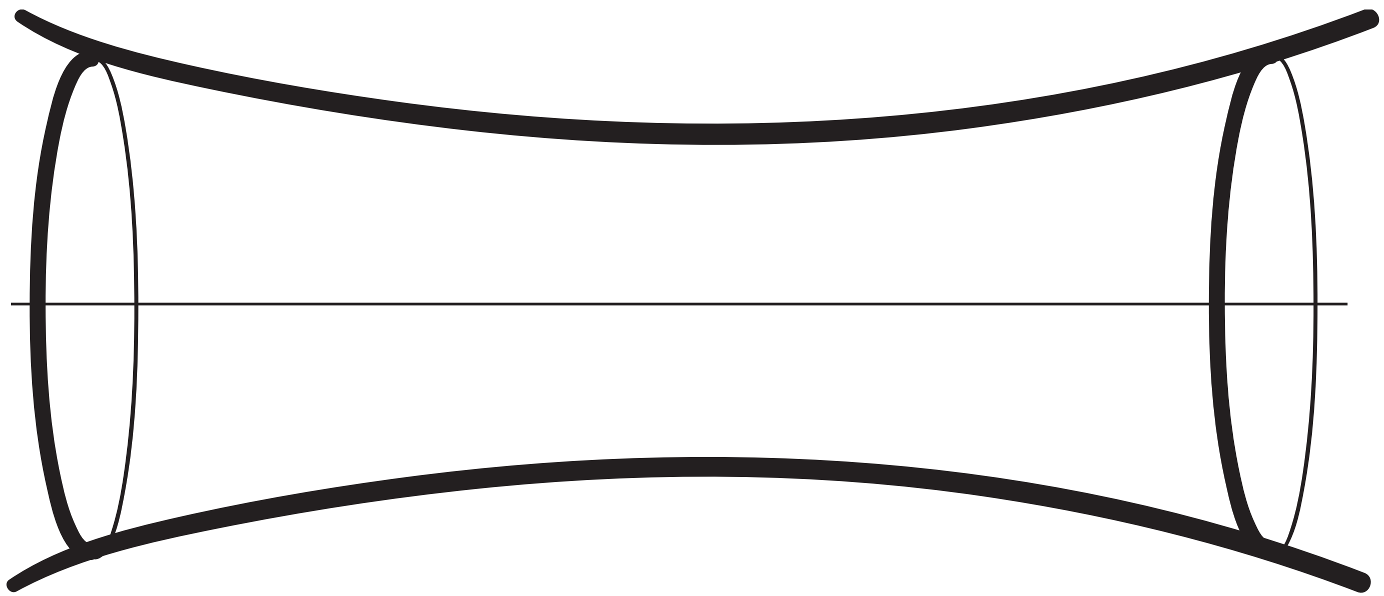
Choice of the family of simple profiles

In a steady flow it is natural to use Falkner Skan

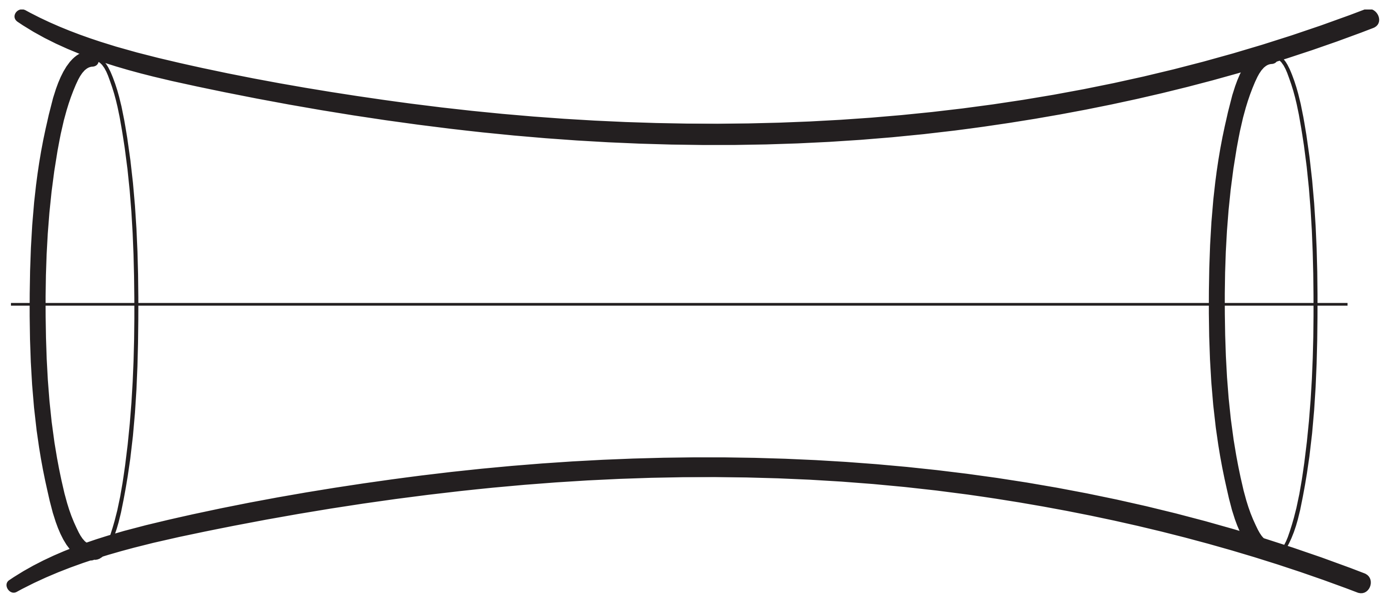
Interactive Boundary Layer



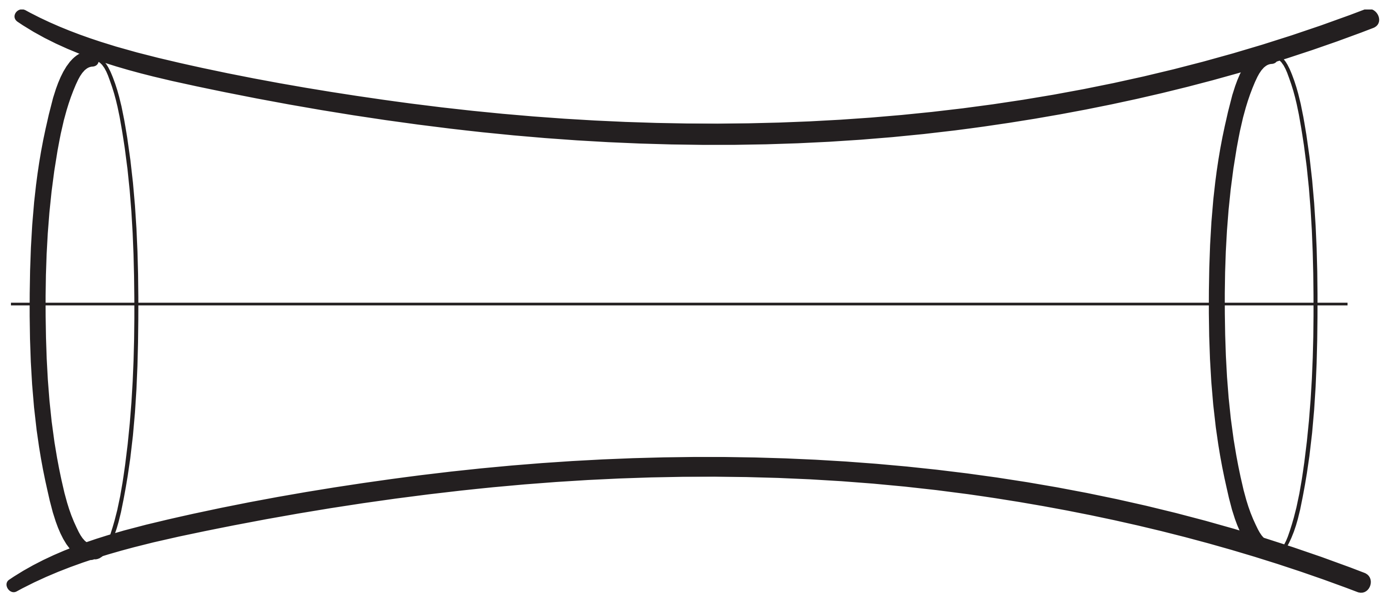
IBL is included in RNSP



RNSP includes usual 1D equations
RNSP includes Womersley profiles
RNSP includes Boundary Layer Theories (IBL)



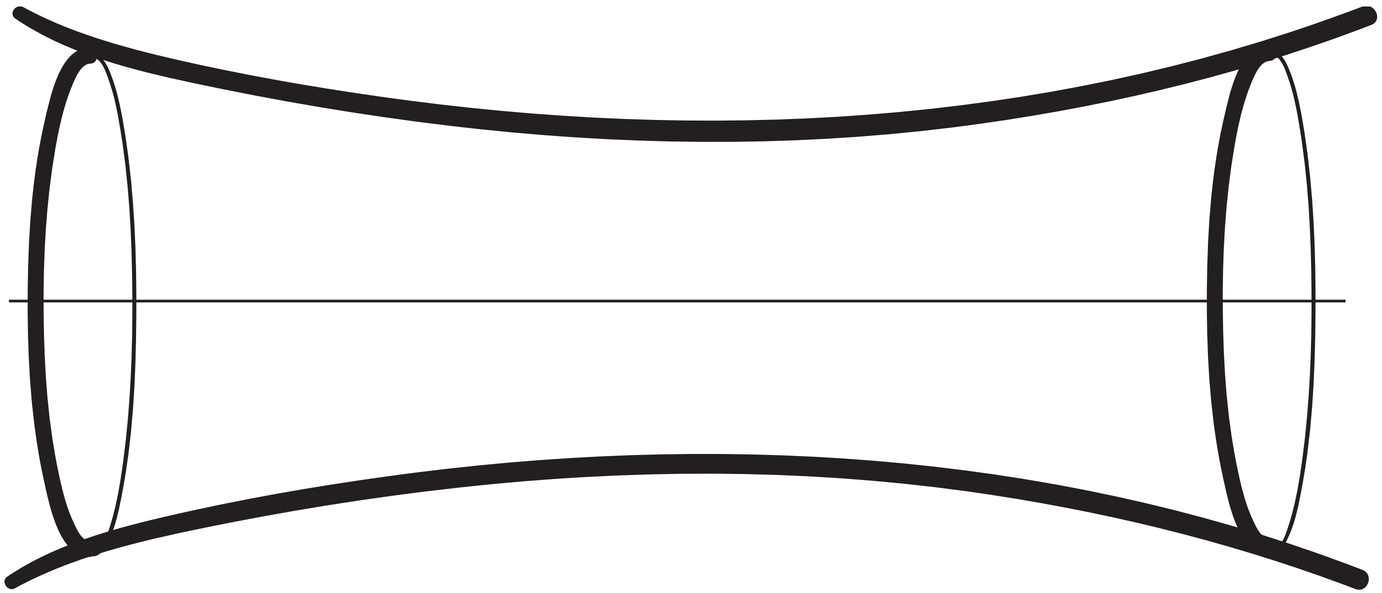
Comparisons



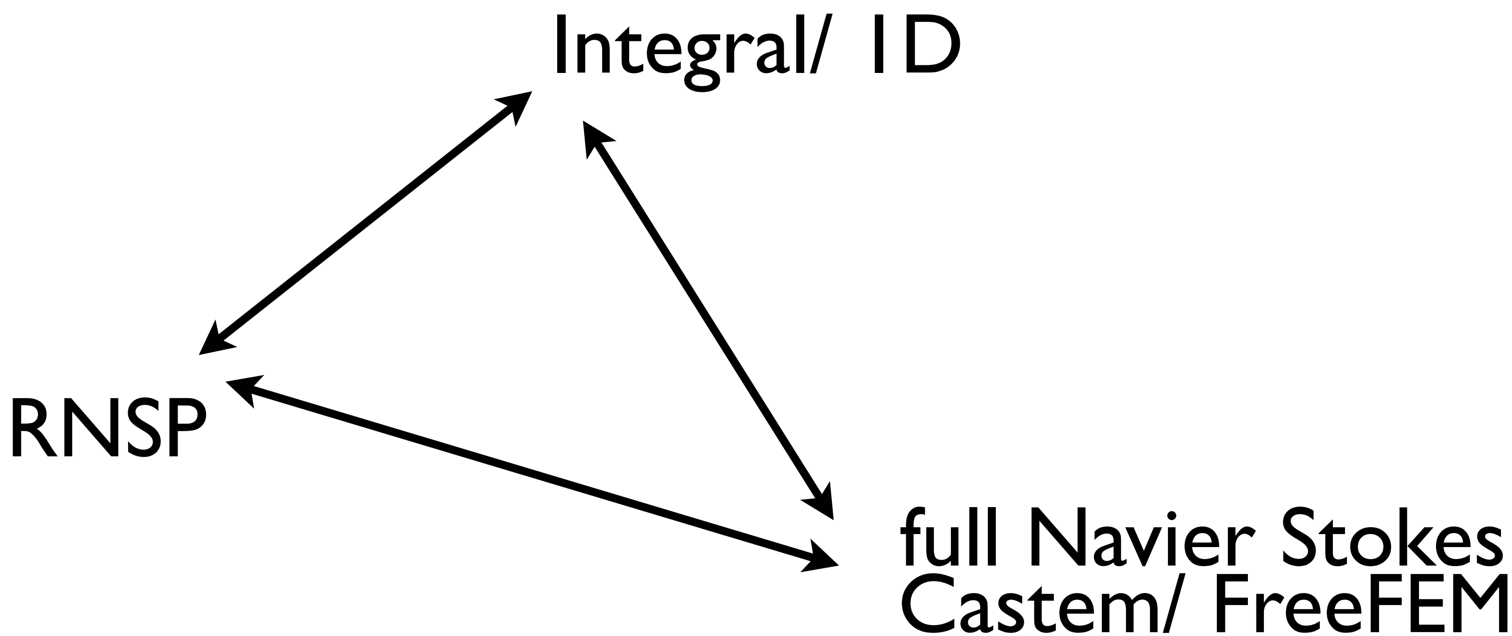
Comparisons

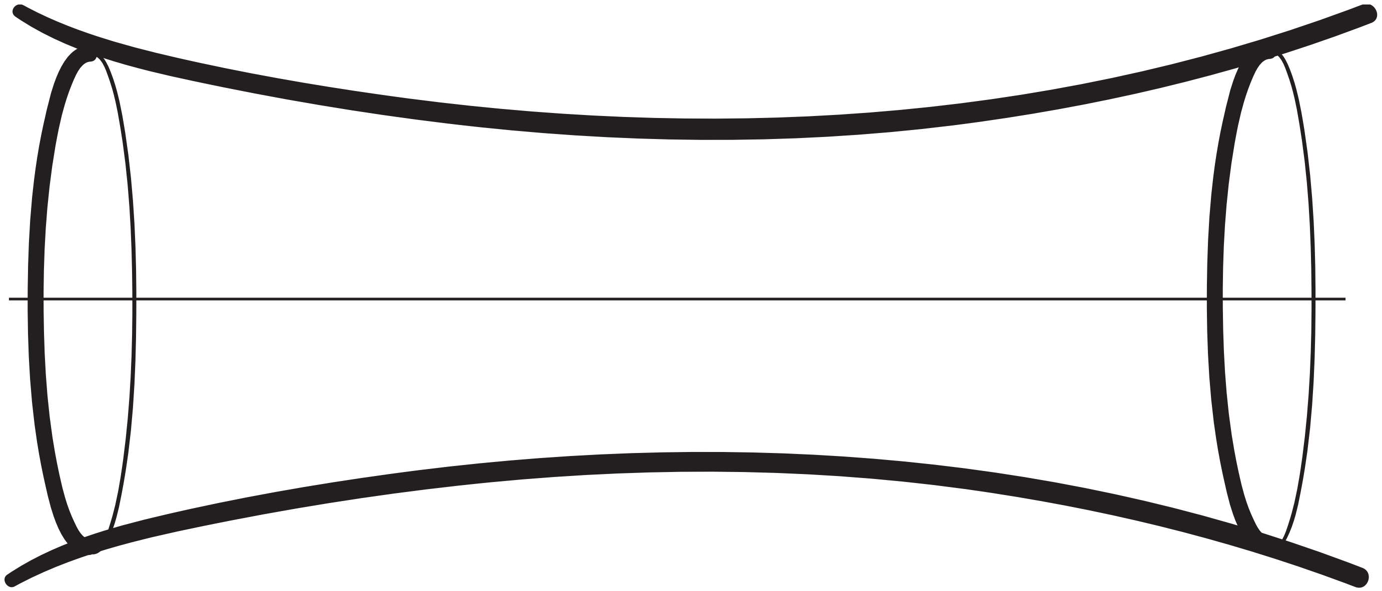
Integral/ 1D

full Navier Stokes
Castem/ FreeFEM

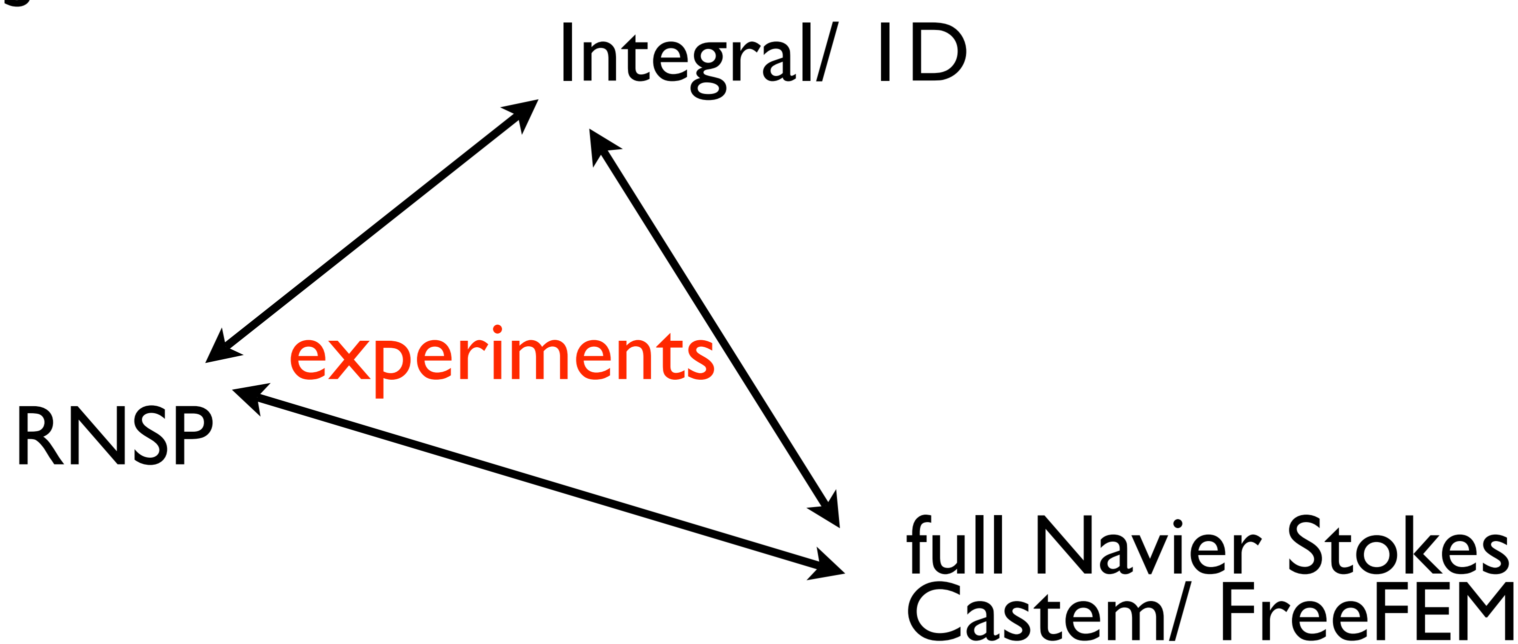


Comparisons

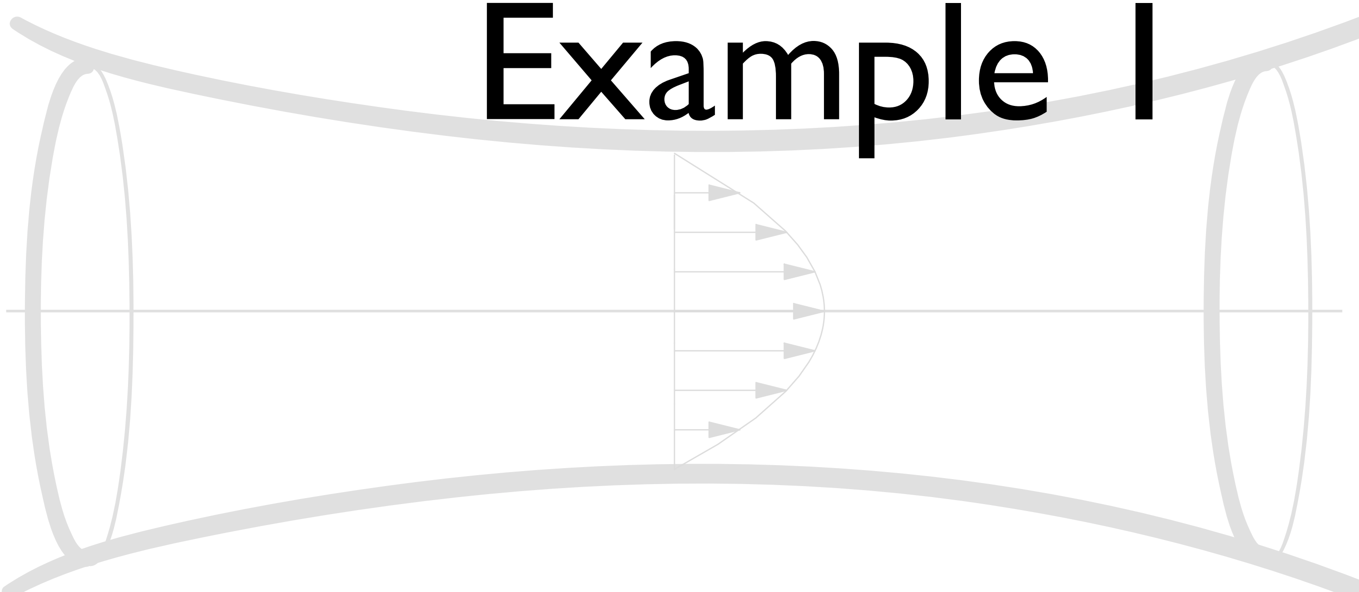




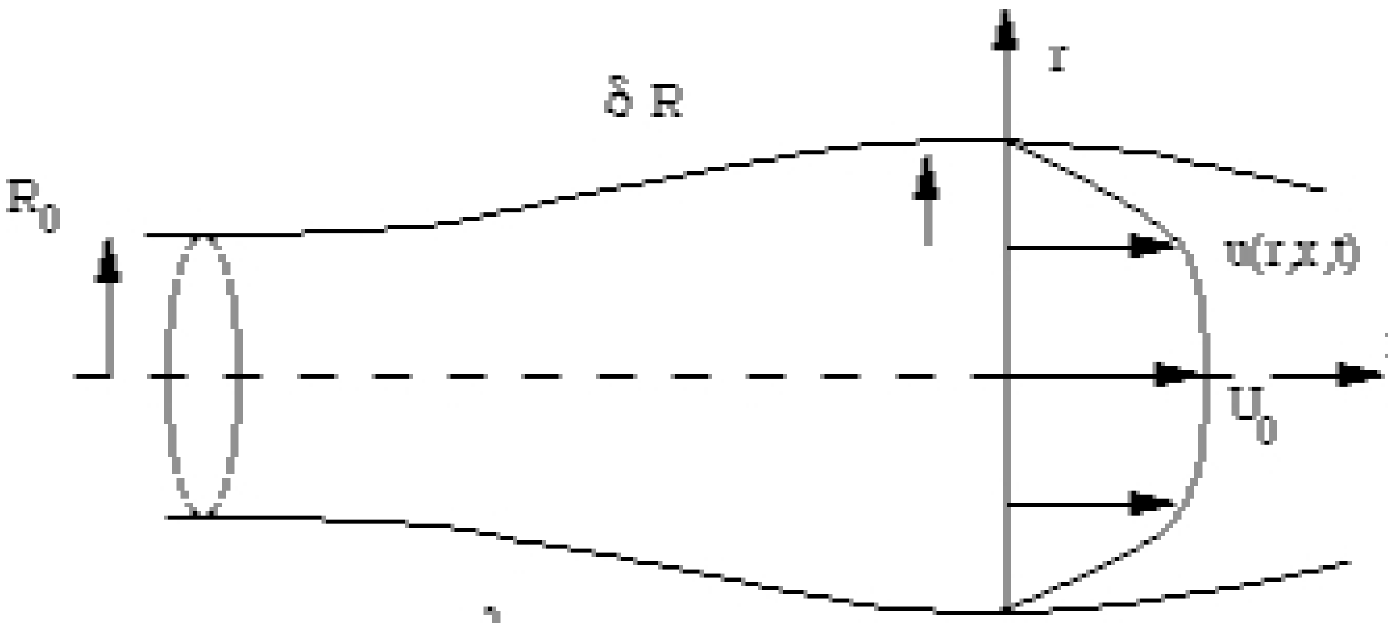
Comparisons



Example 1



flow in arteries



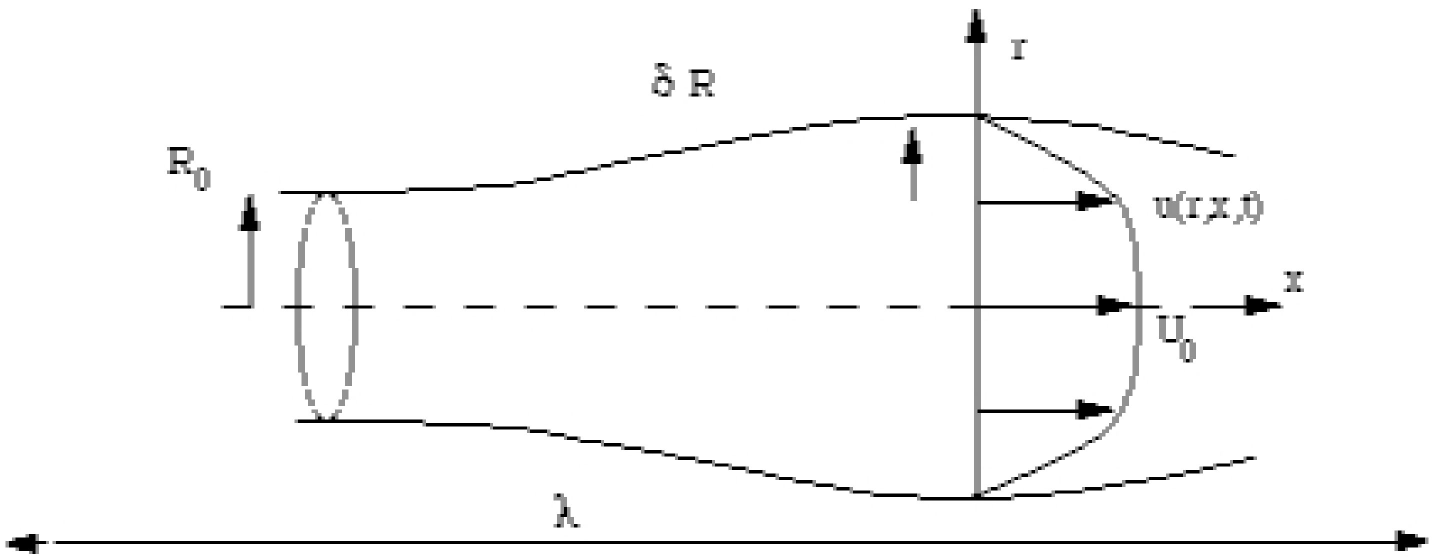
$$\frac{\partial}{\partial x}u + \frac{\partial}{r\partial r}rv = 0,$$

$$\frac{\partial u}{\partial t} + \varepsilon_2\left(u\frac{\partial}{\partial x}u + v\frac{\partial}{\partial r}u\right) = -\frac{\partial p}{\partial x} + \frac{2\pi}{\alpha^2}\frac{\partial}{r\partial r}\left(r\frac{\partial}{\partial r}u\right), 0 = -\frac{\partial p}{\partial r}.$$

$$\varepsilon_2 = \frac{\delta R}{R_0}, \qquad \alpha = R_0\sqrt{\frac{2\pi/T}{\nu}}$$

introducing wall elasticity: $p(x,t) = k(R(x,t) - R_0)$

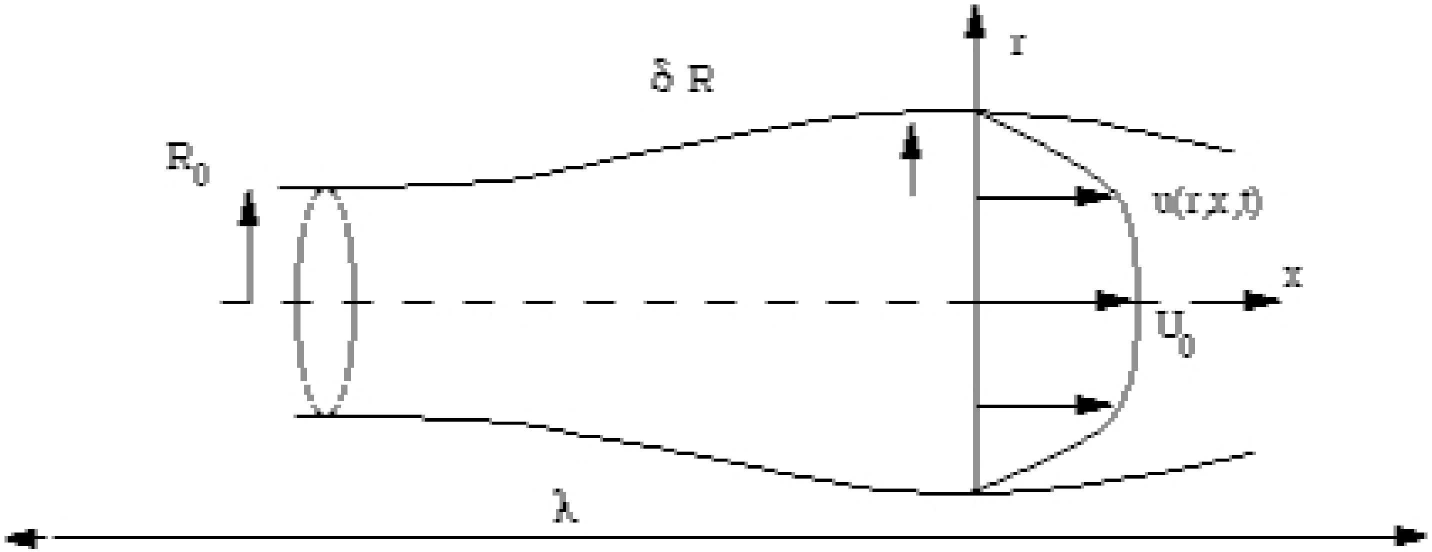
+ The boundary conditions: here hyperbolical ($R(x_{in},t)$ and $R(x_{out},t)$) given



week coupling

$$\frac{u^{n+1}-u^{n}}{\Delta t}+NL(u^{n})=-\frac{\partial p^{n}}{\rho\partial x}+\nu\frac{\partial}{r\partial r}r\frac{\partial u^{n+1}}{\partial r}$$

$$v^{n+1}(R^{n})=-\int_{0}^{R^{n}}\frac{r}{R^{n}}\frac{\partial u^{n+1}}{\partial x}dr$$

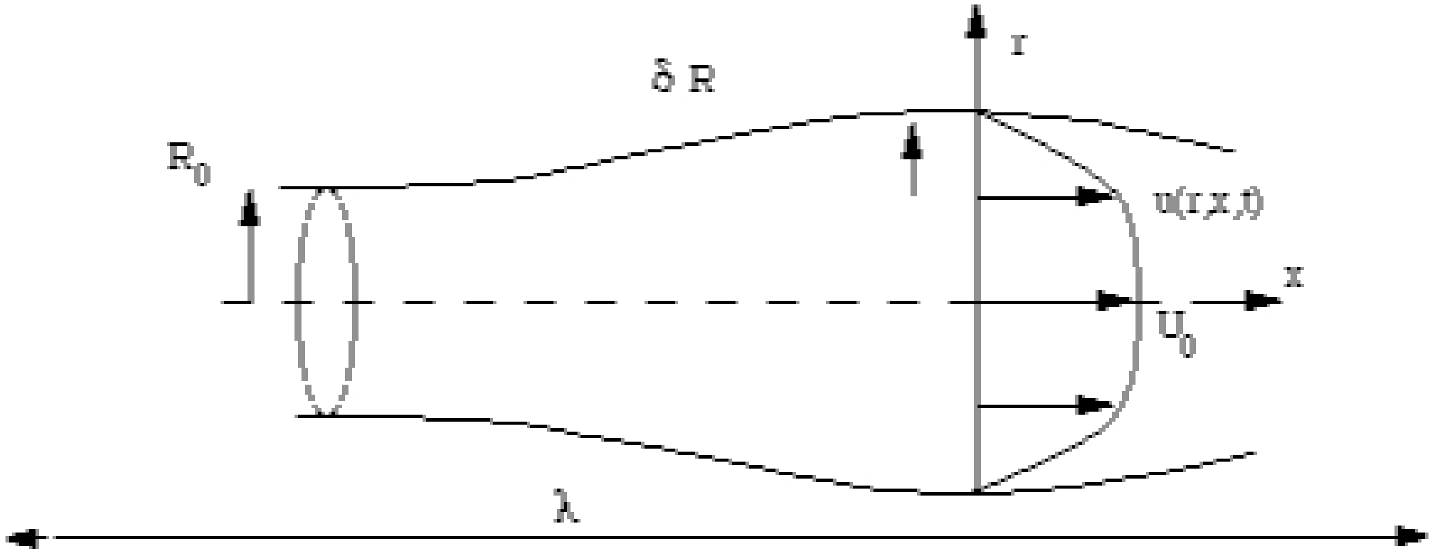


week coupling

$$\frac{u^{n+1}-u^{n}}{\Delta t}+NL(u^{n})=-\frac{\partial p^{n}}{\rho\partial x}+\nu\frac{\partial}{r\partial r}r\frac{\partial u^{n+1}}{\partial r}$$

$$v^{n+1}(R^{n})=-\int_{0}^{R^{n}}\frac{r}{R^{n}}\frac{\partial u^{n+1}}{\partial x}dr$$

$$R^{n+1}=R^{n}+v^{n+1}(R^{n})\Delta t$$

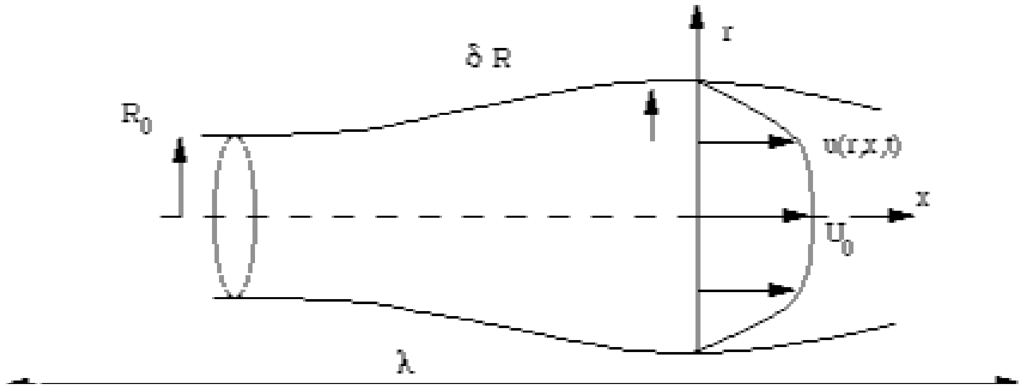


week coupling

$$\frac{u^{n+1}-u^n}{\Delta t}+NL(u^n)=-\frac{\partial p^n}{\rho\partial x}+\nu\frac{\partial}{r\partial r}r\frac{\partial u^{n+1}}{\partial r}$$

$$v^{n+1}(R^n)=-\int_0^{R^n}\frac{r}{R^n}\frac{\partial u^{n+1}}{\partial x}dr$$

$$R^{n+1}=R^n+v^{n+1}(R^n)\Delta t \qquad p^{n+1}=k(R^{n+1}-R_0)$$

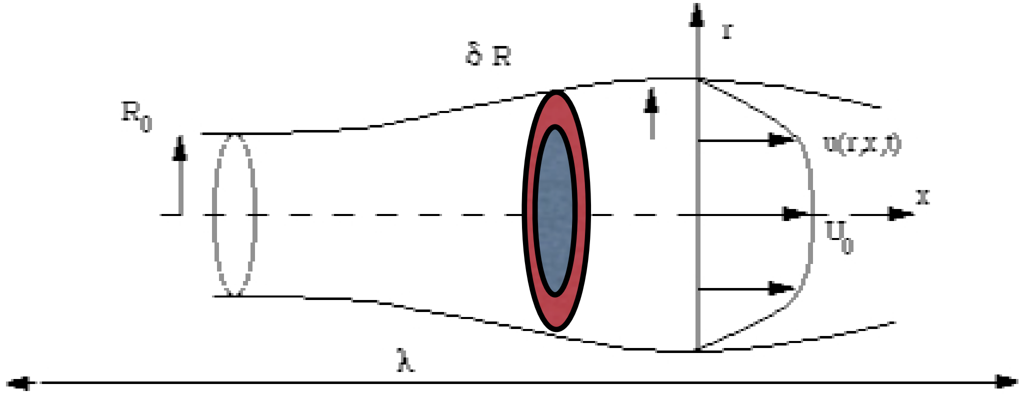


Flow in an elastic artery: integral relations

- new integral equations: adapting Von Kármán integral methods

The key is to integrate the equations with respect to the variable $\eta = r/R$ from the centre of the pipe to the wall $(0 \leq \eta \leq 1)$.

specific integral system improved compared to the classical ones



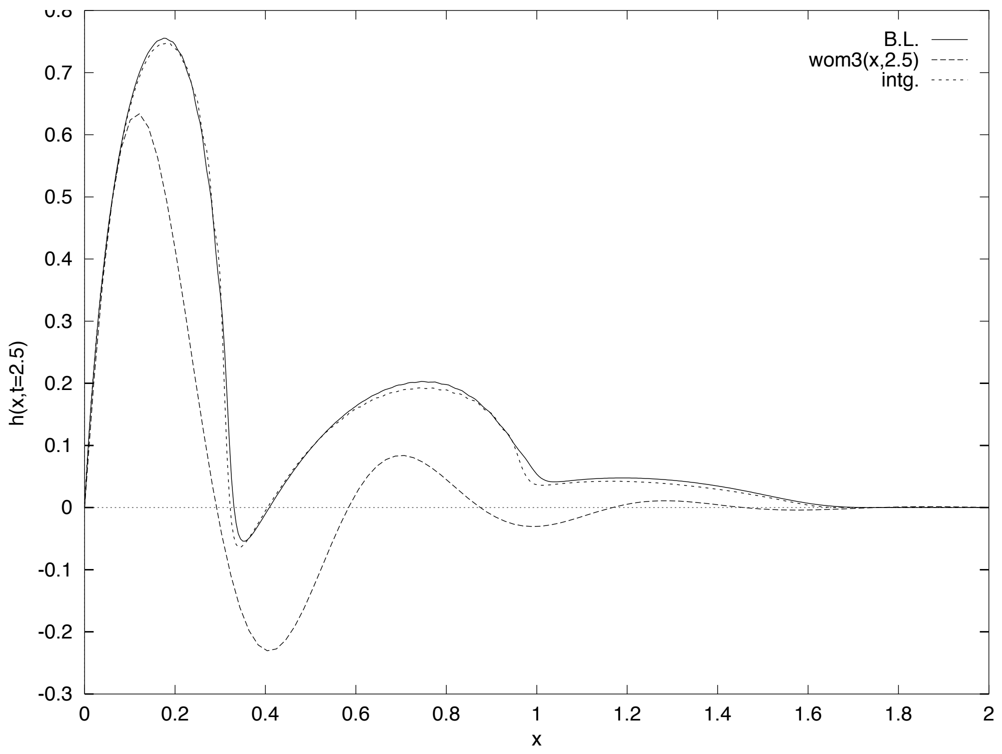
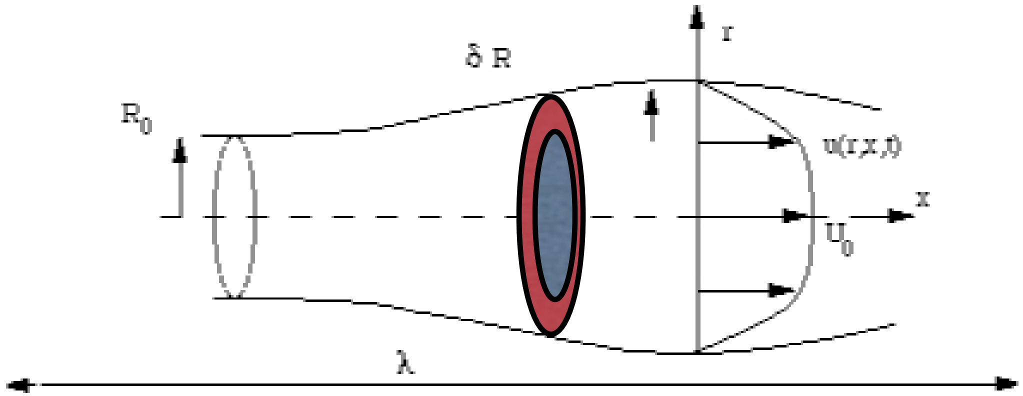
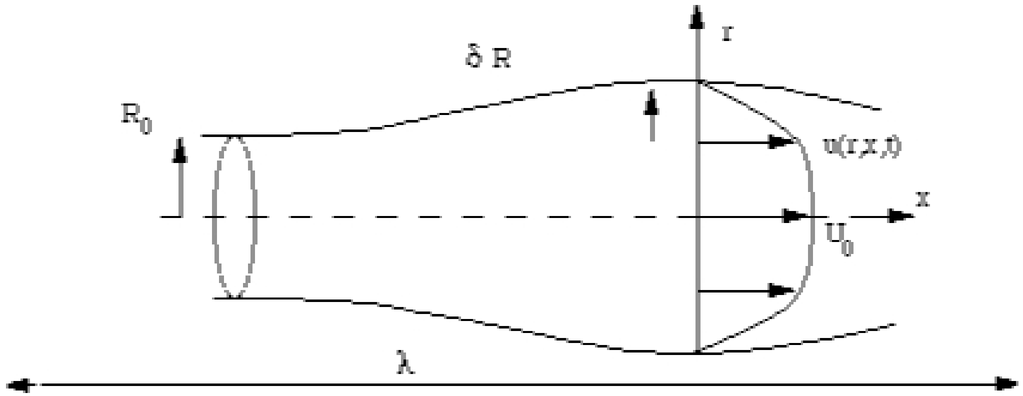


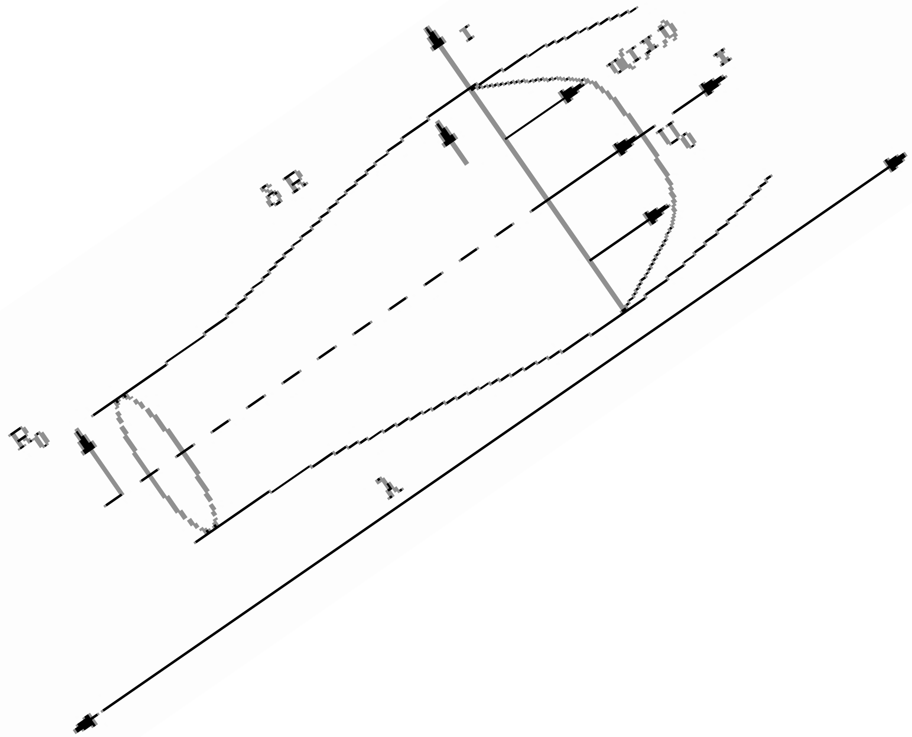
Figure 1: The displacement of the wall ($h(x, t = 2.5)$) as a function of x is plotted here at time $t = 2.5$. The dashed line (wom3(x,2.5)) is the Womersley solution (reference), the solid line (B.L.) is the result of the Boundary Layer code and the dots (intg) are the results of the integral method ($\alpha = 3$, $k_1 = 1$, $k_2 = 0$ and $\varepsilon_2 = 0.2$).



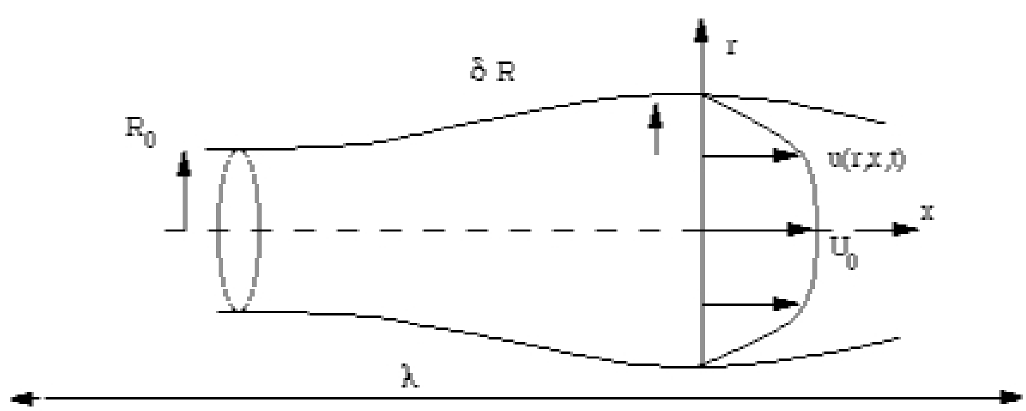
Constructed an inverse method



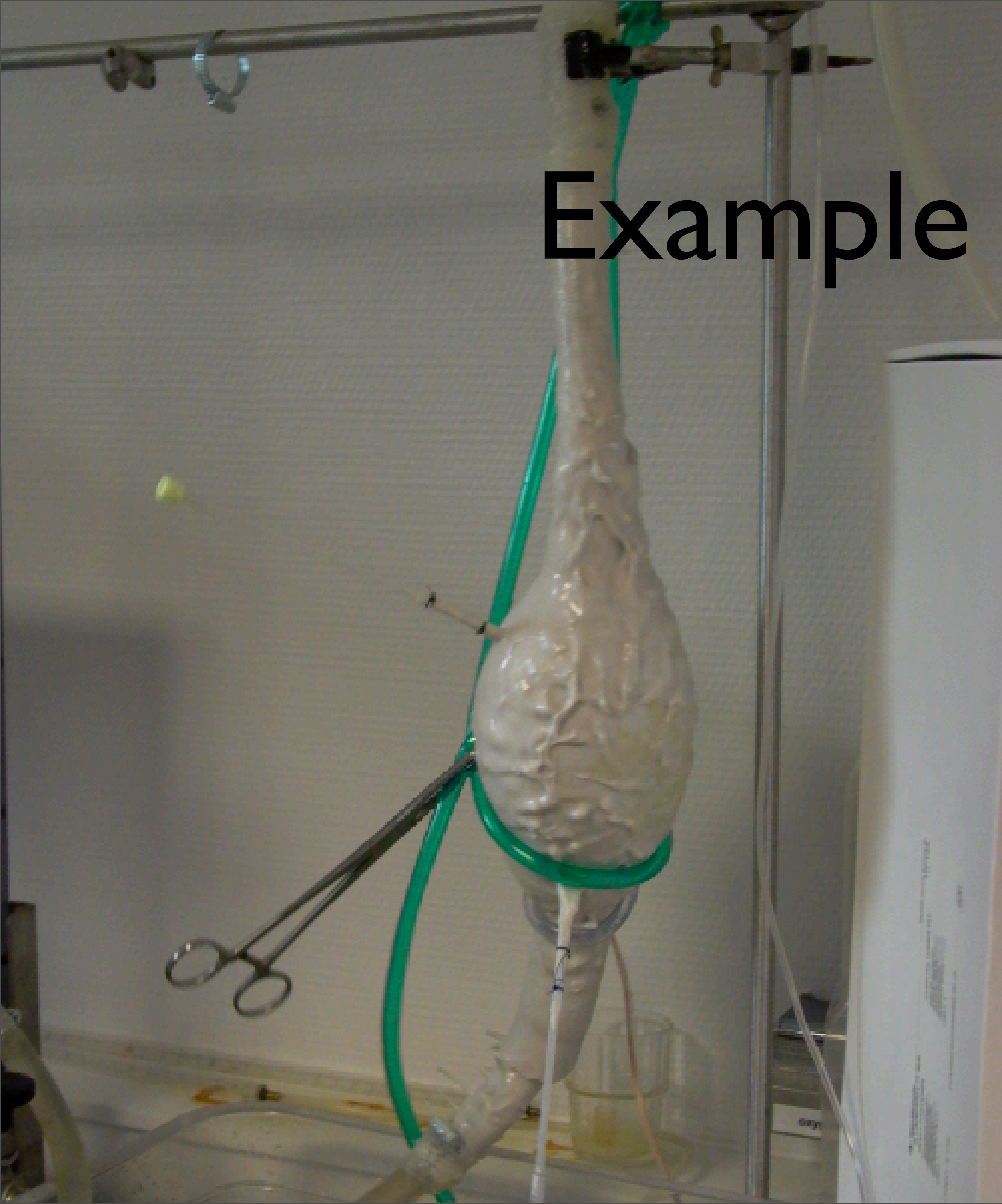
r
δ R
R0
u(r,x,t)
x
U0
λ



r
u(r,x,t)
x
U0
δR
R0
λ


r
δ R
R_0
u(r,x,t)
x
U_0
λ

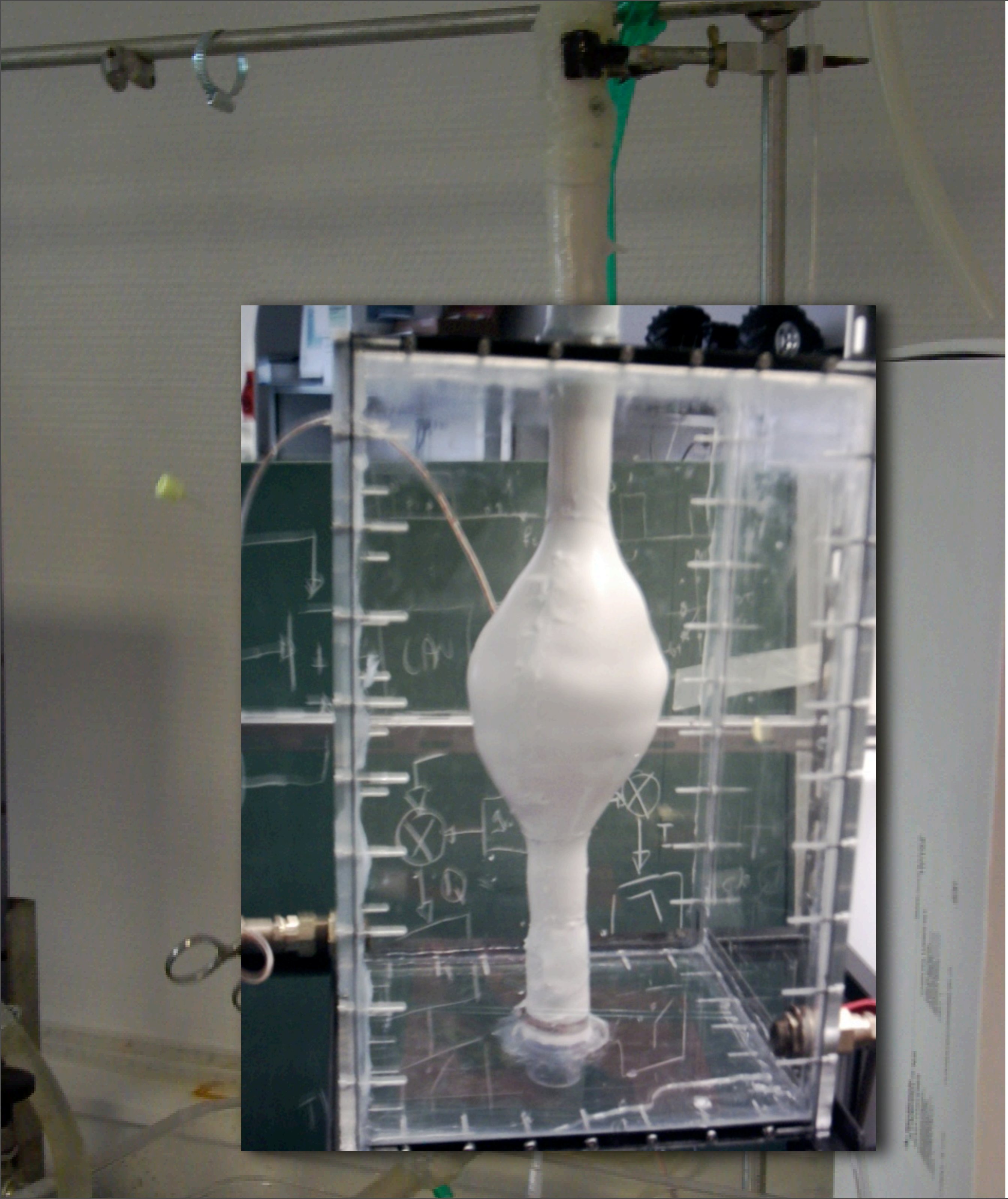
Example 2



Experiment

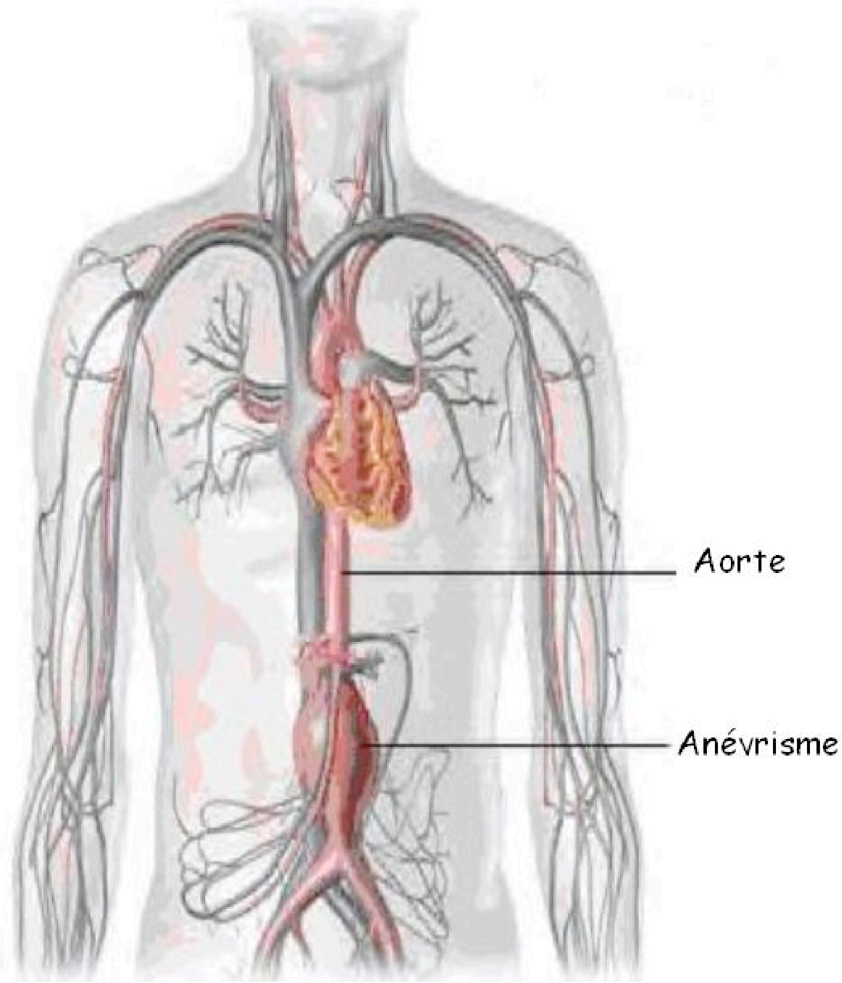
O. Romain,
J. Mazeyrat LISIF PVI

P. Leprince, M. Kerouia
Hospital Pitié Salpétrière



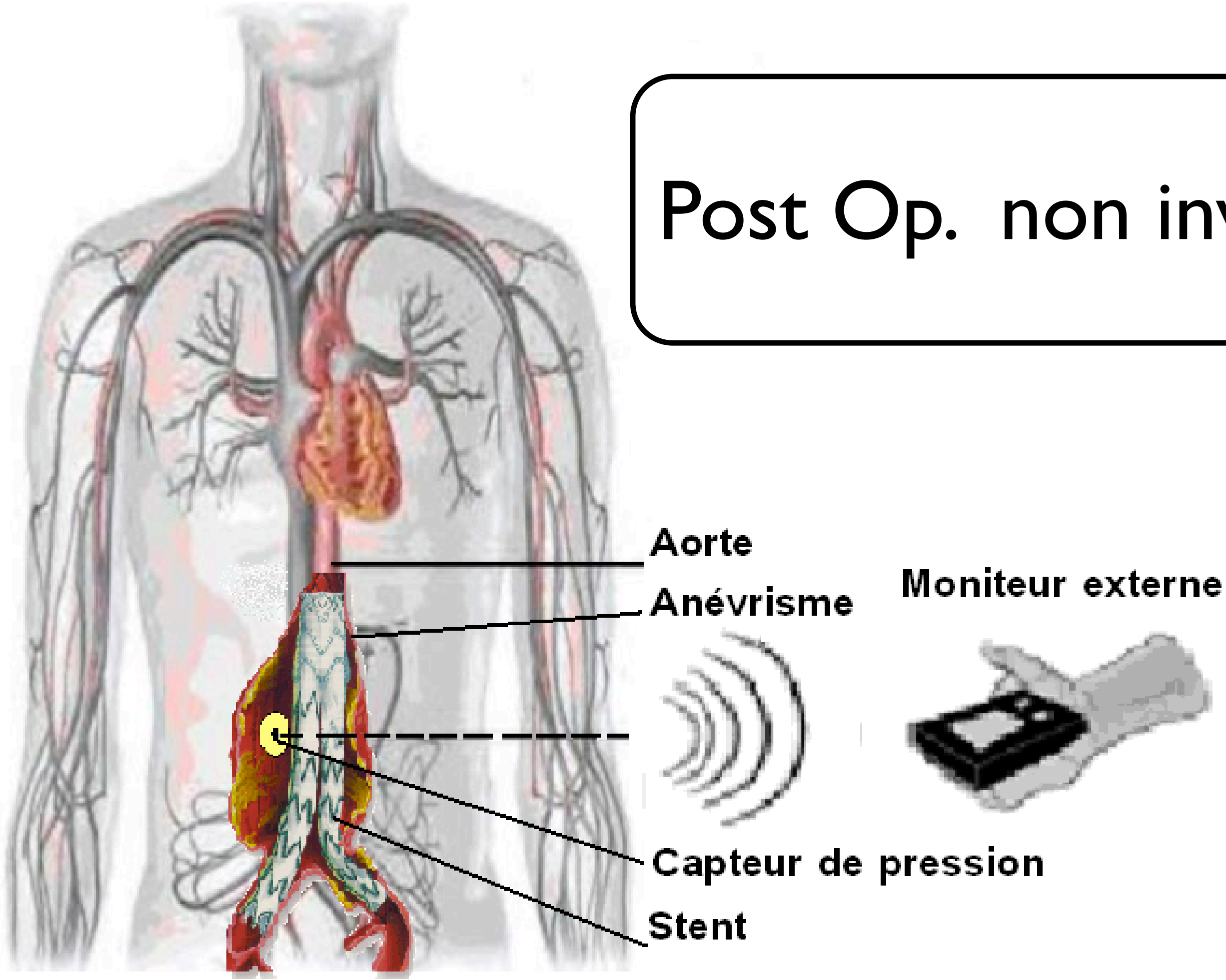
Modelling

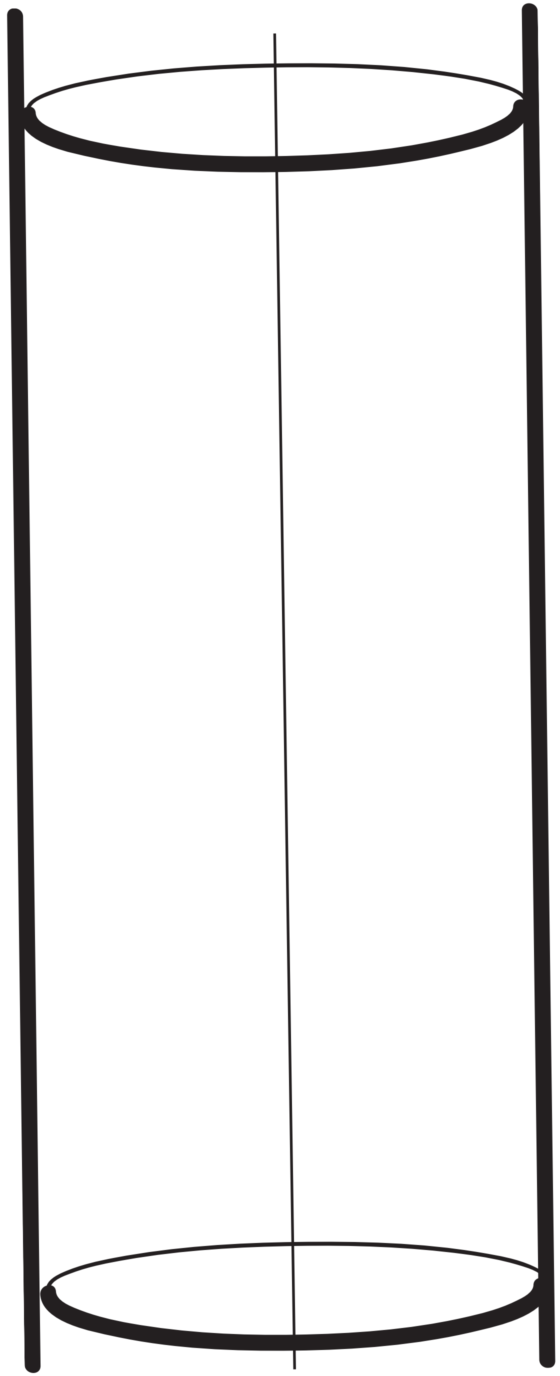
J. Frelat
M. Destrade
(LMM)

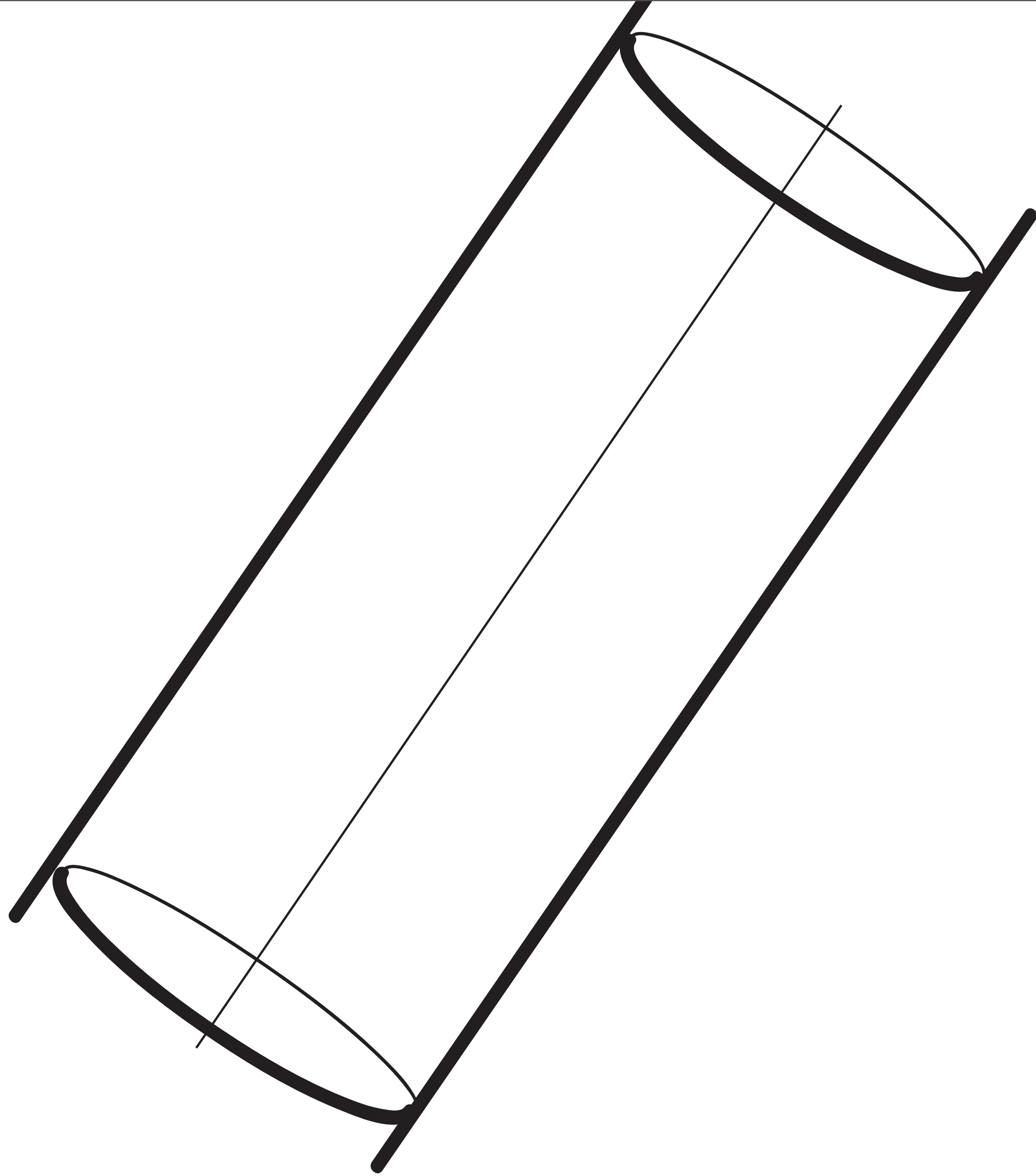

Aorte
Anévrisme

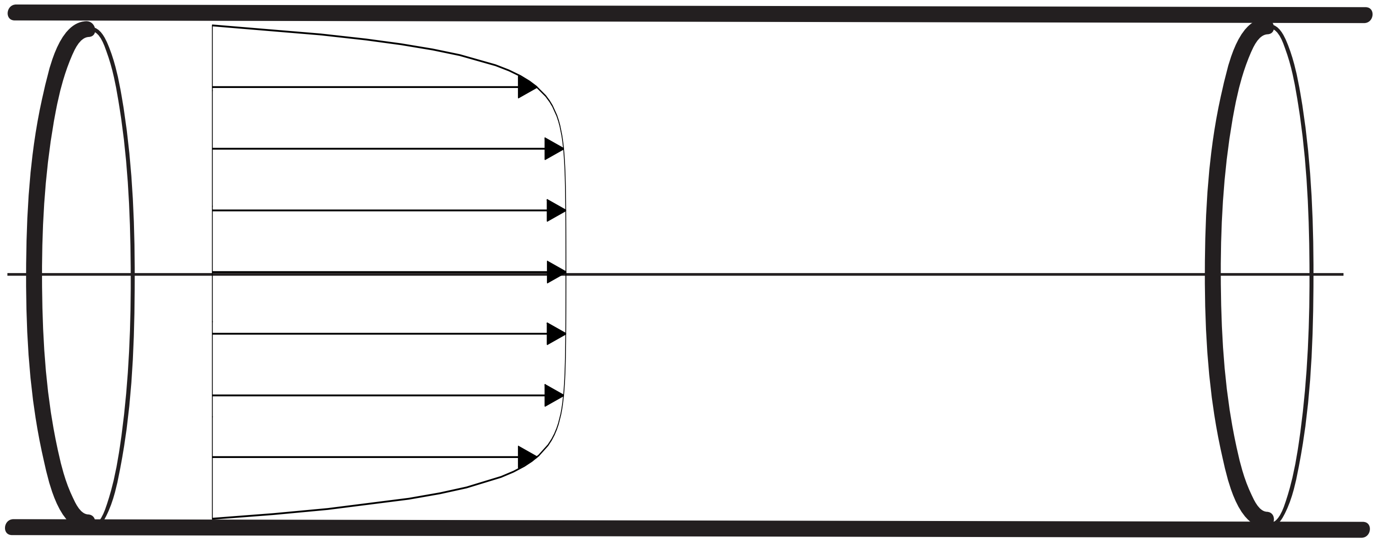
O. Romain, J. Mazeyrat LISIF PVI

Post Op. non invasive survey

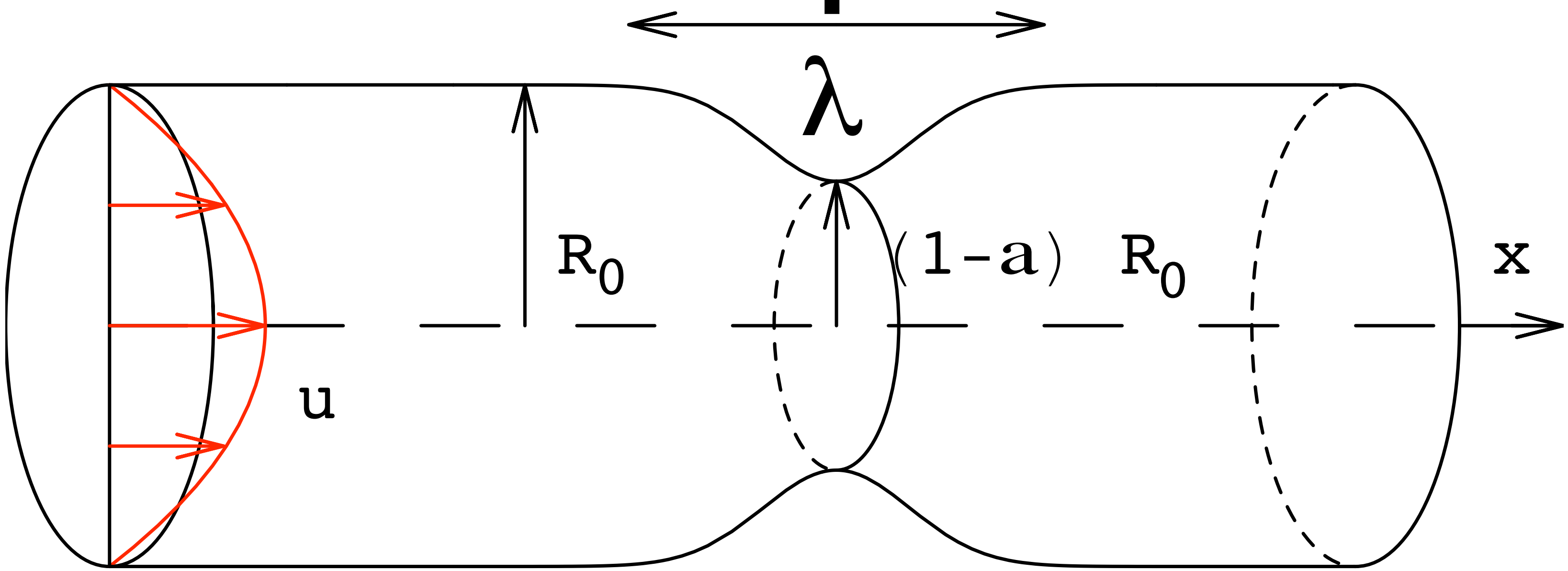








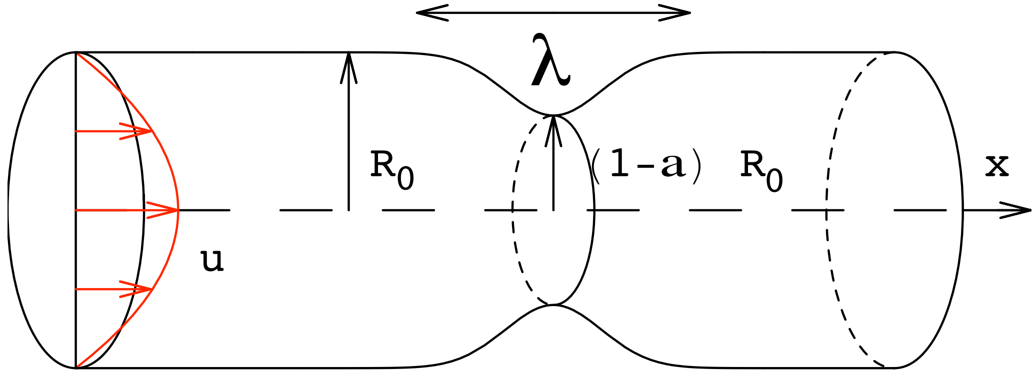
Example 3



- Flow in a stenozed vessel
- steady, rigid wall

Sylvie Lorthois (IMFT) Toulouse

RNSP Scales

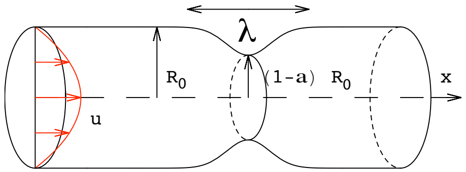


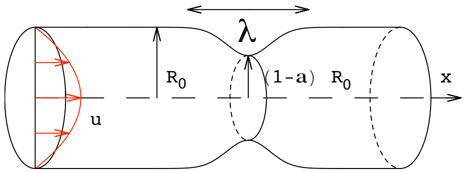
Using:

$x^* = xR_0Re$, $r^* = rR_0$, $u^* = U_0u$, $v^* = \frac{U_0}{Re}v$,
$p^* = p_0^* + \rho_0 U_0^2 p$ and $\tau^* = \frac{\rho U_0^2}{Re}\tau$

the following partial differential system is obtained from Navier Stokes as $Re \to \infty$:

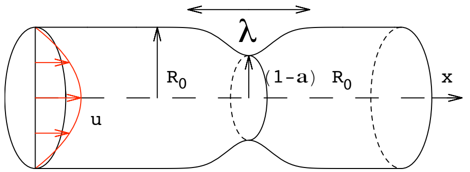
RNSP: Reduced Navier Stokes/ Prandtl System





RNSP: Reduced Navier Stokes/ Prandtl System

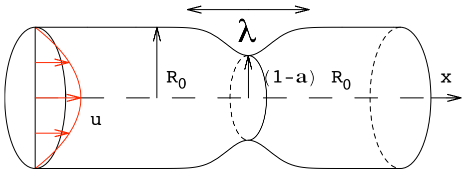
$$
\begin{aligned}
\frac{\partial}{\partial x}u + \frac{\partial}{r\partial r}rv &= 0,\\
(u\frac{\partial}{\partial x}u + v\frac{\partial}{\partial r}u) &= -\frac{\partial p}{\partial x} + \frac{\partial}{r\partial r}(r\frac{\partial}{\partial r}u),\\
0 &= -\frac{\partial p}{\partial r}.
\end{aligned}
$$



RNSP: Reduced Navier Stokes/ Prandtl System

$$
\begin{aligned}
\frac{\partial}{\partial x}u + \frac{\partial}{r\partial r}rv &= 0, \\
(u\frac{\partial}{\partial x}u + v\frac{\partial}{\partial r}u) &= -\frac{\partial p}{\partial x} + \frac{\partial}{r\partial r}(r\frac{\partial}{\partial r}u), \\
0 &= -\frac{\partial p}{\partial r}.
\end{aligned}
$$

$+$ The boundary conditions.



RNSP: Reduced Navier Stokes/ Prandtl System

$$
\begin{aligned}
\frac{\partial}{\partial x}u + \frac{\partial}{r\partial r}rv &= 0,\\
(u\frac{\partial}{\partial x}u + v\frac{\partial}{\partial r}u) &= -\frac{\partial p}{\partial x} + \frac{\partial}{r\partial r}(r\frac{\partial}{\partial r}u),\\
0 &= -\frac{\partial p}{\partial r}.
\end{aligned}
$$

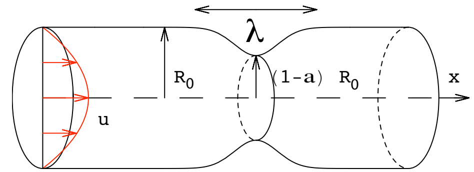
- axial symmetry ($\partial_r u = 0$ and $v = 0$ at $r = 0$),

- no slip condition at the wall ($u = v = 0$ at $r = 1 - f(x)$),

- the entry velocity profiles ($u(0, r)$ and $v(0, r)$) are given

- *no* output condition in $x_{out} = \frac{x^*_{out}}{R_0 Re}$

- streamwise marching, even when flow separation.



RNSP: Reduced Navier Stokes/ Prandtl System

$$
\begin{aligned}
\frac{\partial}{\partial x}u + \frac{\partial}{r\partial r}rv &= 0, \\
(u\frac{\partial}{\partial x}u + v\frac{\partial}{\partial r}u) &= -\frac{\partial p}{\partial x} + \frac{\partial}{r\partial r}(r\frac{\partial}{\partial r}u), \\
0 &= -\frac{\partial p}{\partial r}.
\end{aligned}
$$

Parabolic Problem - Marching Problem

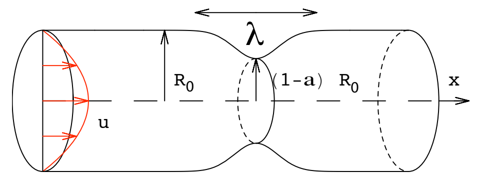
- axial symmetry ($\partial_r u = 0$ and $v = 0$ at $r = 0$),

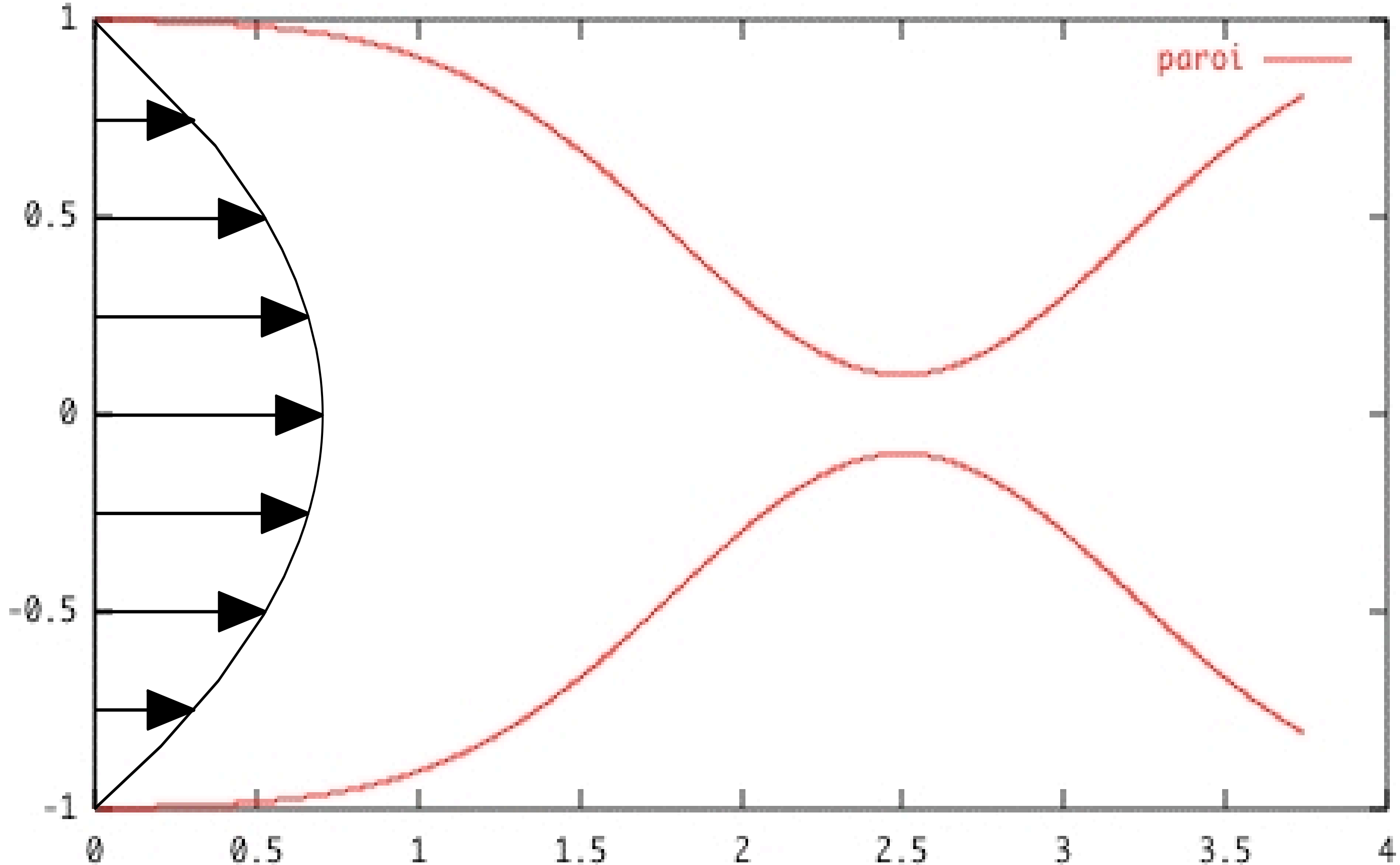
- no slip condition at the wall ($u = v = 0$ at $r = 1 - f(x)$),

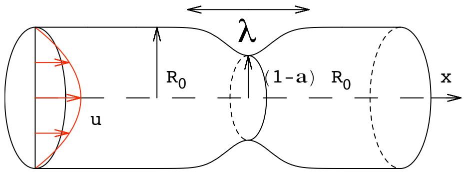
- the entry velocity profiles ($u(0, r)$ and $v(0, r)$) are given

- *no* output condition in $x_{out} = \frac{x^*_{out}}{R_0 Re}$

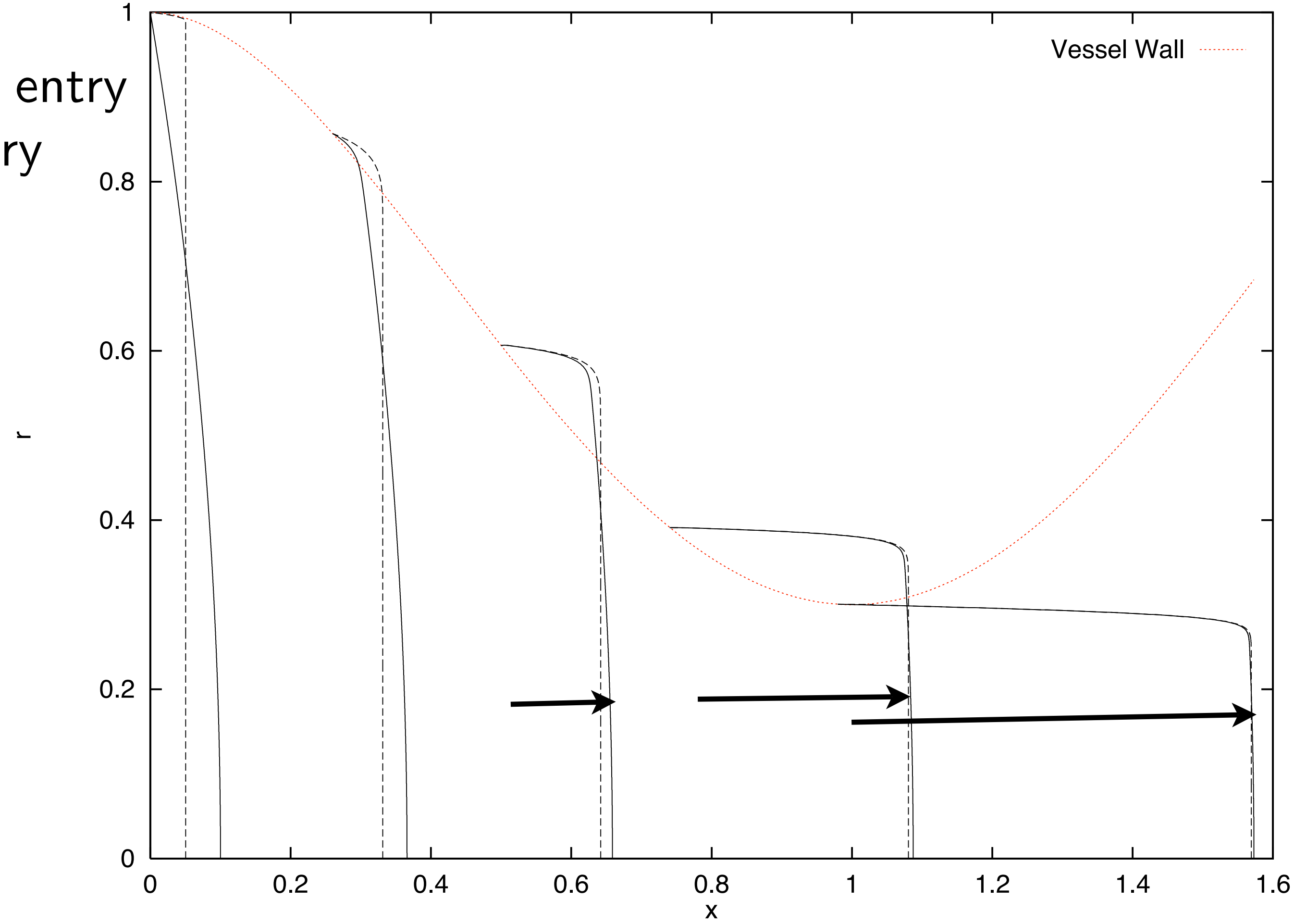
- streamwise marching, even when flow separation.

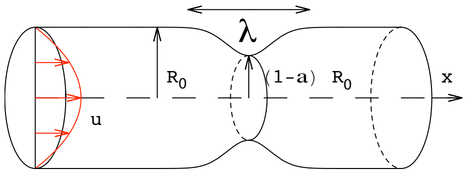

λ
R0
(1-a) R0
x
u


paroi
1
0.5
0
-0.5
-1
0
0.5
1
1.5
2
2.5
3
3.5
4

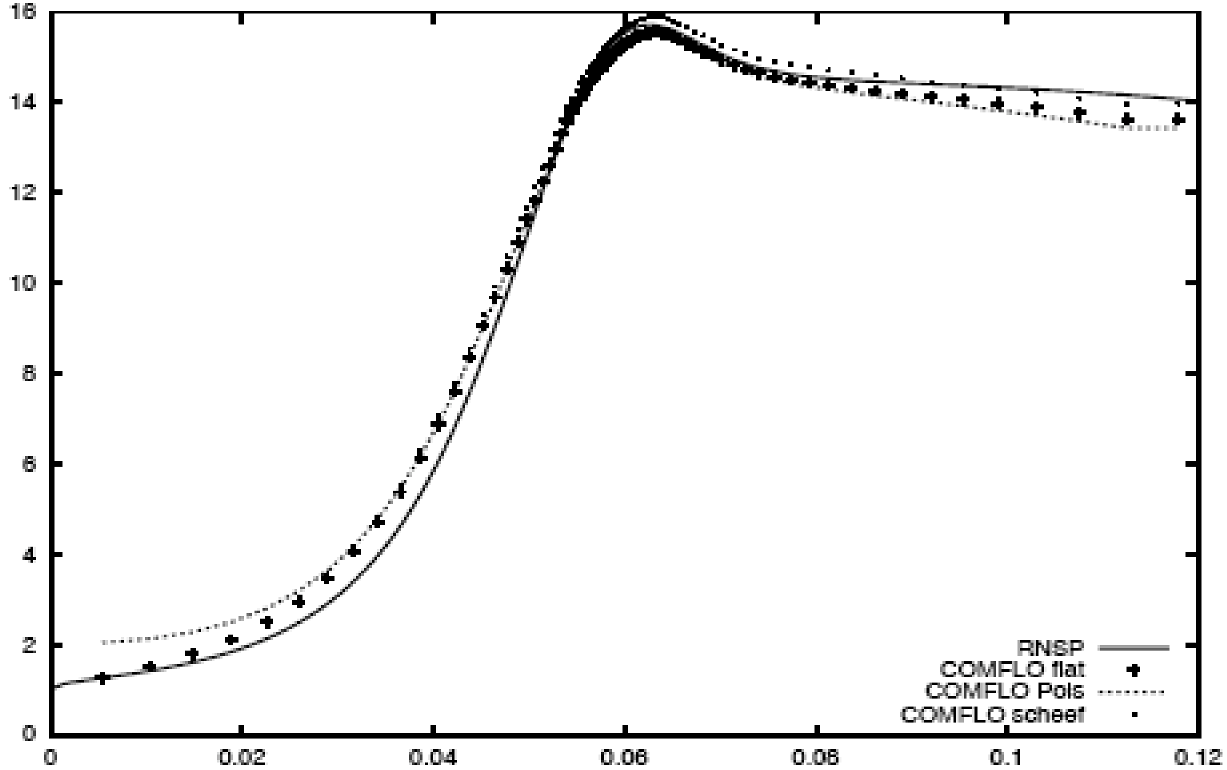


Evolution of the velocity profile along the convergent part of a 70% stenosis $(Re = 500)$;
solid line: Poiseuille entry
broken line: flat entry

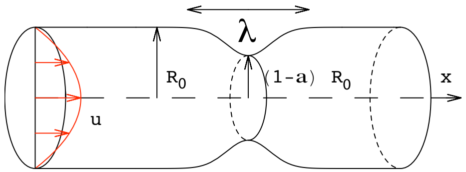




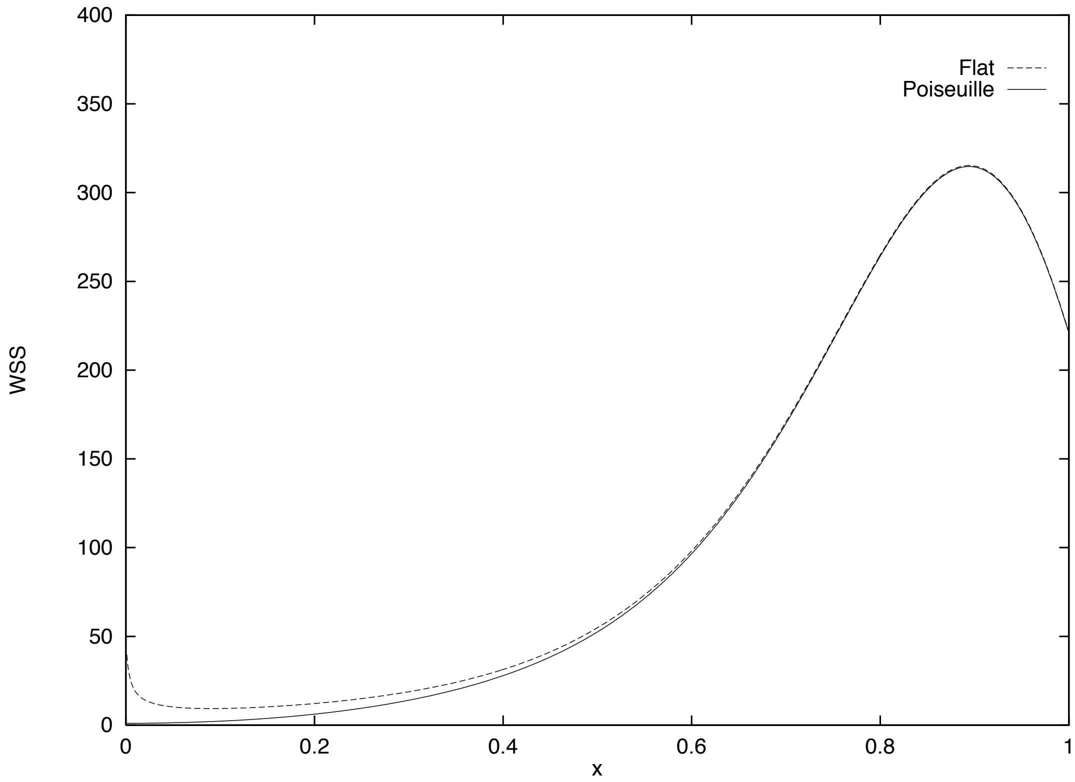
Testing asymmetry in the entry profile



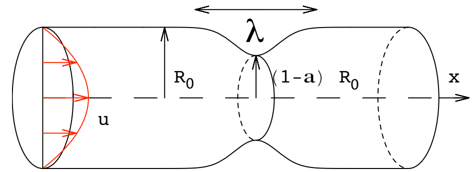
The velocities in the middle for Comflo and RNS.
Comflo uses here 50X50X100 points. Dimensionless scales!

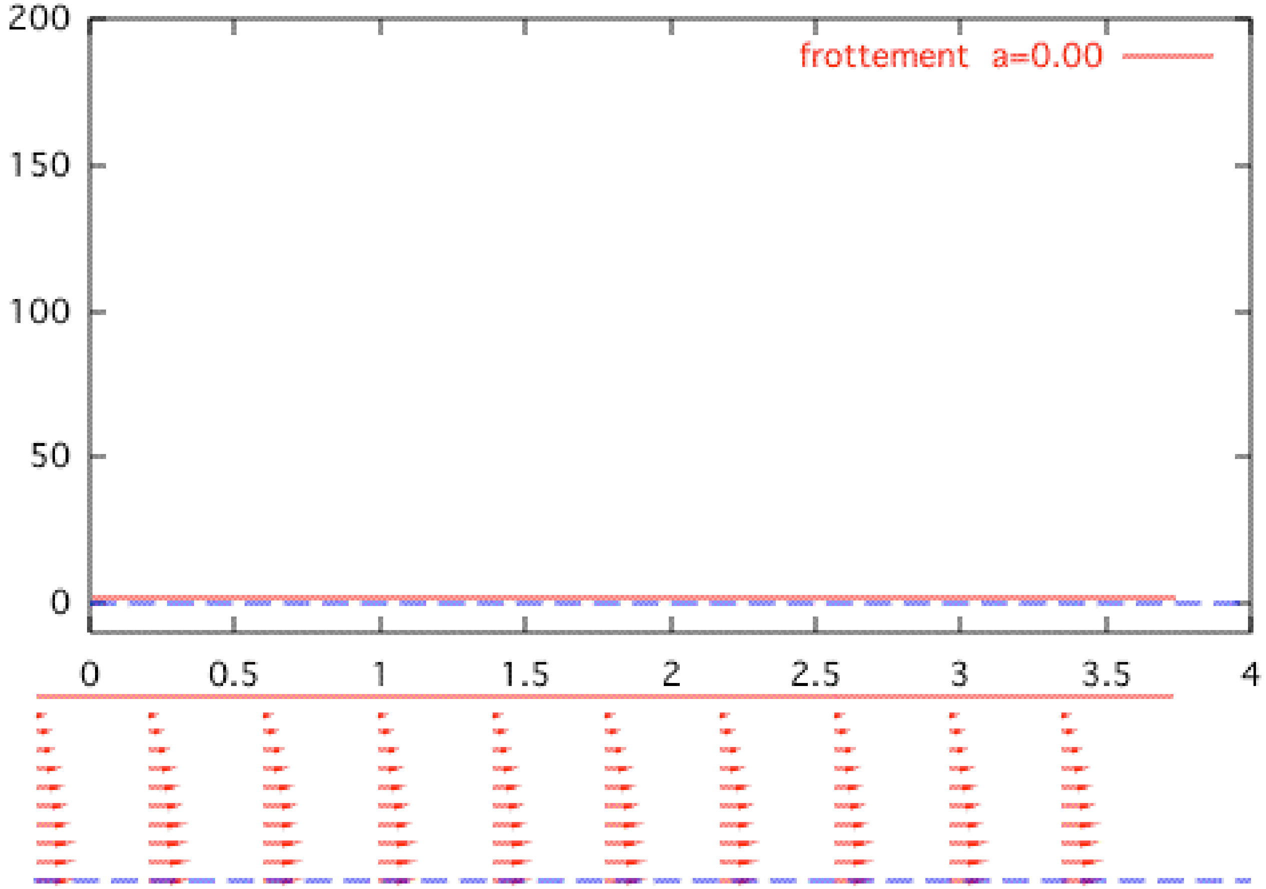


Wall Shear Stress

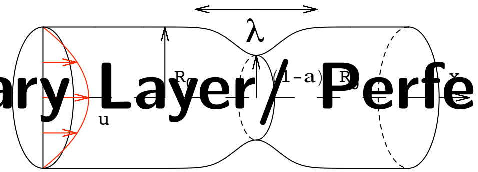


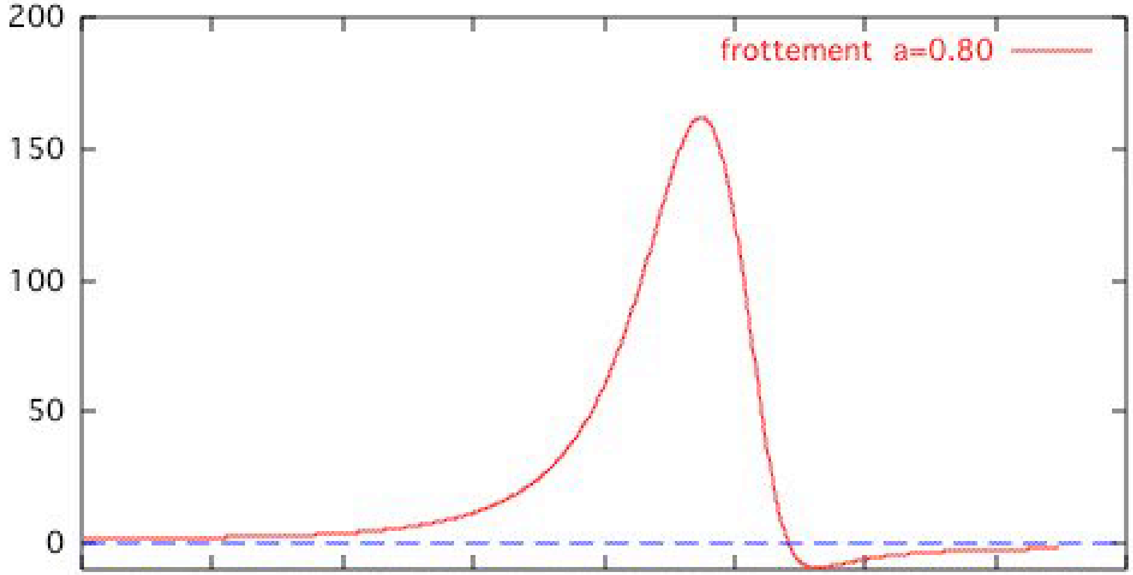
Evolution of the WSS distribution along the convergent part of a 70% stenosis $(Re = 500)$; solid line: Poiseuille entry profile ; broken line: flat entry profile.


λ
R0
(1-a) R0
x
u


200
150
100
50
0
frottement a=0.00
0
0.5
1
1.5
2
2.5
3
3.5
4

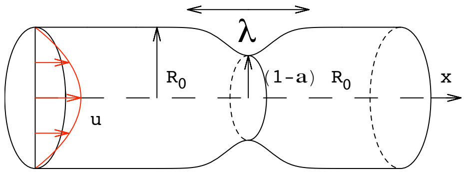
Boundary Layer / Perfect Fluid



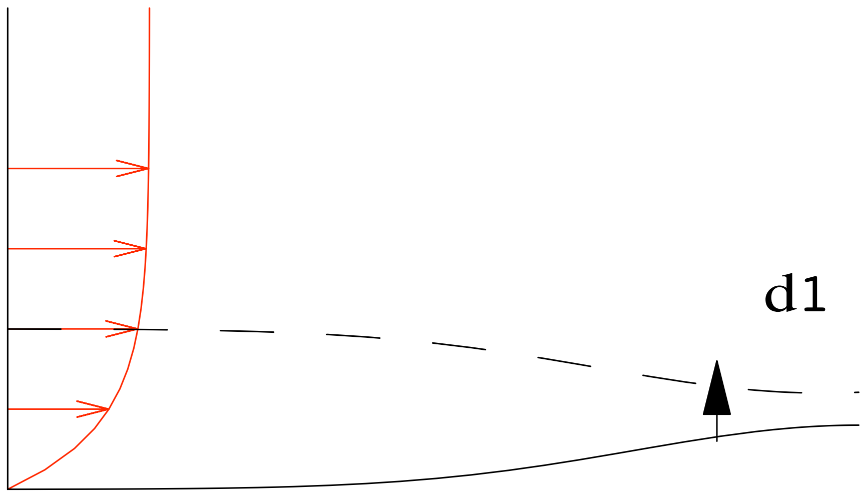


thin boundary layer

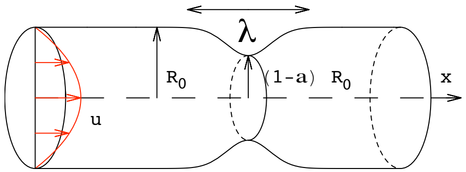
Inviscid core



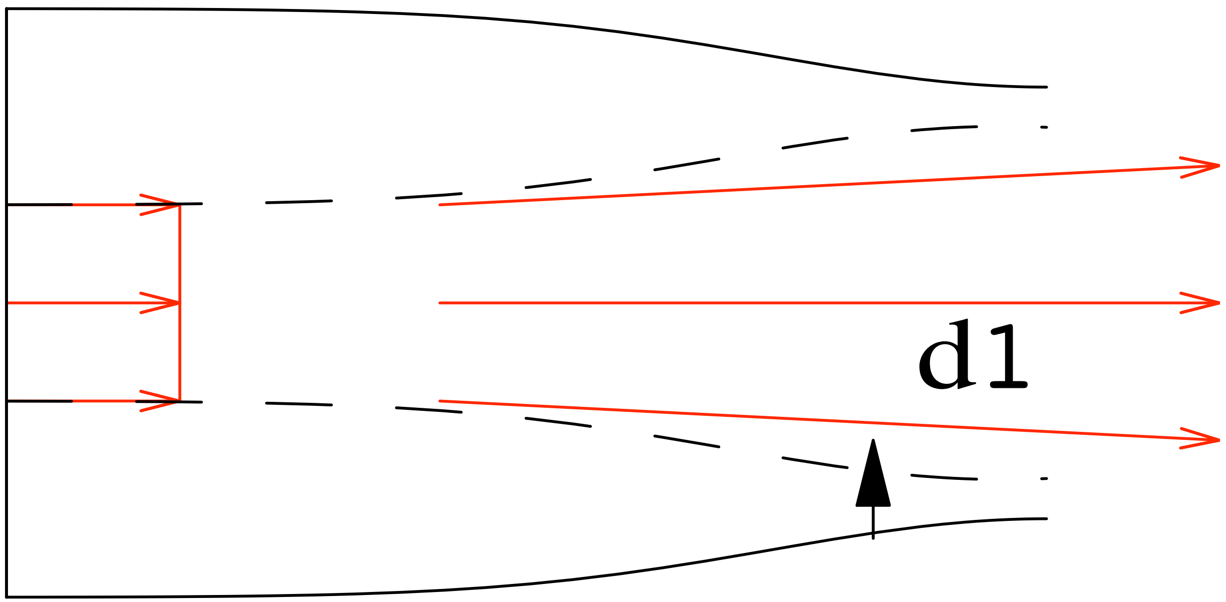
Boundary Layer/ Perfect Fluid



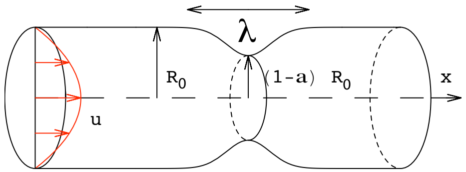
The boundary layer is generated near the wall
δ_1 is the displacement thickness.



Boundary Layer/ Perfect Fluid



The displacement thickness acts as a "new" wall!
→Interacting Boundary Layer (IBL)



IBL integral: 1D equation

$$\frac{d}{d\bar{x}}(\frac{\bar{\delta}_1}{H}) = \bar{\delta}_1(1+\frac{2}{H})\frac{d\bar{u}_e}{d\bar{x}} + \frac{f_2 H}{\bar{\delta}_1 \bar{u}_e},$$
$$\bar{u}_e = \frac{1}{(R^2 - 2(\lambda/Re)^{-1/2}\bar{\delta}_1)}.$$

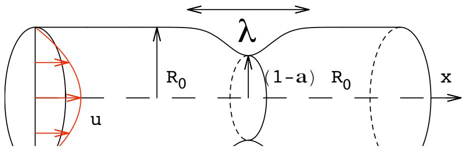
To solve this system, a closure relationship linking H and f_2 to the velocity and the displacement thickness is needed.

Defining $\Lambda_1 = \bar{\delta}_1^2 \frac{d\bar{u}_e}{d\bar{x}}$,

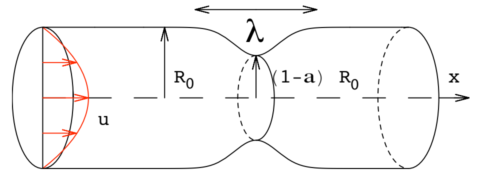
the system is closed from the resolution of the Falkner Skan system as follows:

if $\Lambda_1 < 0.6$ then $H = 2.5905exp(-0.37098\Lambda_1)$, else $H = 2.074$.

From H,f_2 is computed as $f_2 = 1.05(-H^{-1} + 4H^{-2})$.



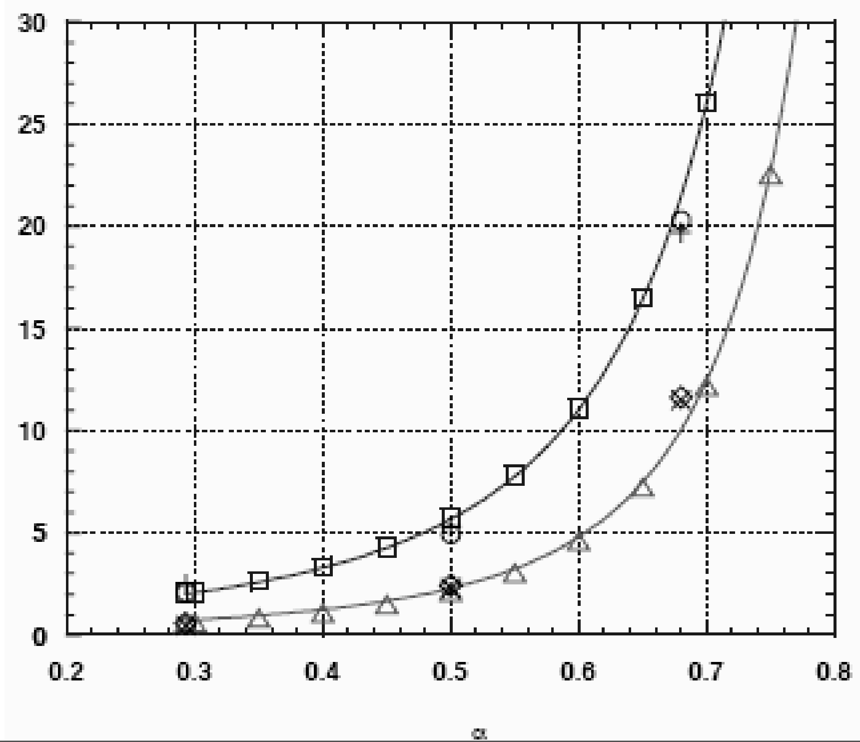
IBL integral: 1D equation Simplified Shear Stress



A simple formula as been settled:

$$WSS = (\mu \frac{\partial u^*}{\partial y^*})/((\mu \frac{4U_0}{R})) \sim .22 \frac{((Re/\lambda)^{1/2} + 3)}{(1-\alpha)^3}$$

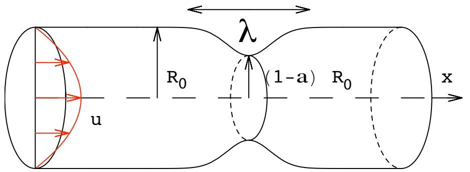
IBL integral: Comparison with Navier Stokes (Siegel et al. 1994)



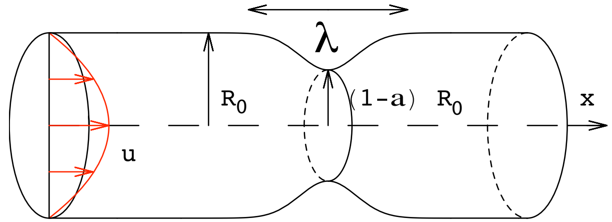
$$WSS = aRe^{1/2} + b$$

Coefficient a and b for the maximum WSS.
solid lines with △ and "square" : coefficient a and b obtained using the IBL integral method ;

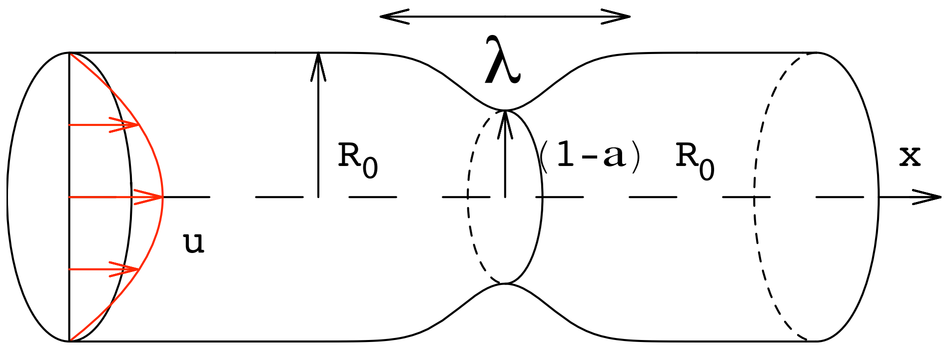
◇ : coefficient a derived from Siegel for $\lambda = 3$;
× : coefficient a derived from Siegel for $\lambda = 6$;
○ : coefficient b derived from Siegel for $\lambda = 3$;
+ : coefficient b derived from Siegel for $\lambda = 6$.

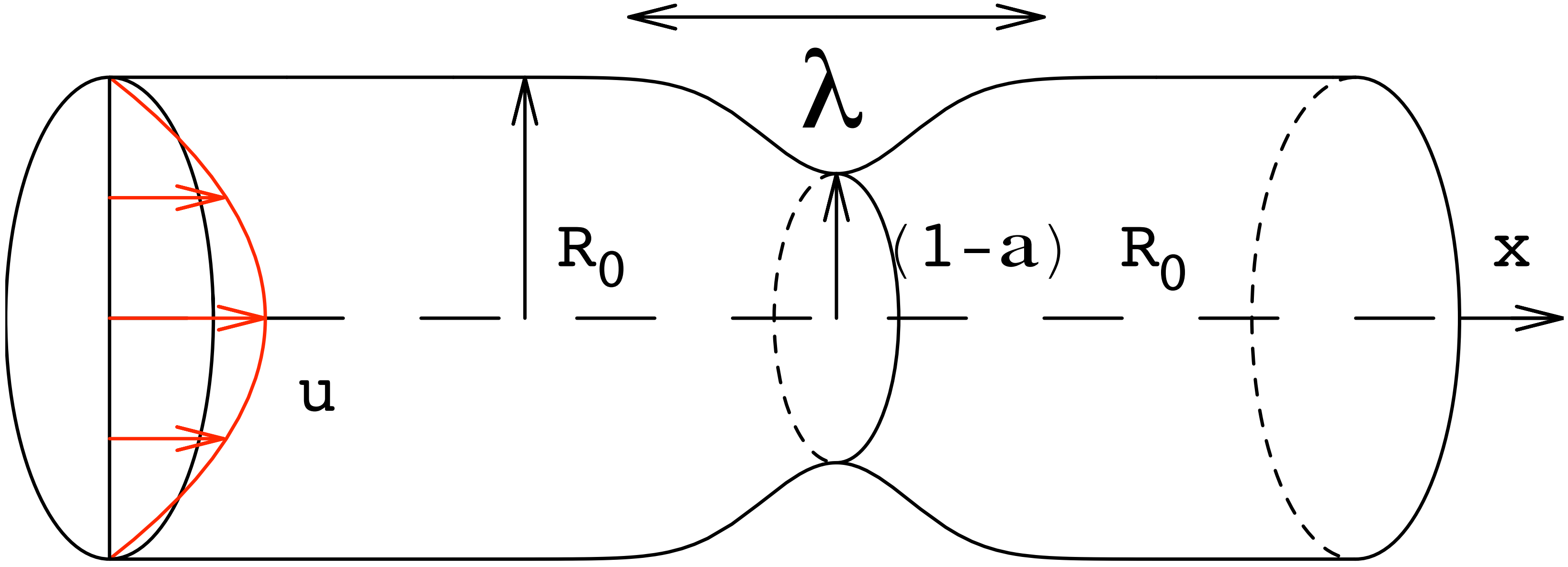


λ
R0
(1-a) R0
x
u



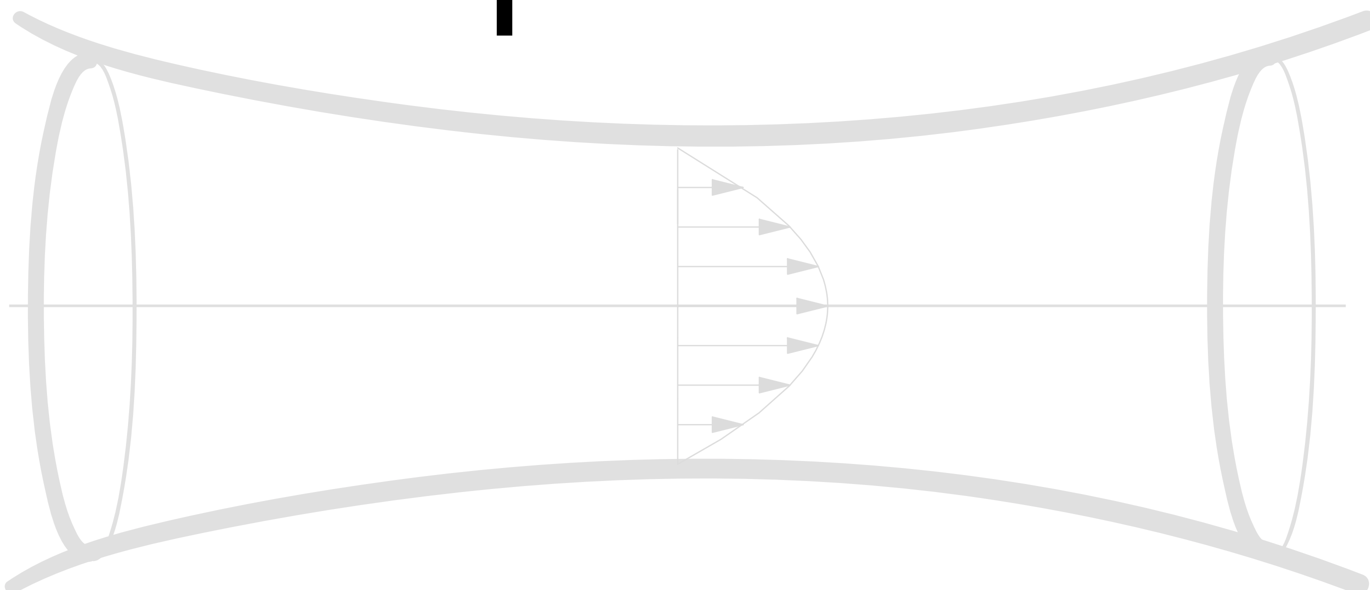
λ
R0
(1-a) R0
x
u


λ
R0
(1-a) R0
x
u



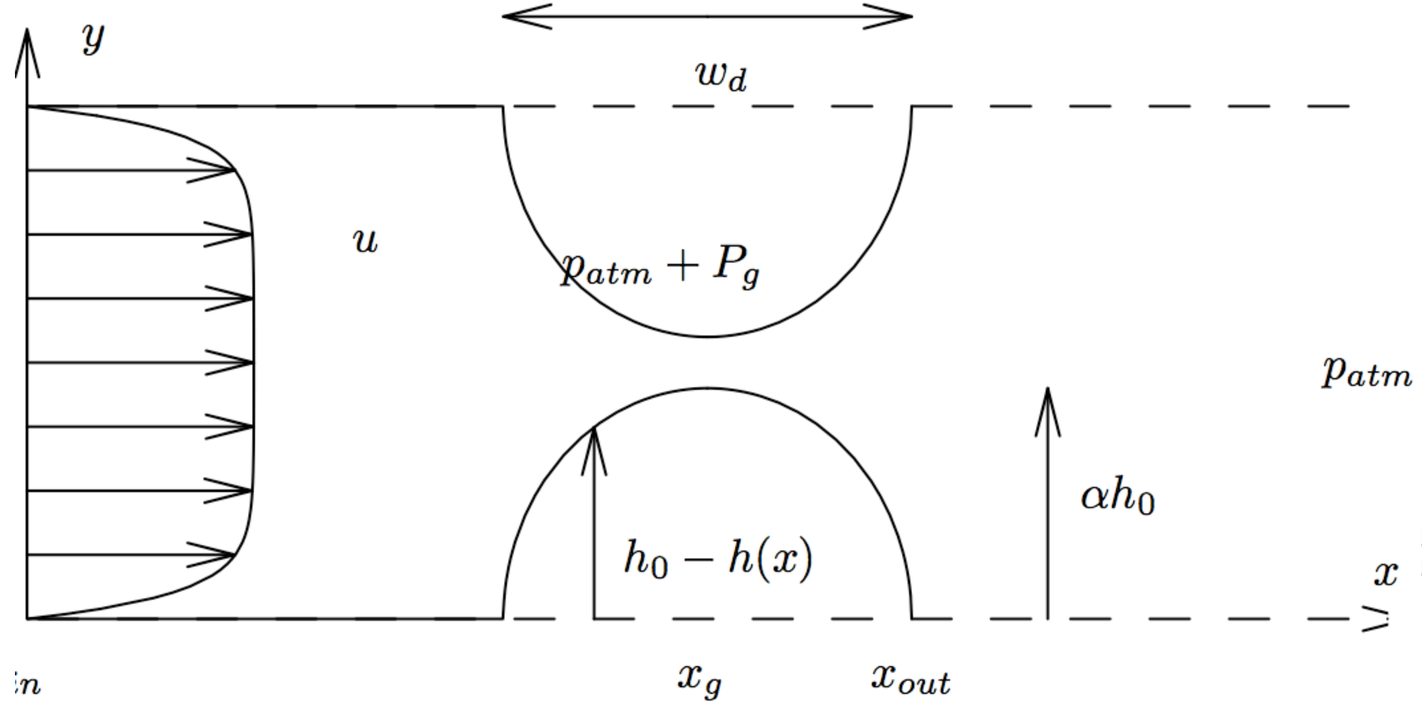
λ
R0
(1-a) R0
x
u

Example 4

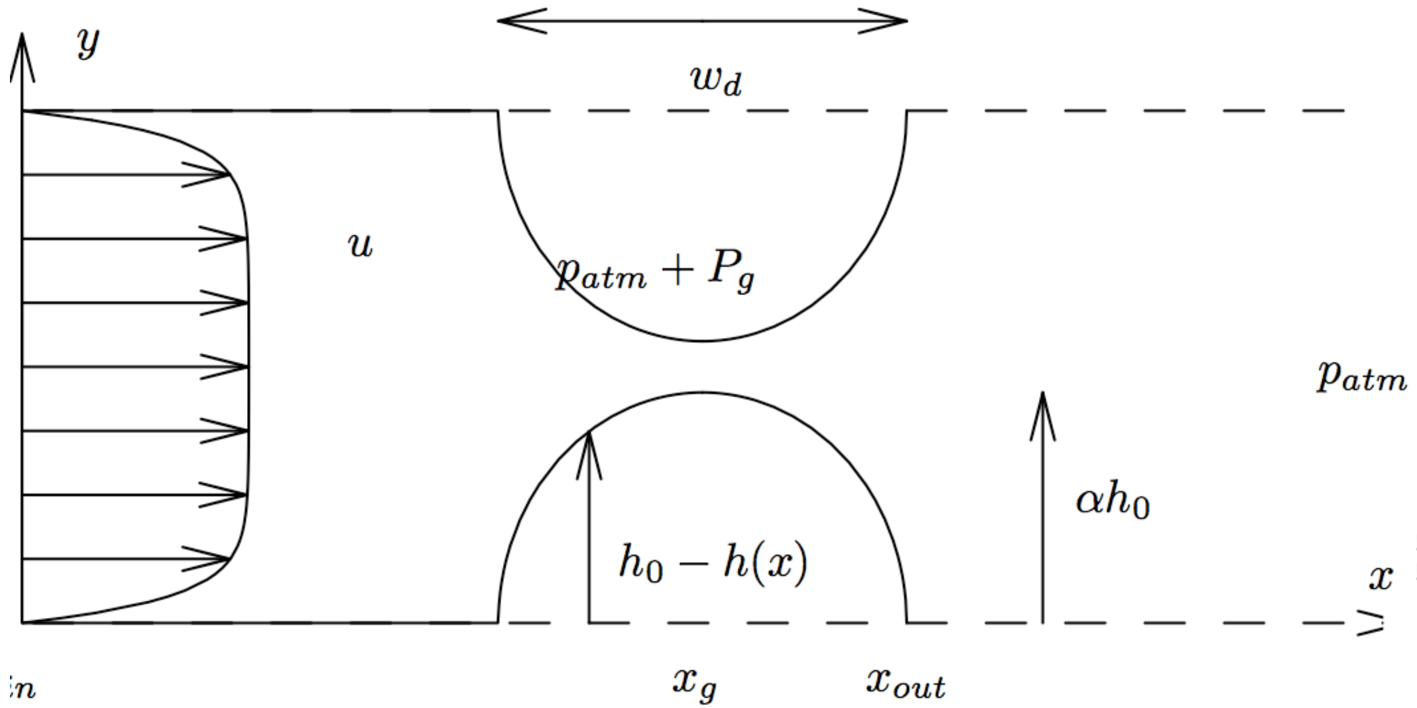


- Flow in a 2D stenozed vessel
- steady, rigid wall

Xavier Pelorson & Annemie van Hirtum
(ICP Grenoble)



- Flow in a stenozed vessel
- steady, rigid wall

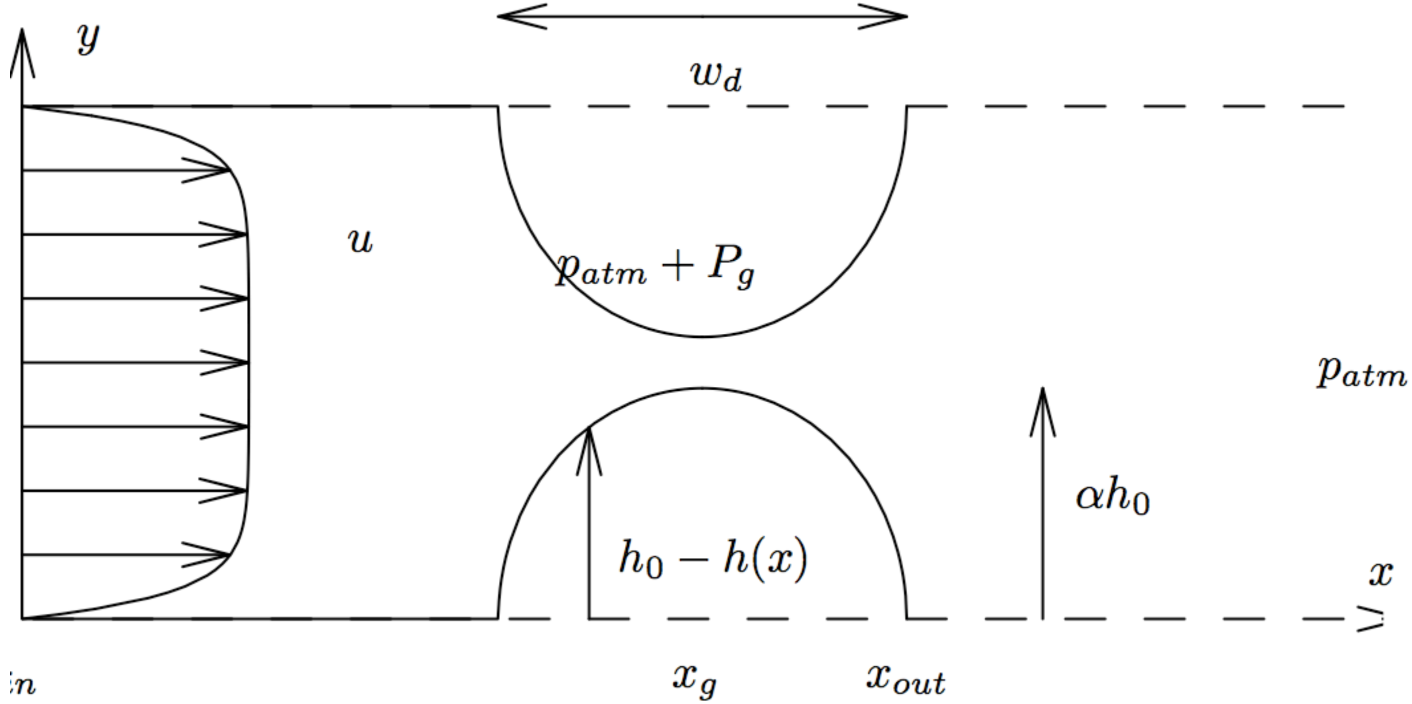


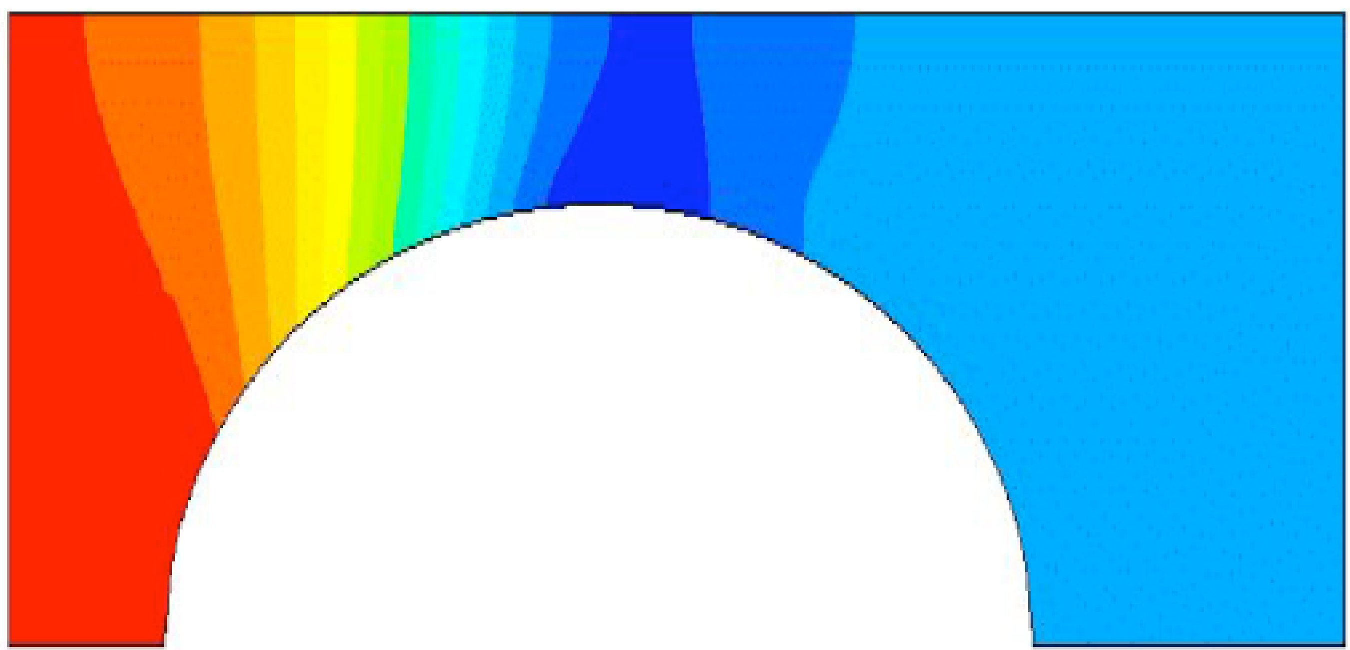
$$\frac{\partial}{\partial x}u + \frac{\partial}{\partial y}v = 0$$

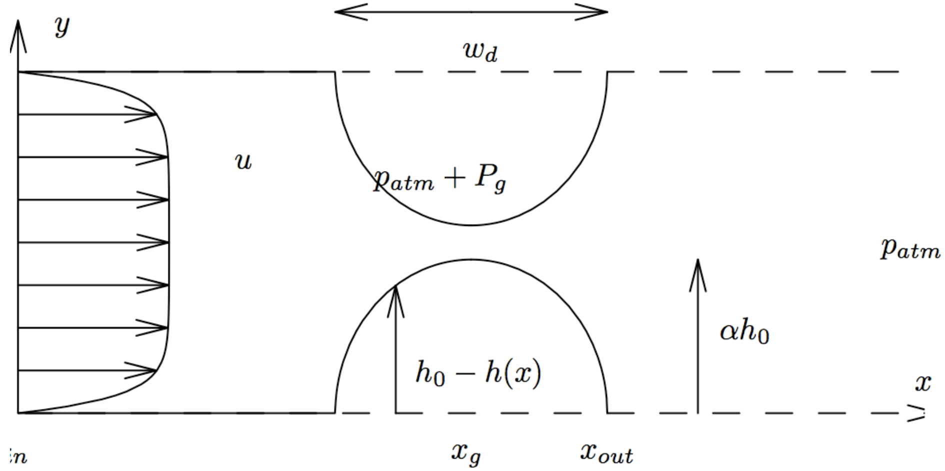
$$u\frac{\partial}{\partial x}u + v\frac{\partial}{\partial y}u = -\frac{\partial}{\partial x}p + \frac{\partial^2}{\partial y^2}u$$

$$0 = -\frac{\partial}{\partial y}p$$

RNSP non dimensional


y
w_d
u
$p_{atm} + P_g$
p_{atm}
αh_0
$h_0 - h(x)$
x
x_g
x_{out}





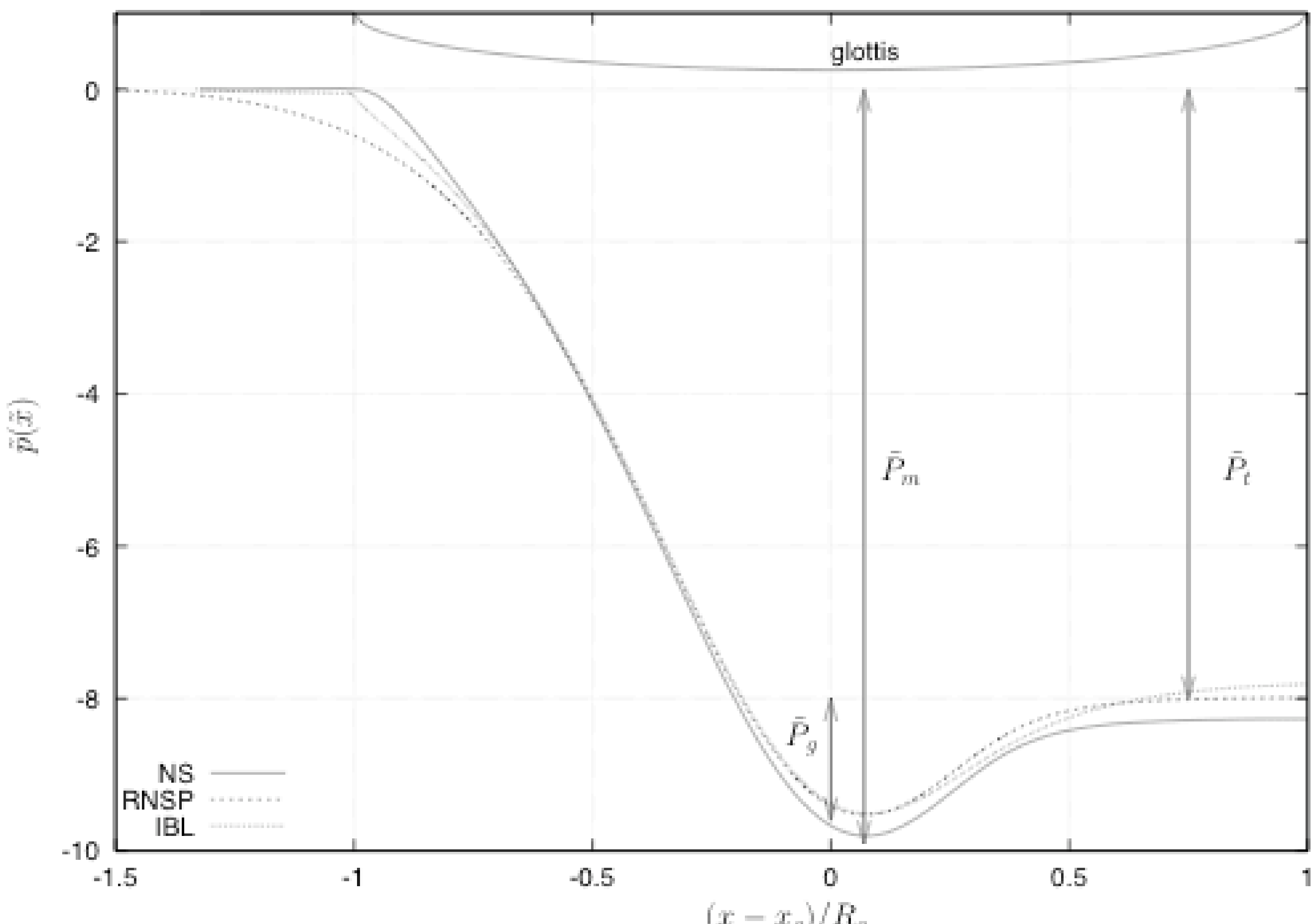
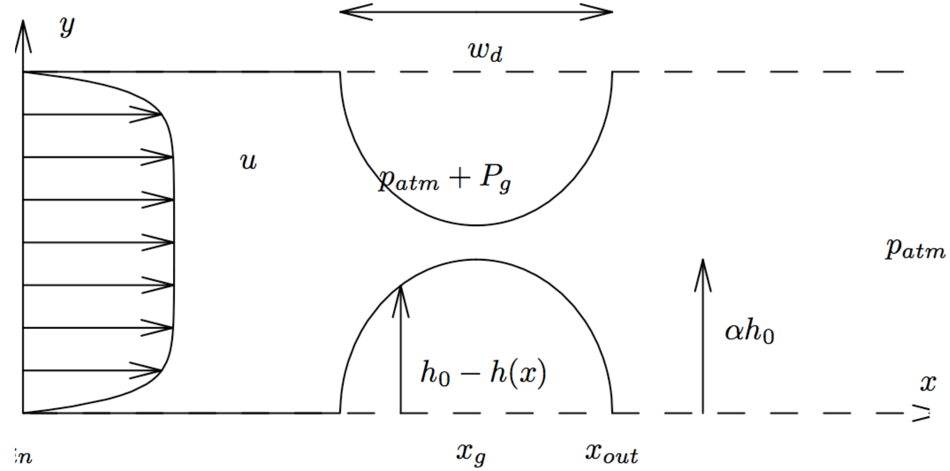


Fig. 2 A comparison between computed non-dimensional pressure for the three models (NS, IBL and RNSP, in this last case the wall has



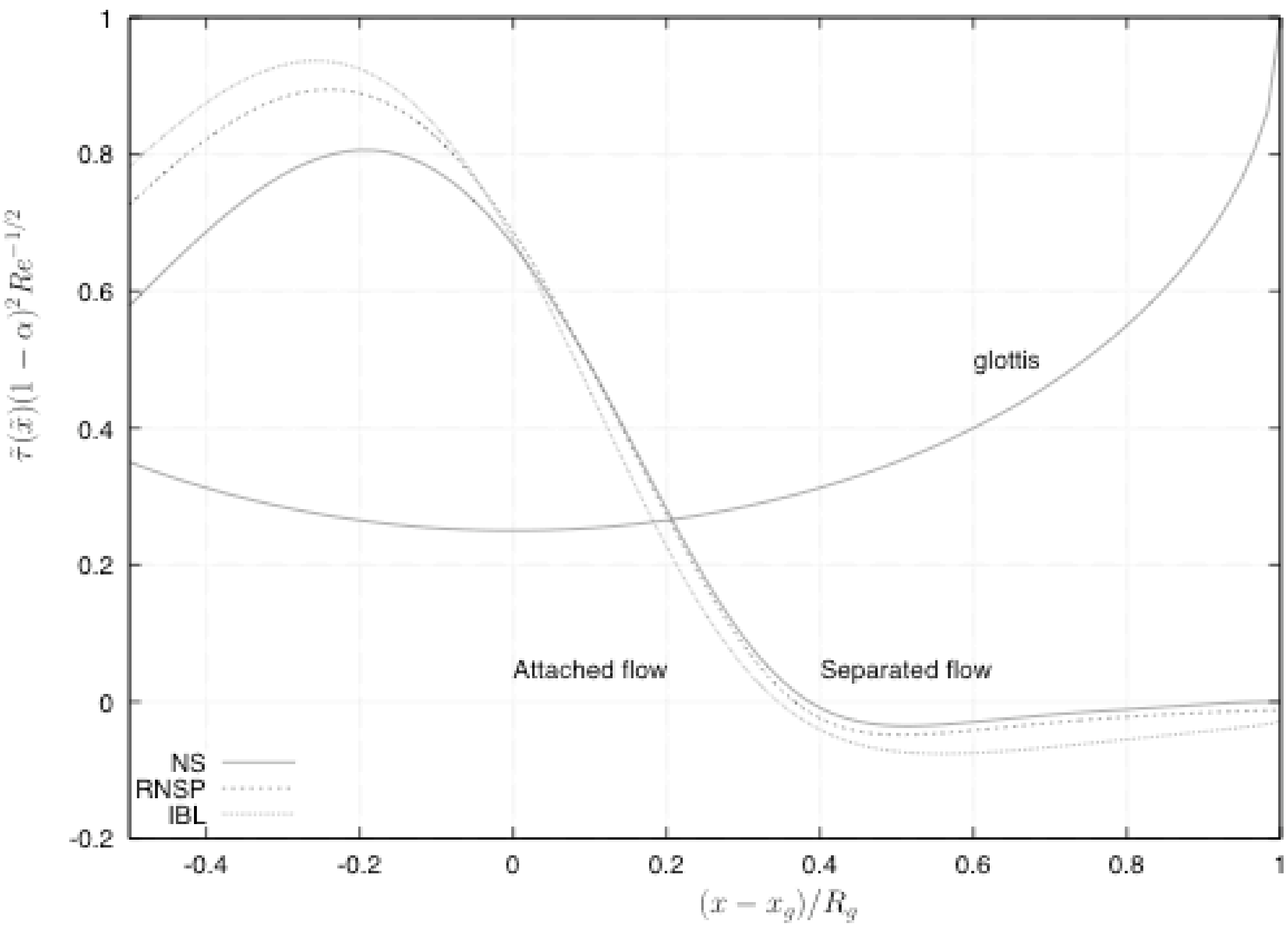
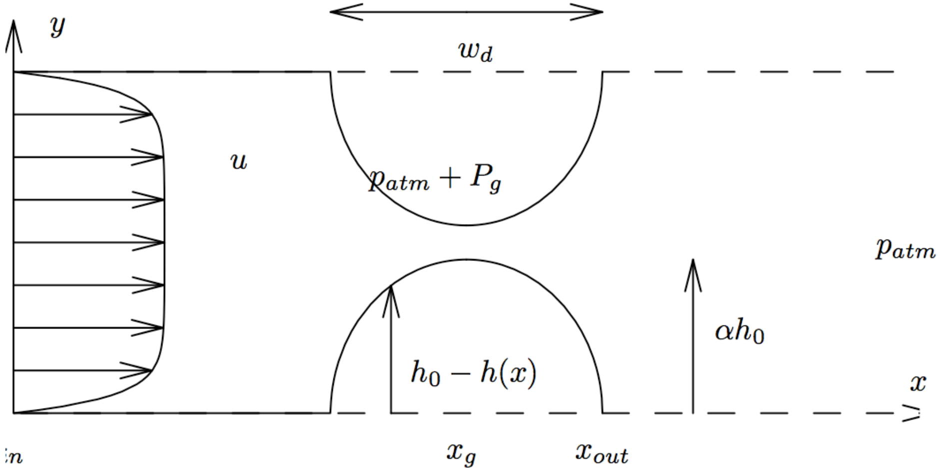


Fig. 4. A comparison between computed skin friction divided by $(0.47+3.07)(1-\alpha)^{-1}/\bar{x}_{..} \simeq (1-\alpha)^{-2} Re^{1/2}$ for the three models



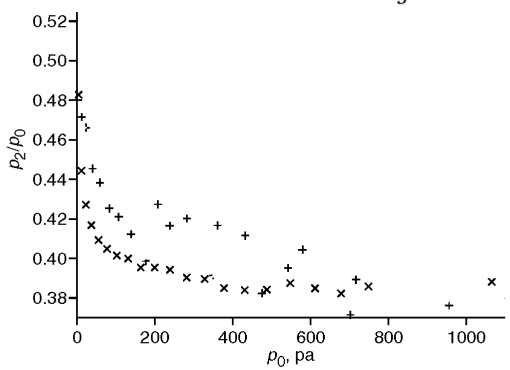


Fig. 11 *Normalised pressure at position p_2 for $h_{min} = 1.45$ mm: (+) measured data, (▷) Thwaites and (×) RNSP*

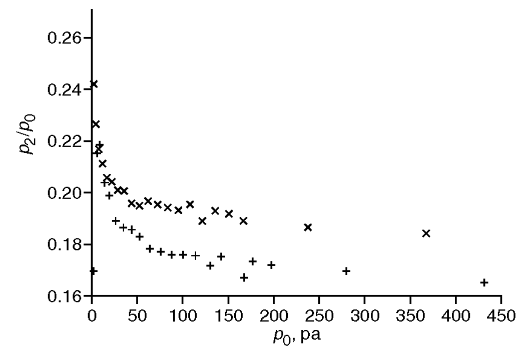


Fig. 14 *Normalised pressure at position p_2 for $h_{min} = 3.00$ mm: (+) measured data, (▷) Thwaites and (×) RNSP*

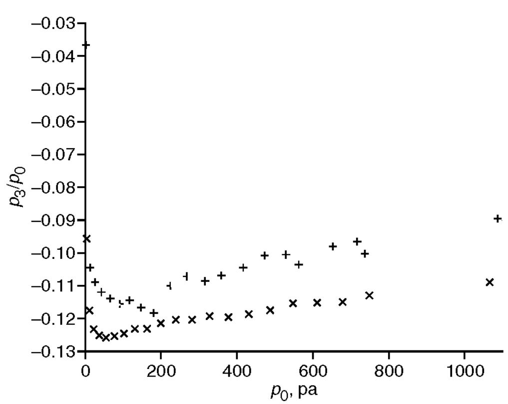


Fig. 12 *Normalised pressure position p_3 for $h_{min} = 1.45$ mm: (+) measured data, (▷) Thwaites and (×) RNSP*

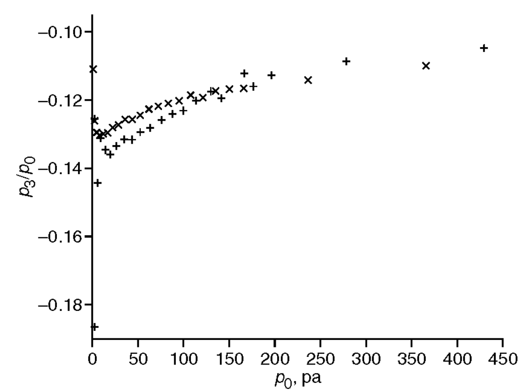
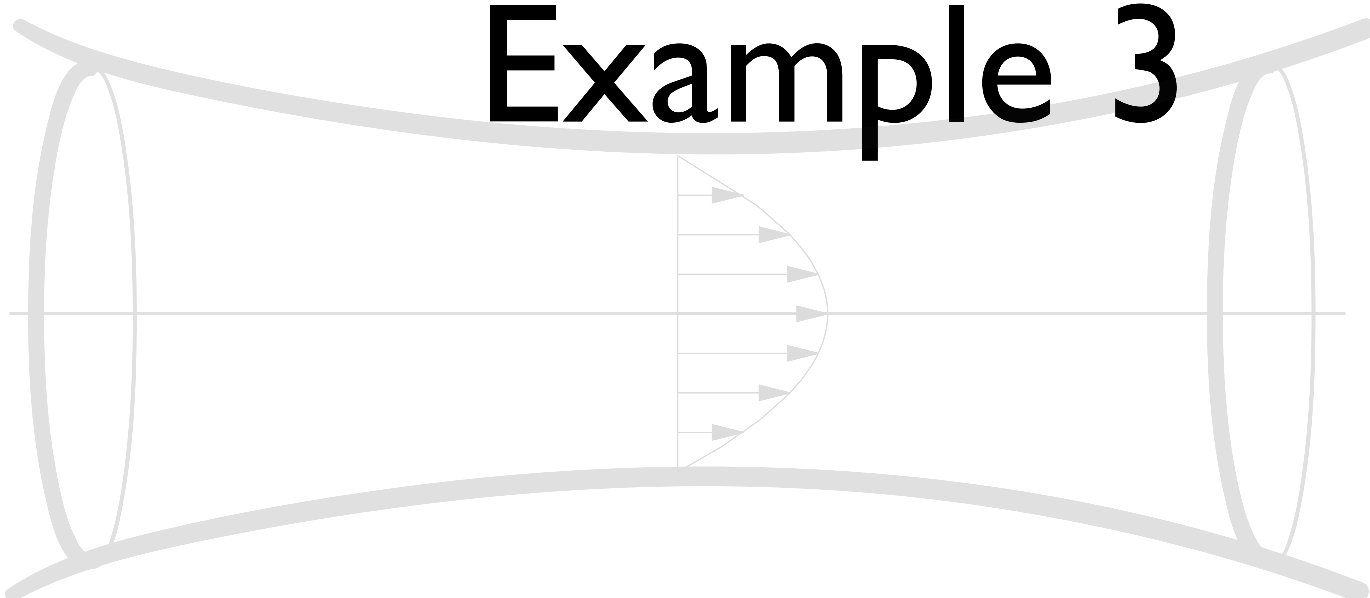


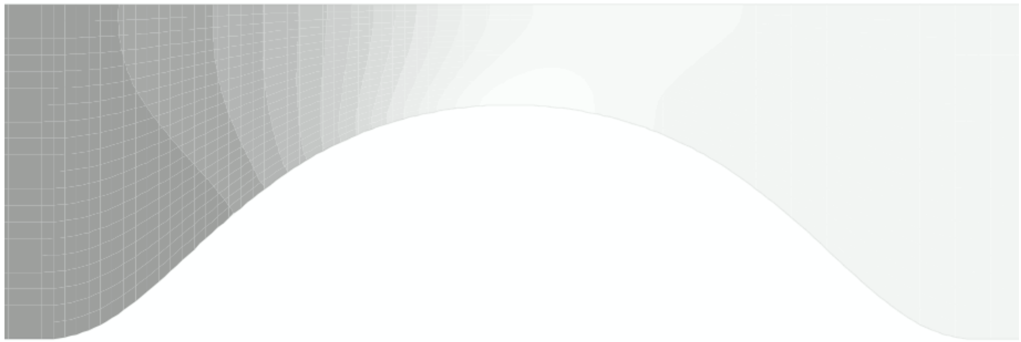
Fig. 15 *Normalised pressure at position p_3 for $h_{min} = 3.00$ mm: (+) measured data, (▷) Thwaites and (×) RNSP*

Example 3



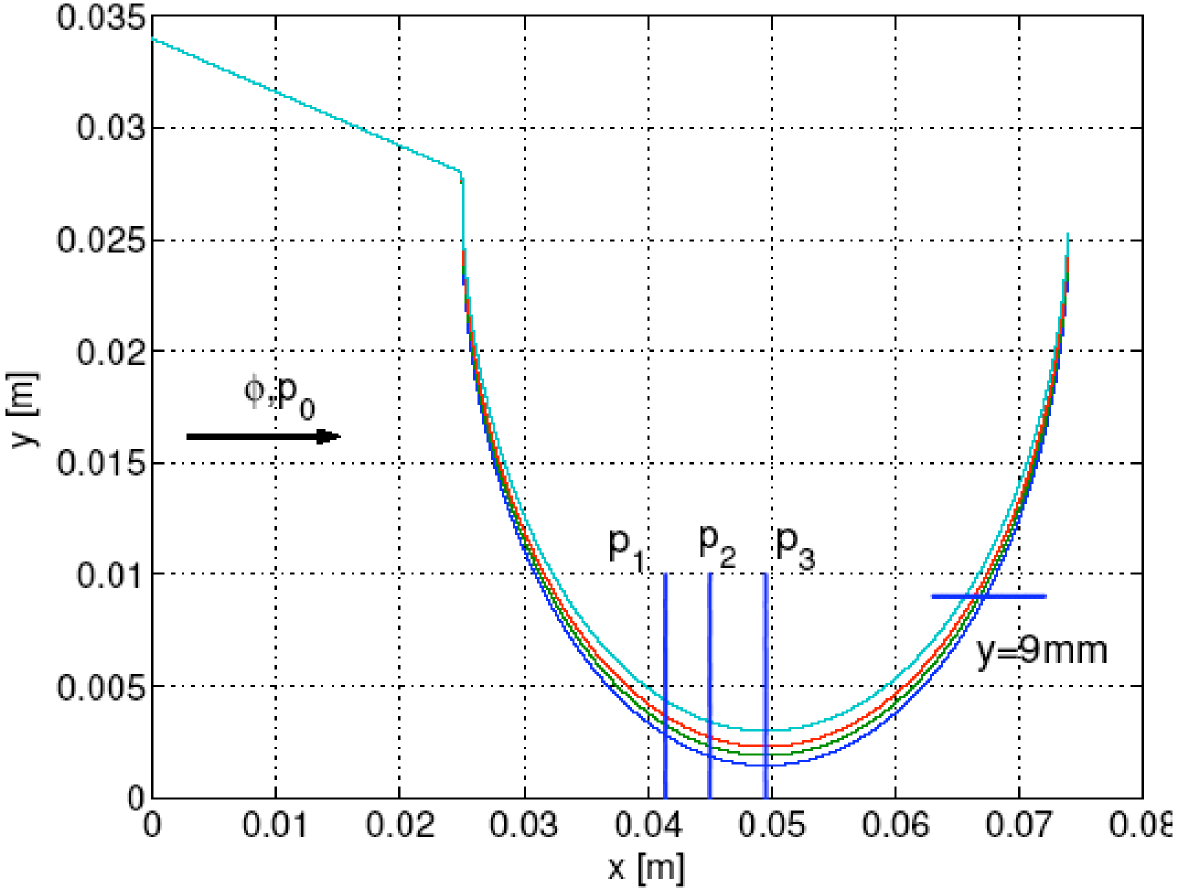
- Flow in a stenozed vessel
- steady, rigid wall
- non symetrical case

non symmetrical case



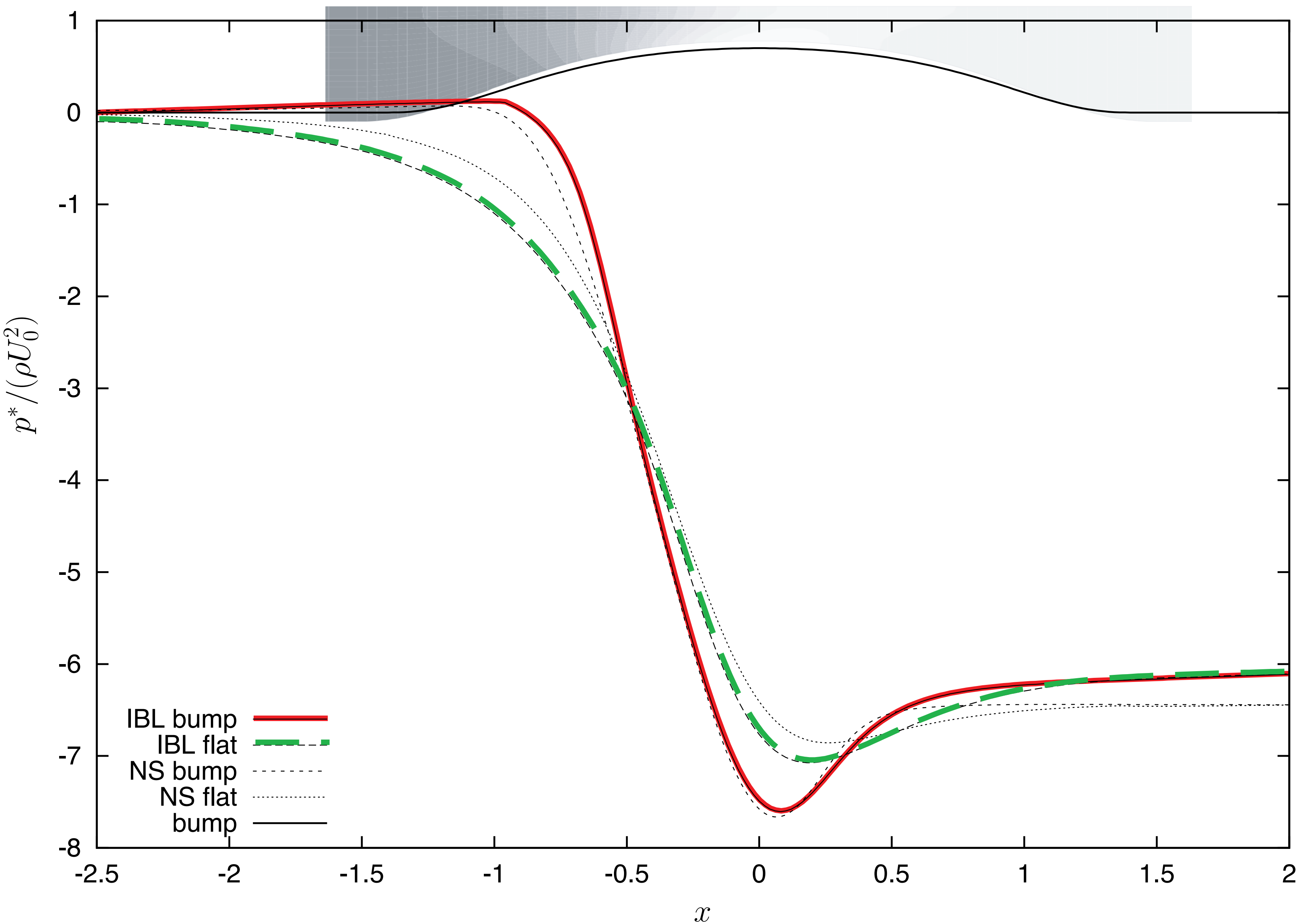
- RNSP
- modified integral method to take into account the transverse variation of pressure
- NS

PAROIS RIGIDES

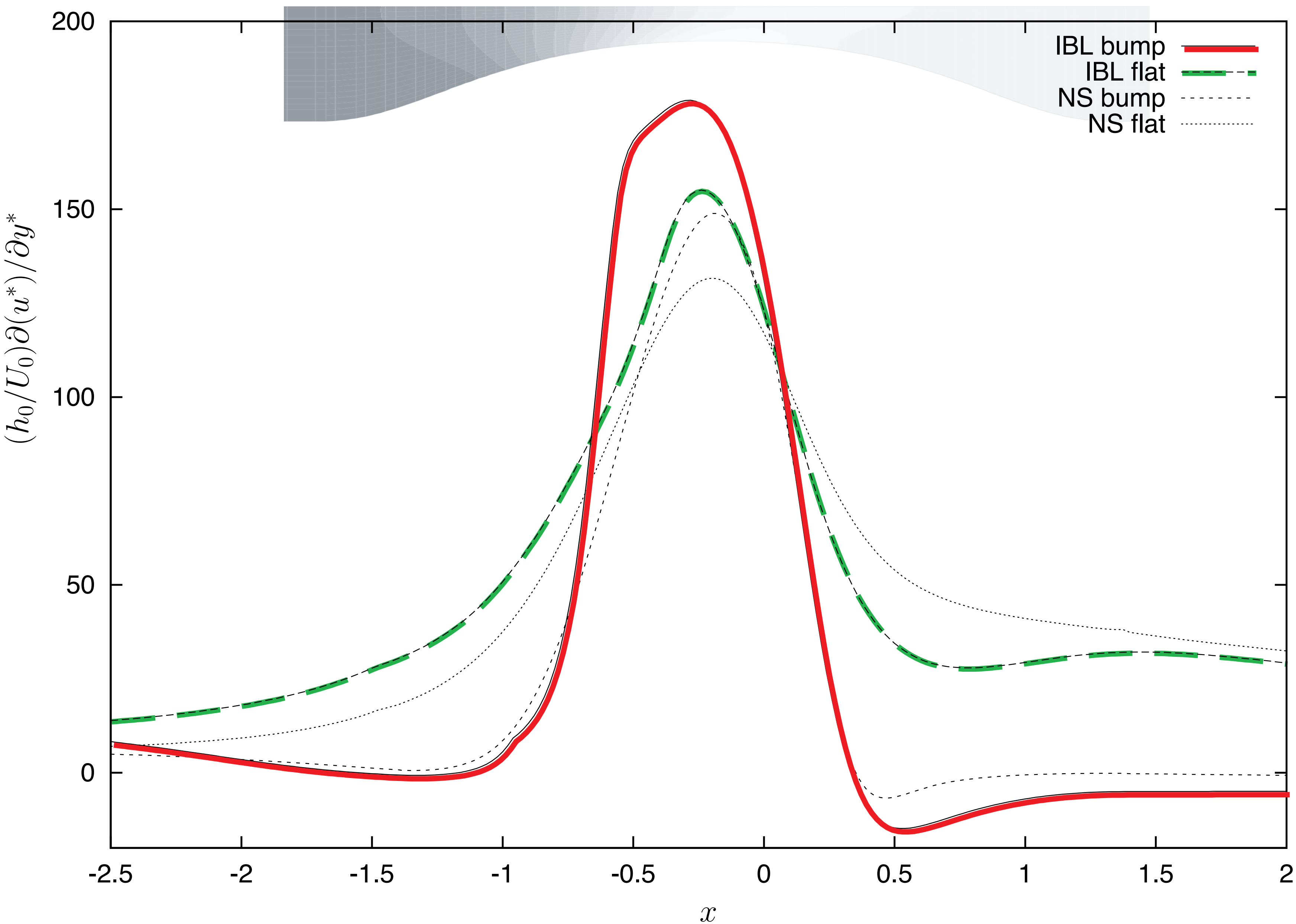


non symmetrical case

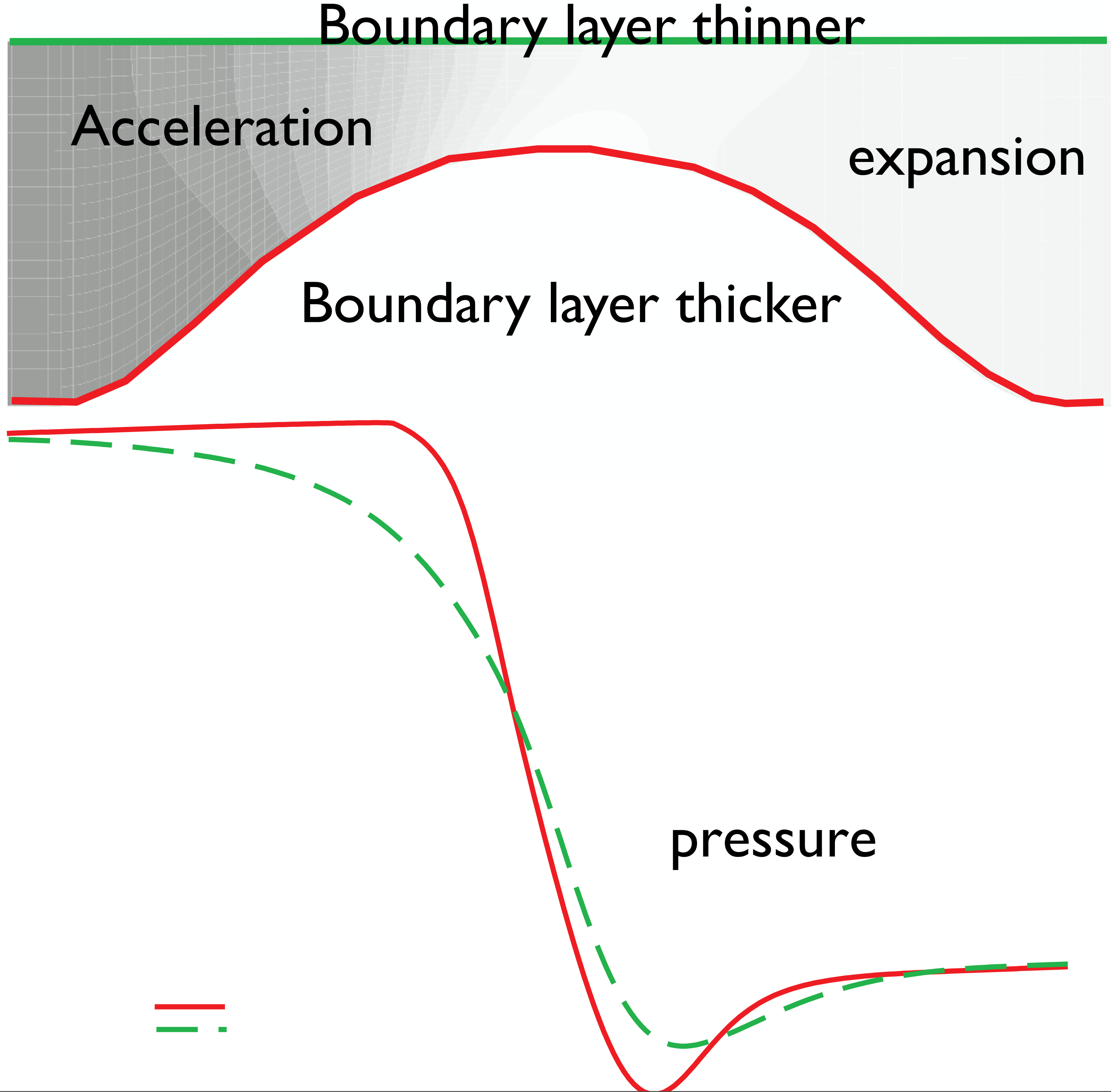
- RNSP
- modified integral method to take into account the transverse variation of pressure
- NS



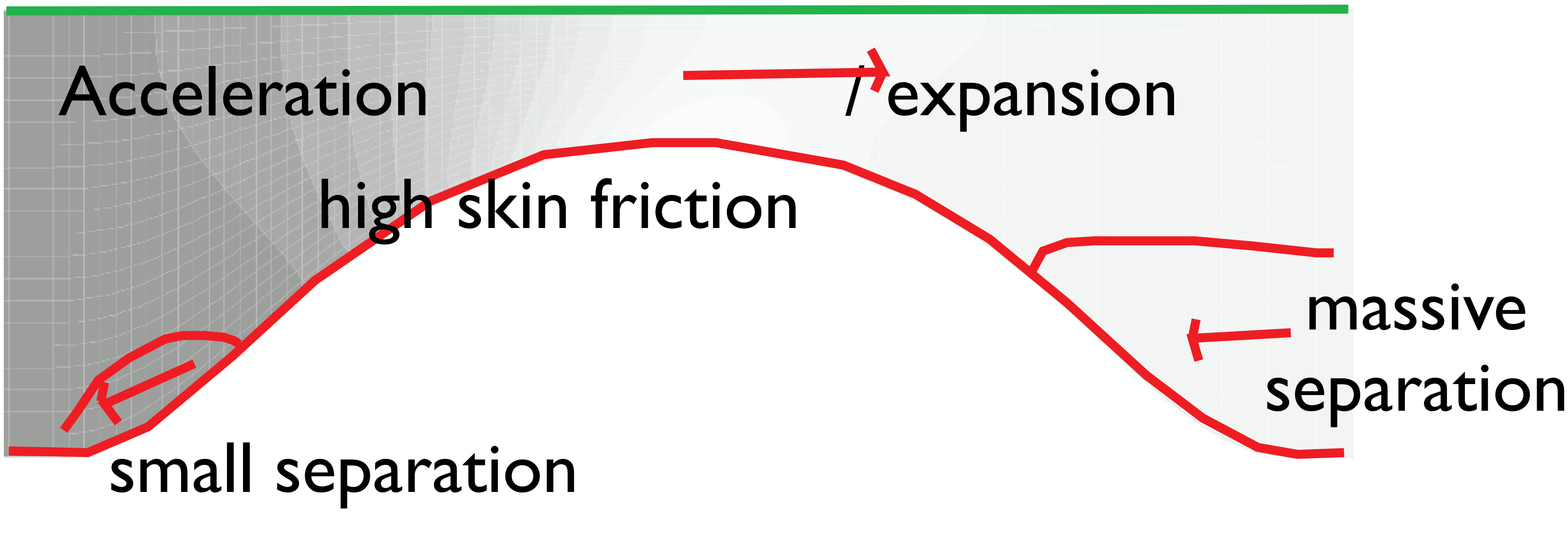
1
0
-1
-2
-3
-4
-5
-6
-7
-8
$p^*/(\rho U_0^2)$
-2.5
-2
-1.5
-1
-0.5
0
0.5
1
1.5
2
x
IBL bump
IBL flat
NS bump
NS flat
bump



IBL bump
IBL flat
NS bump
NS flat
$(h_0/U_0)\partial(u^*)/\partial y^*$
x
200
150
100
50
0
-2.5
-2
-1.5
-1
-0.5
0
0.5
1
1.5
2



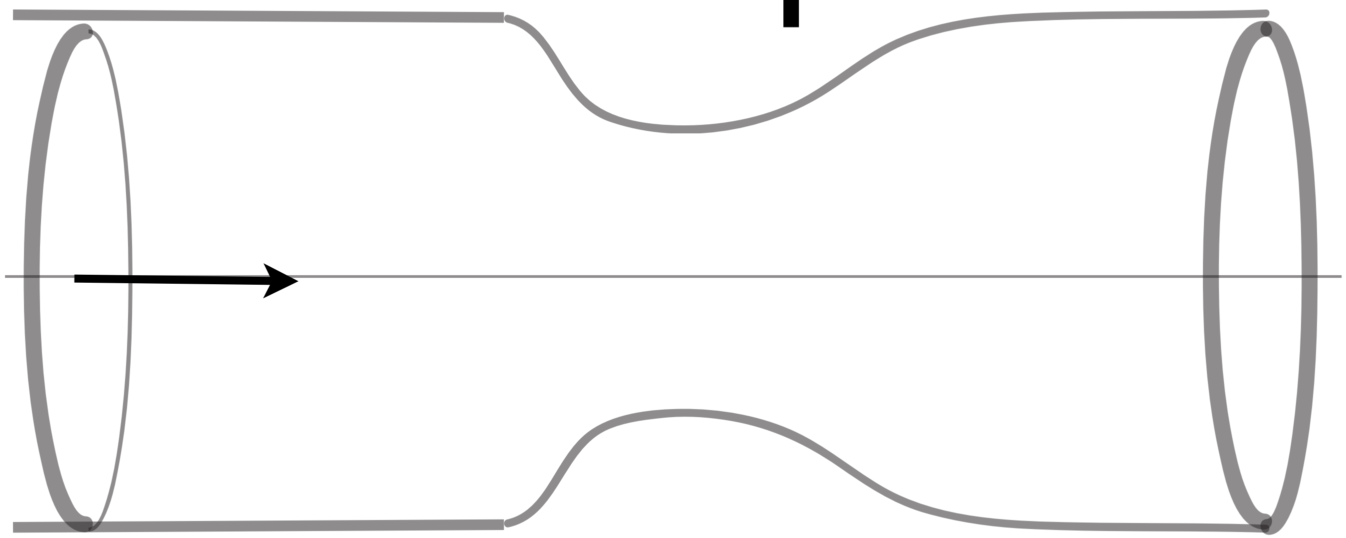
Boundary layer thinner
Acceleration
expansion
Boundary layer thicker
pressure


Acceleration
/ expansion
high skin friction
massive
separation
small separation

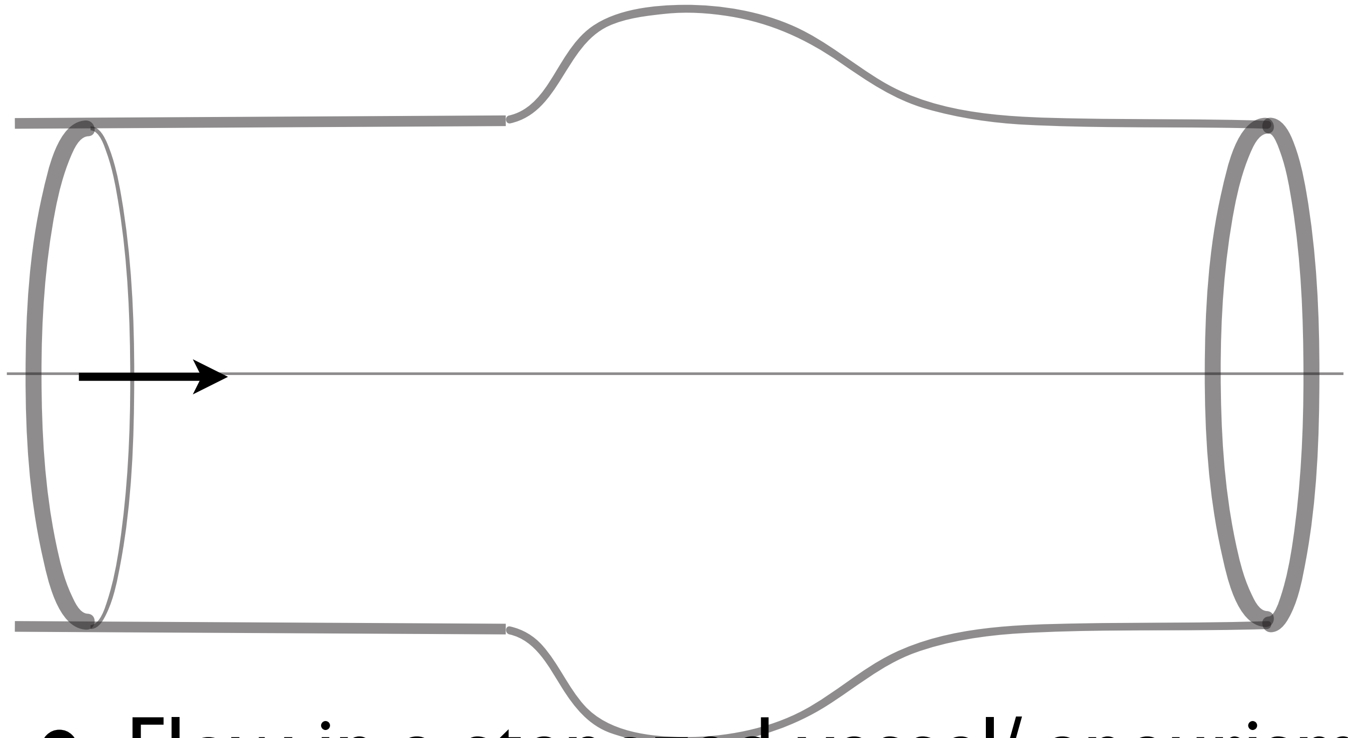
Example 5

- Flow in a stenozed vessel/ aneurism
- unsteady, rigid wall

Example 2

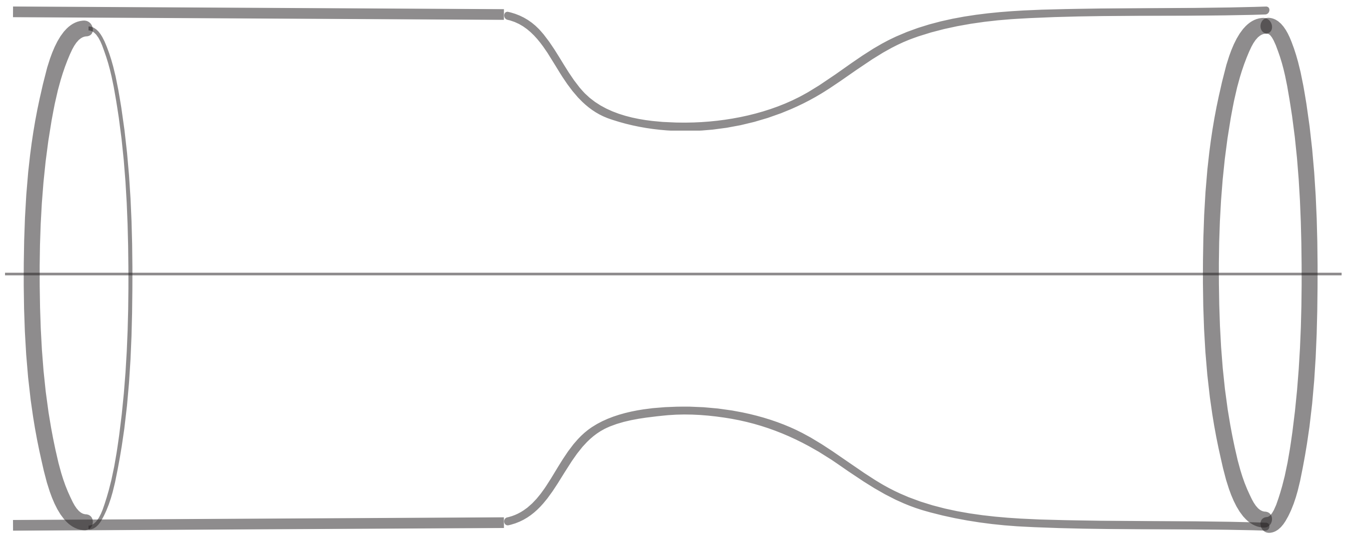


- Flow in a stenozed vessel/
- unsteady, rigid wall

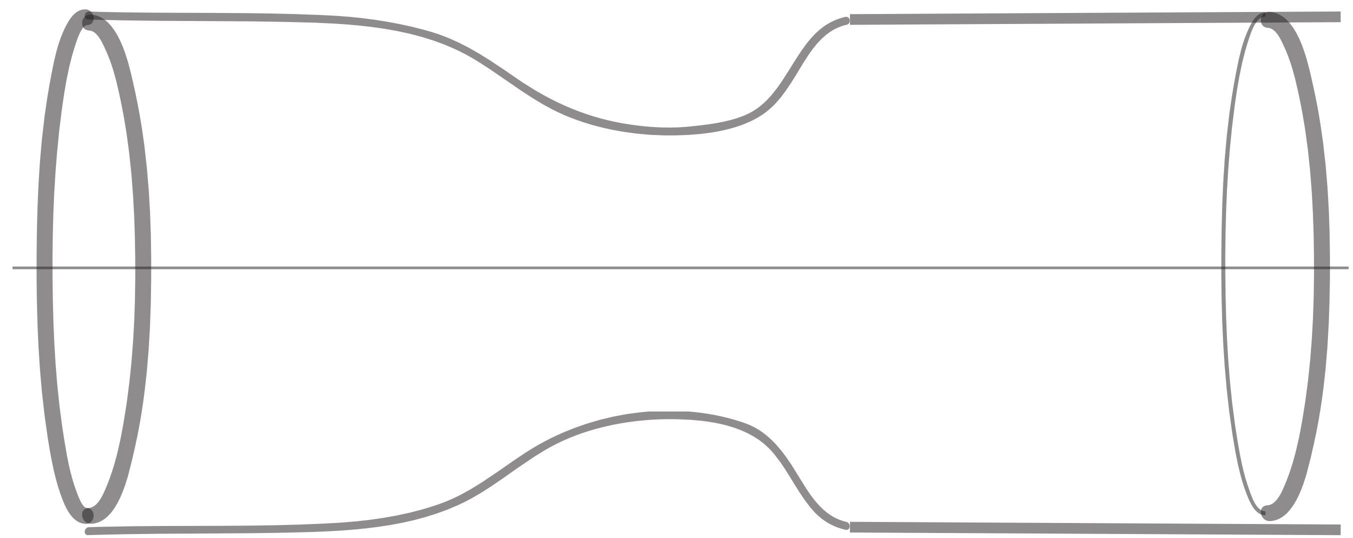


- Flow in a stenozed vessel/ aneurism
- unsteady, rigid wall

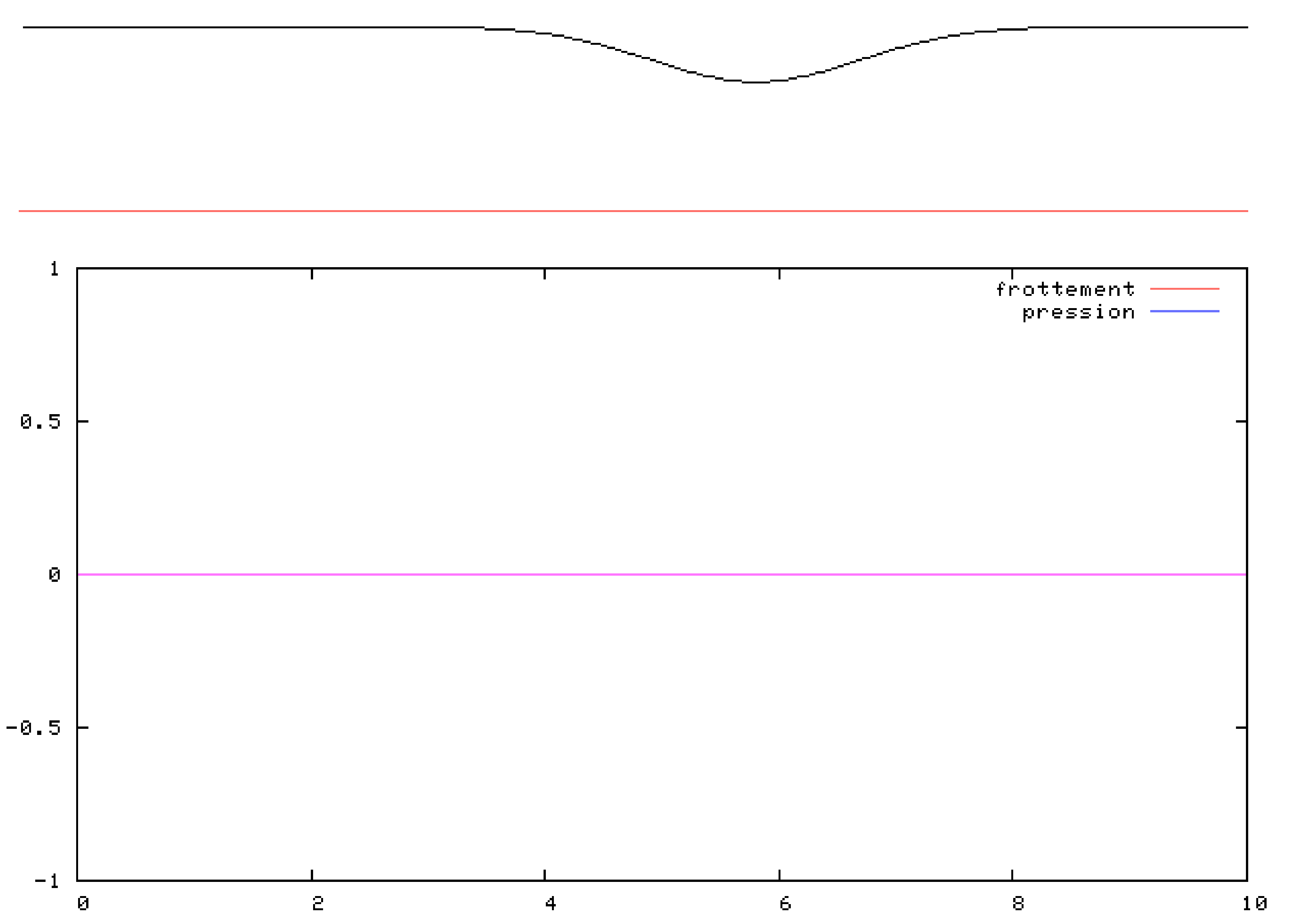
- Stenosis



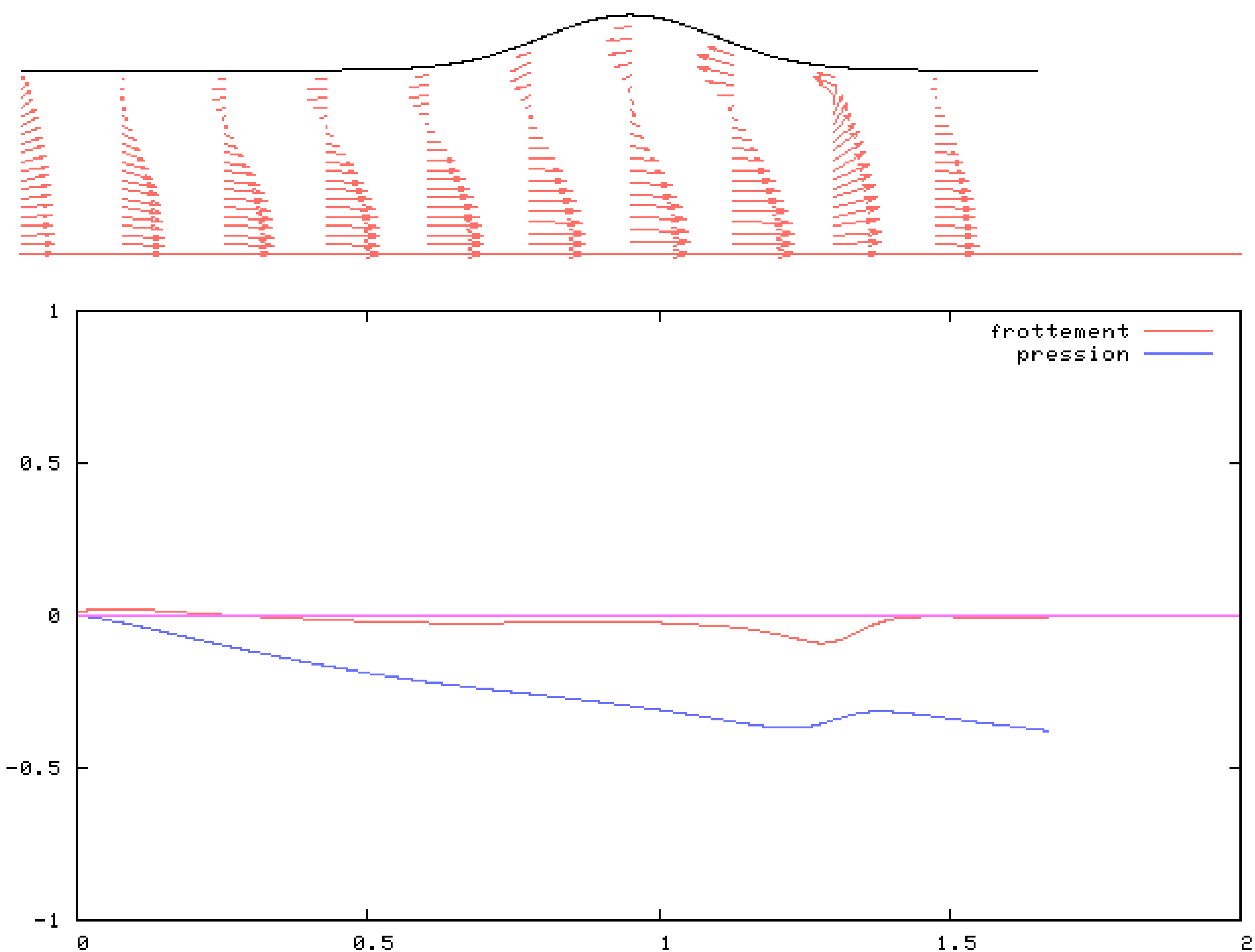
- Stenosis

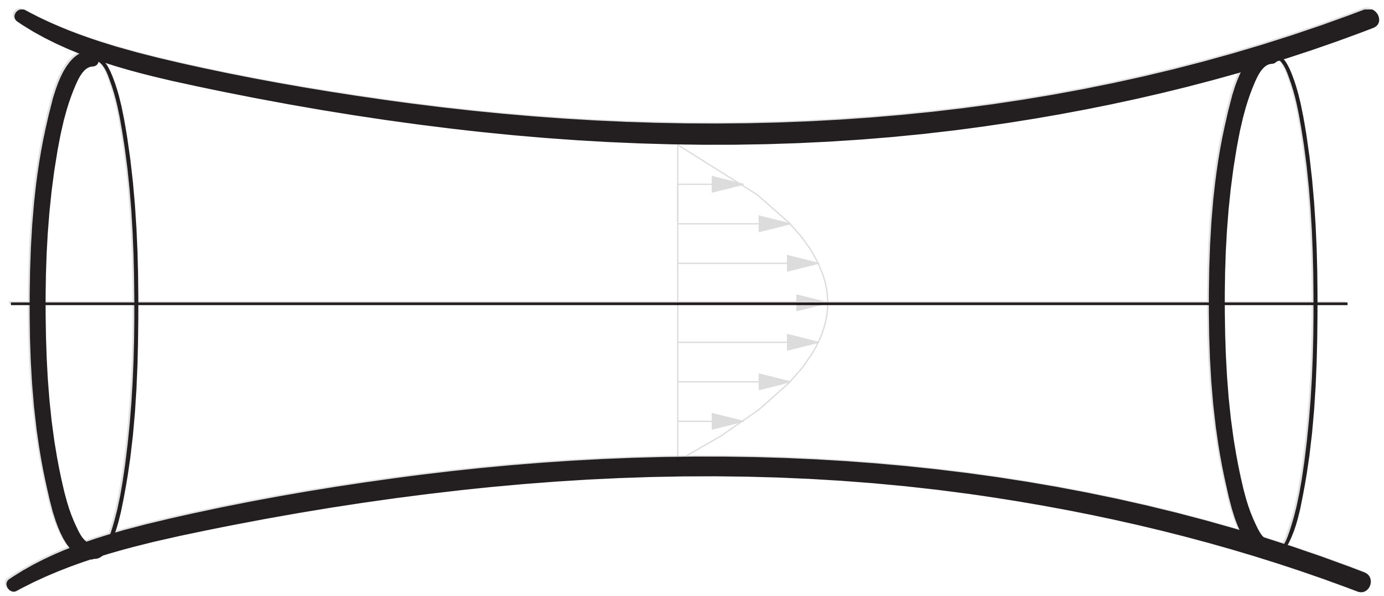


- Stenosis

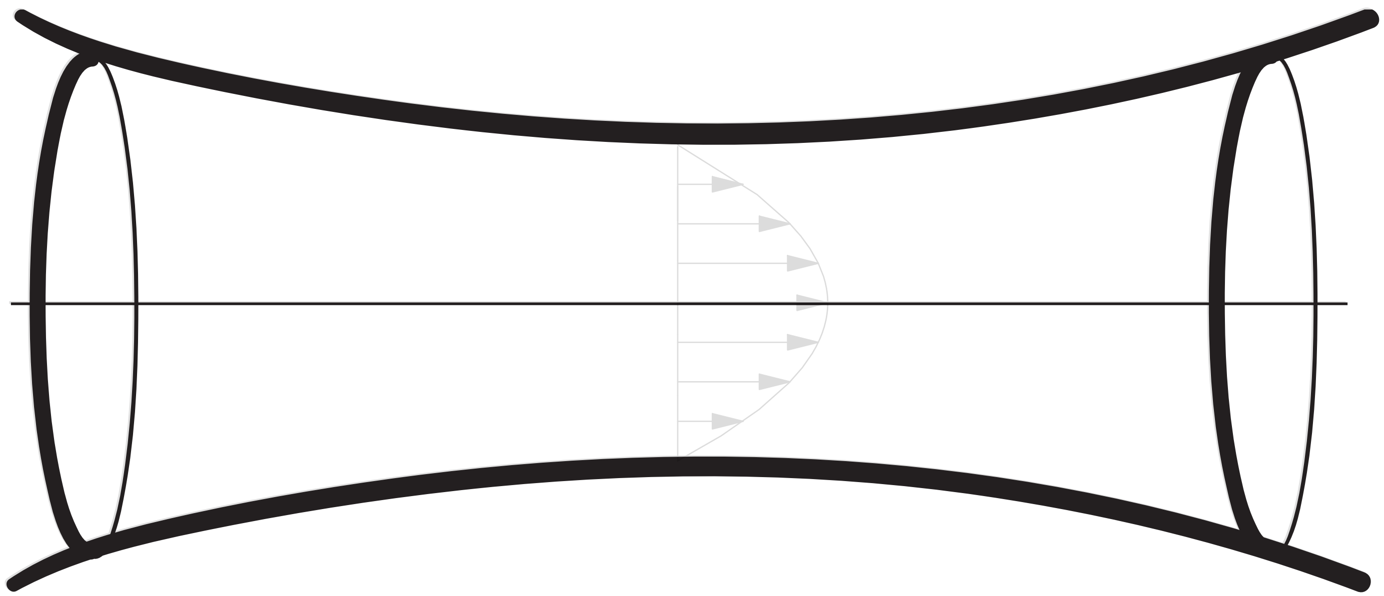


- Aneurysm

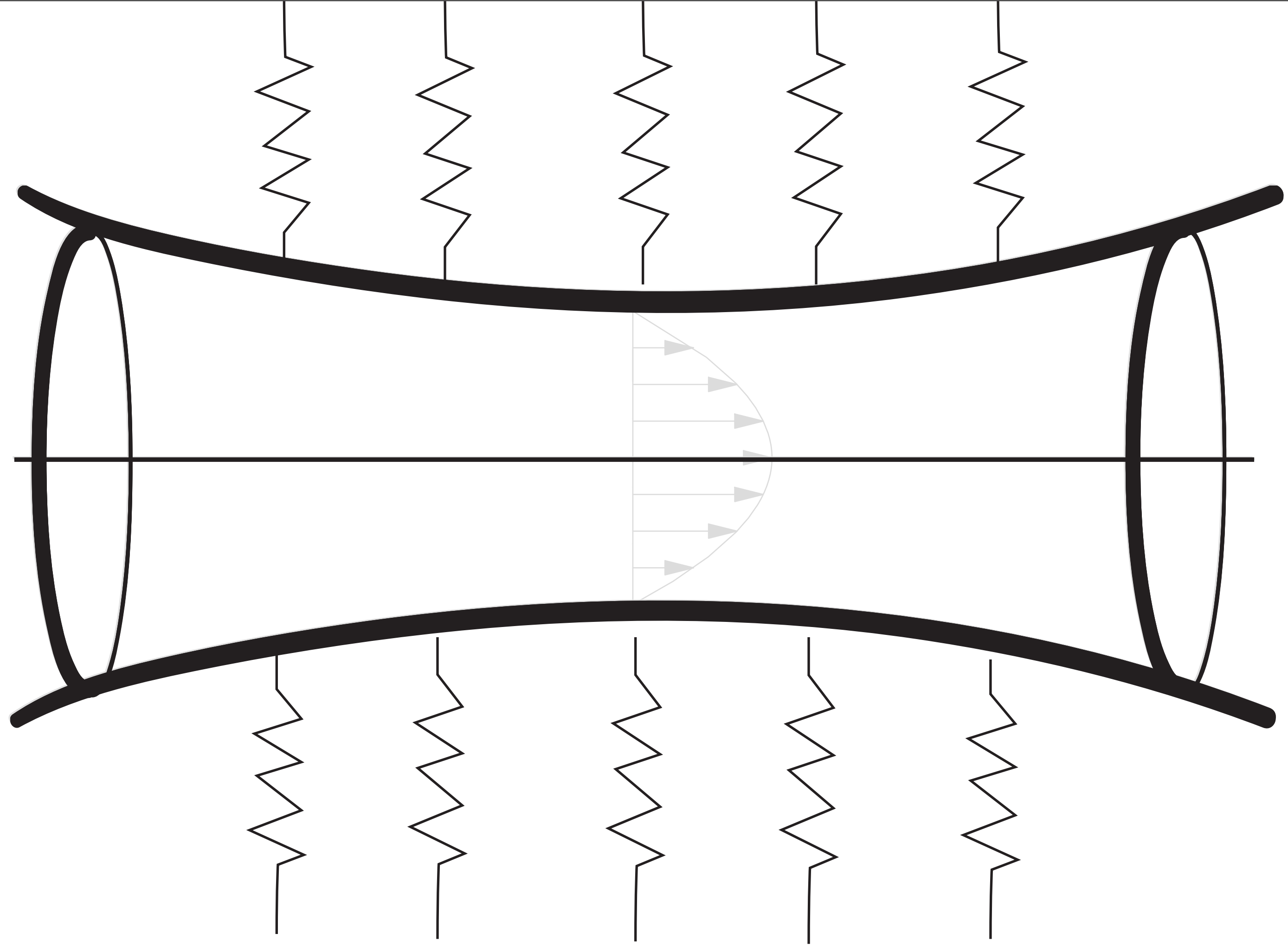




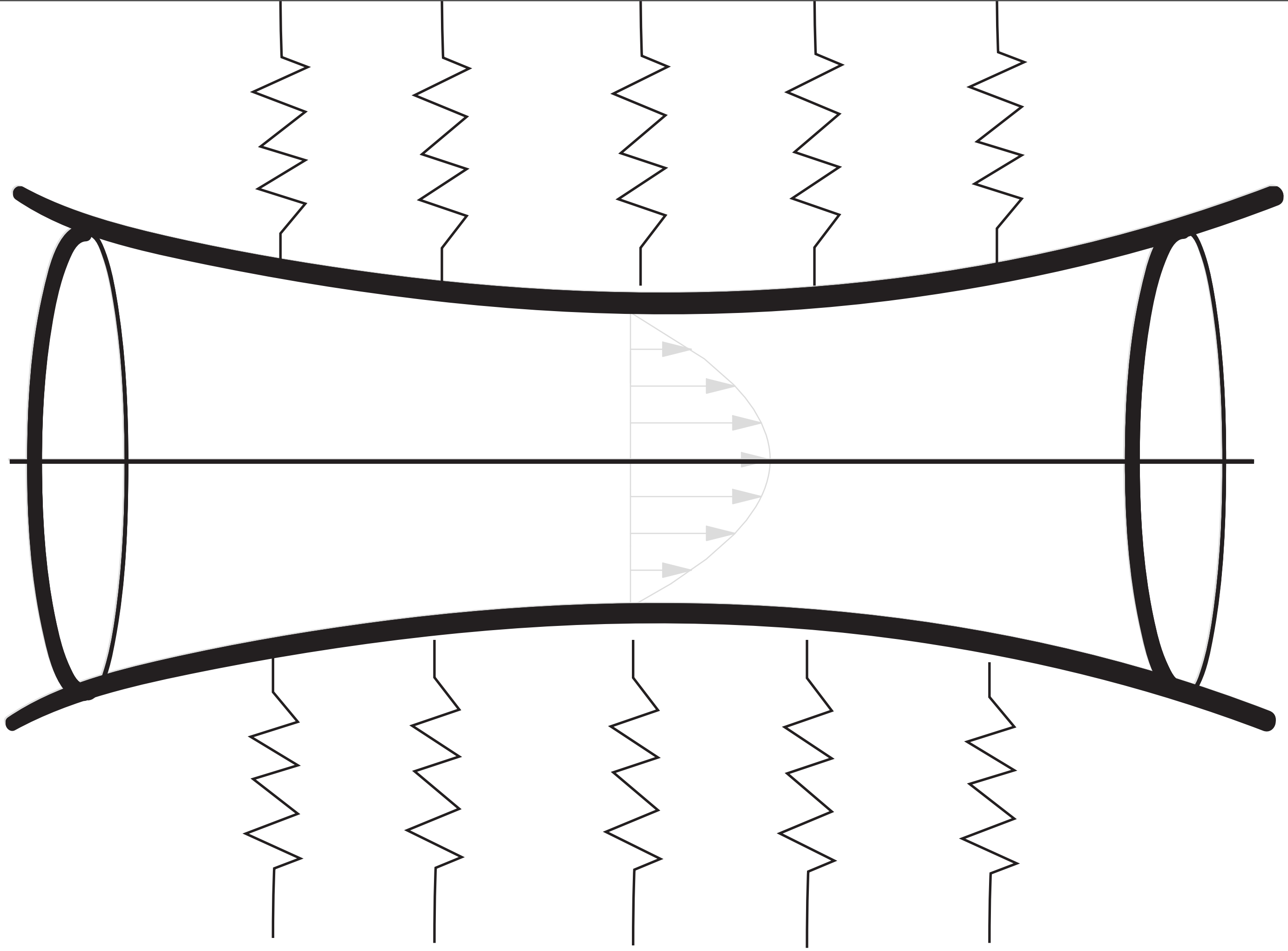
Up to now, the wall was rigid

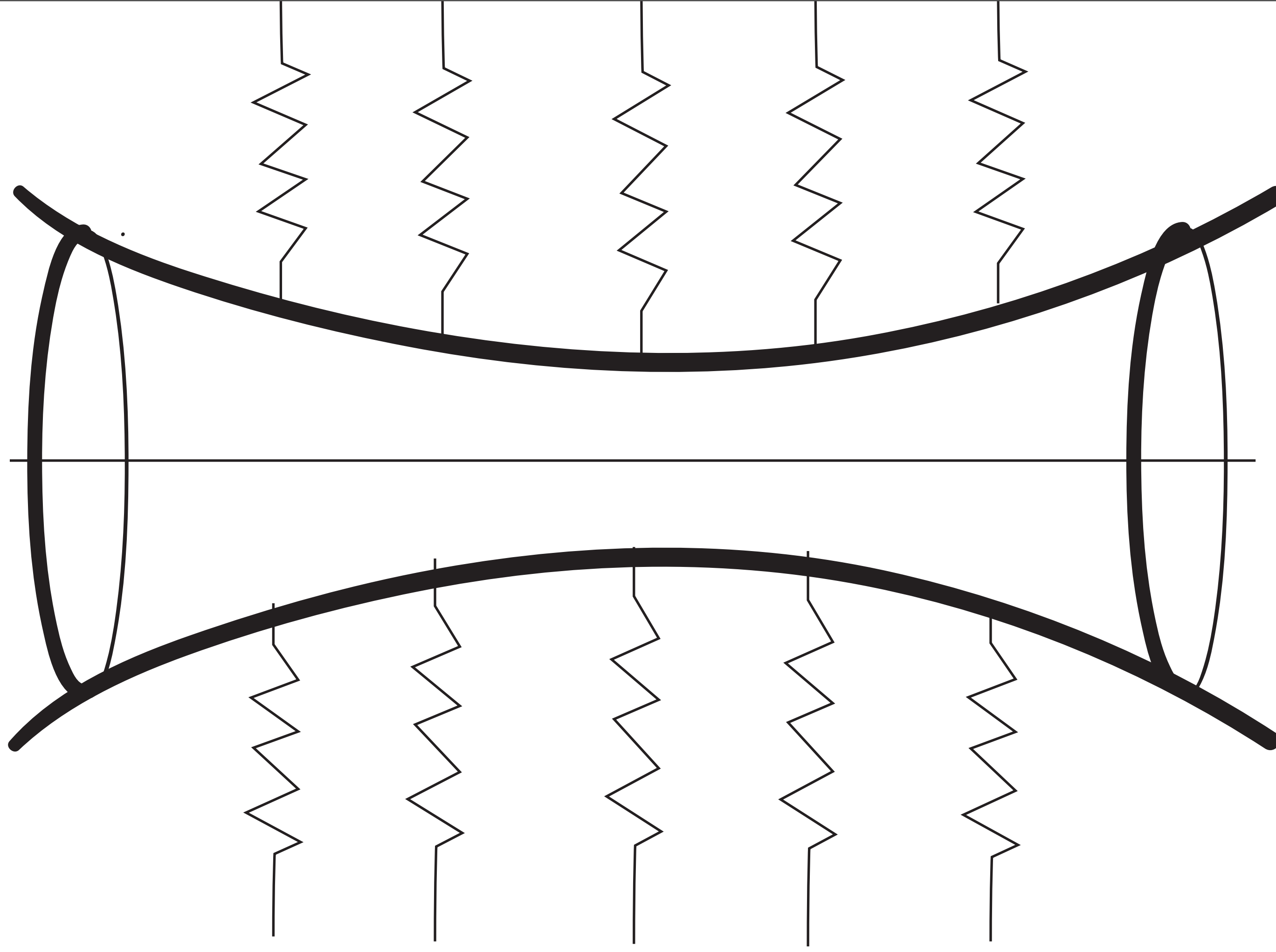


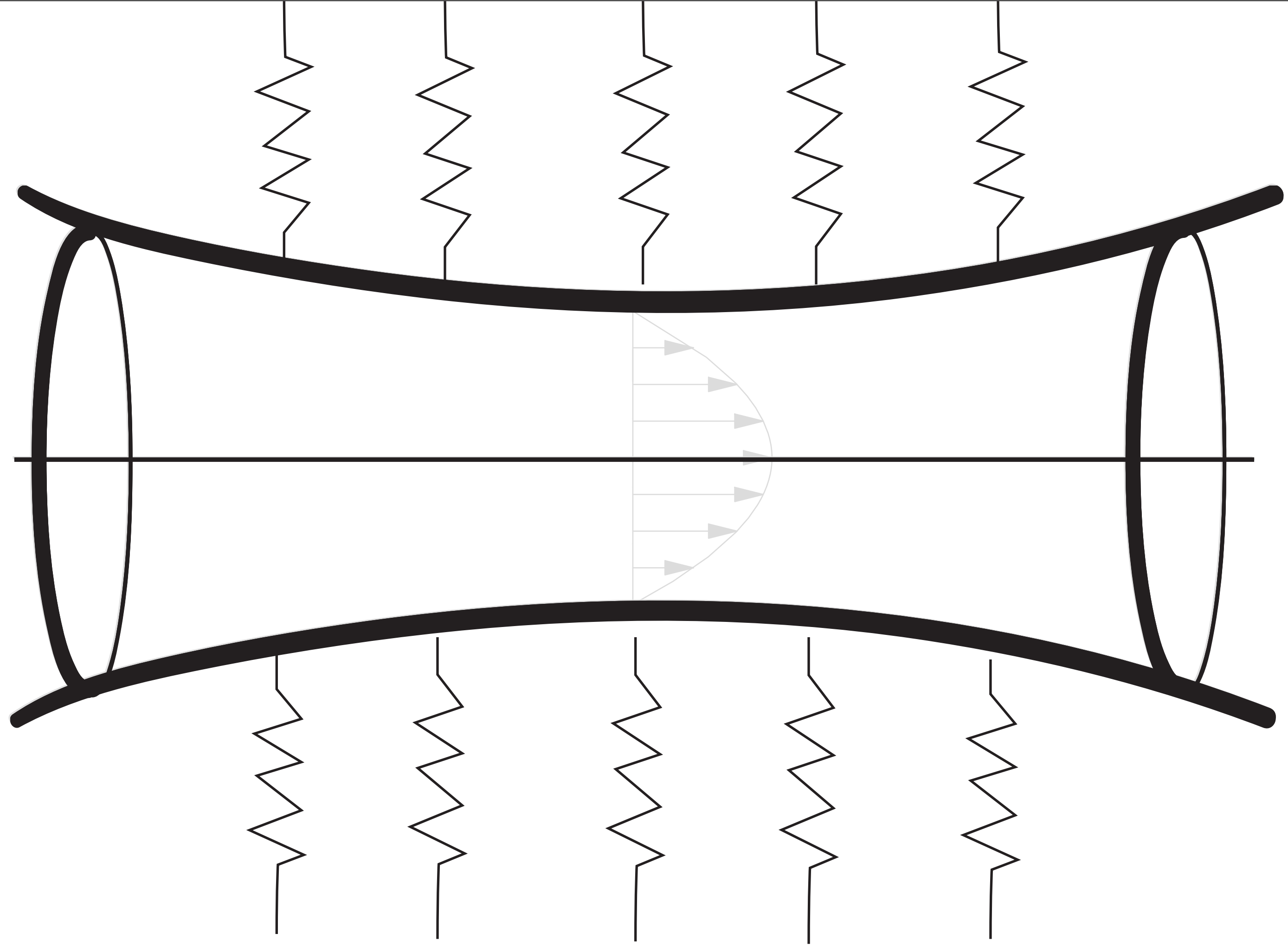
we use a simple elastic model

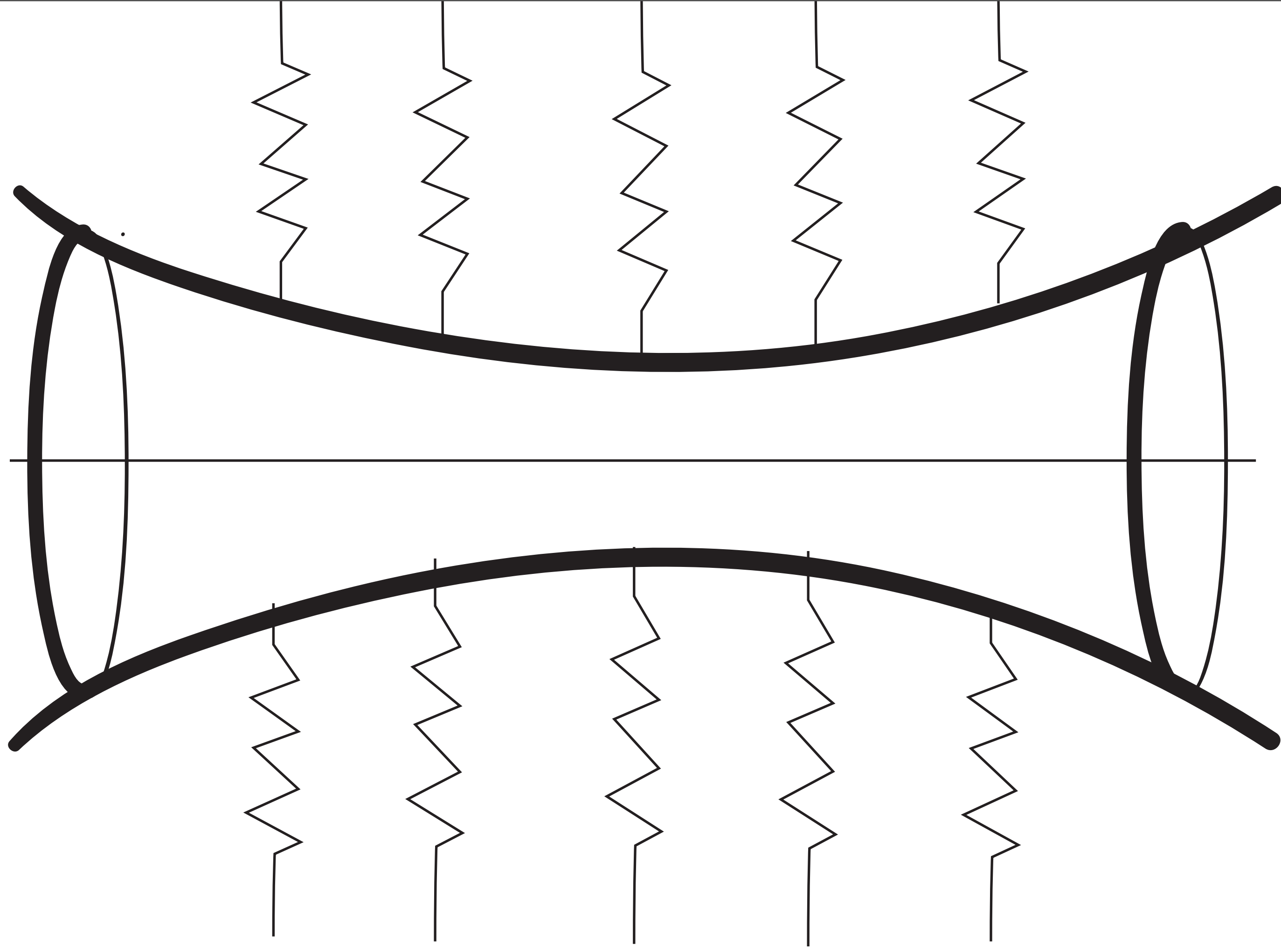


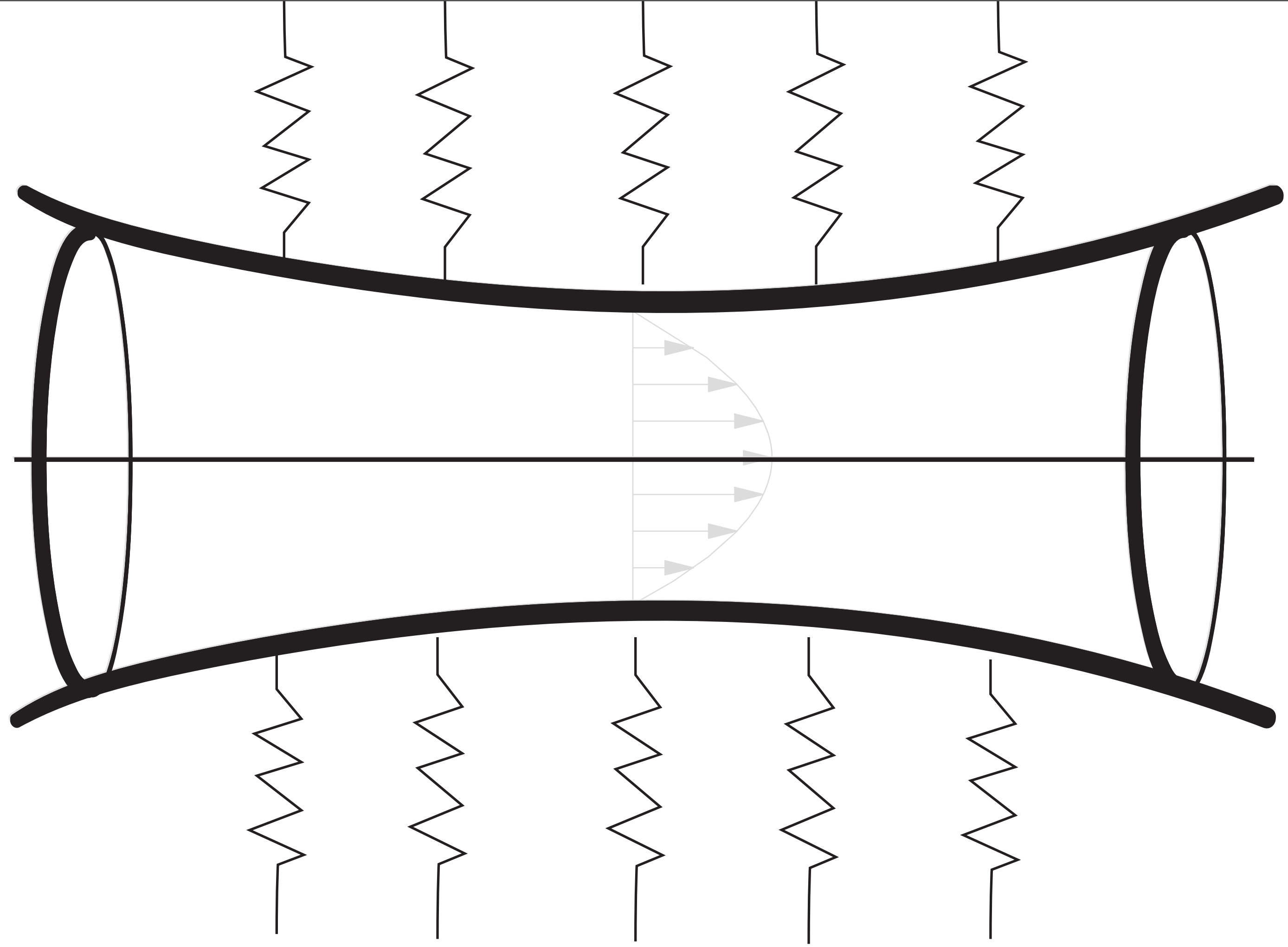
we use a simple elastic model



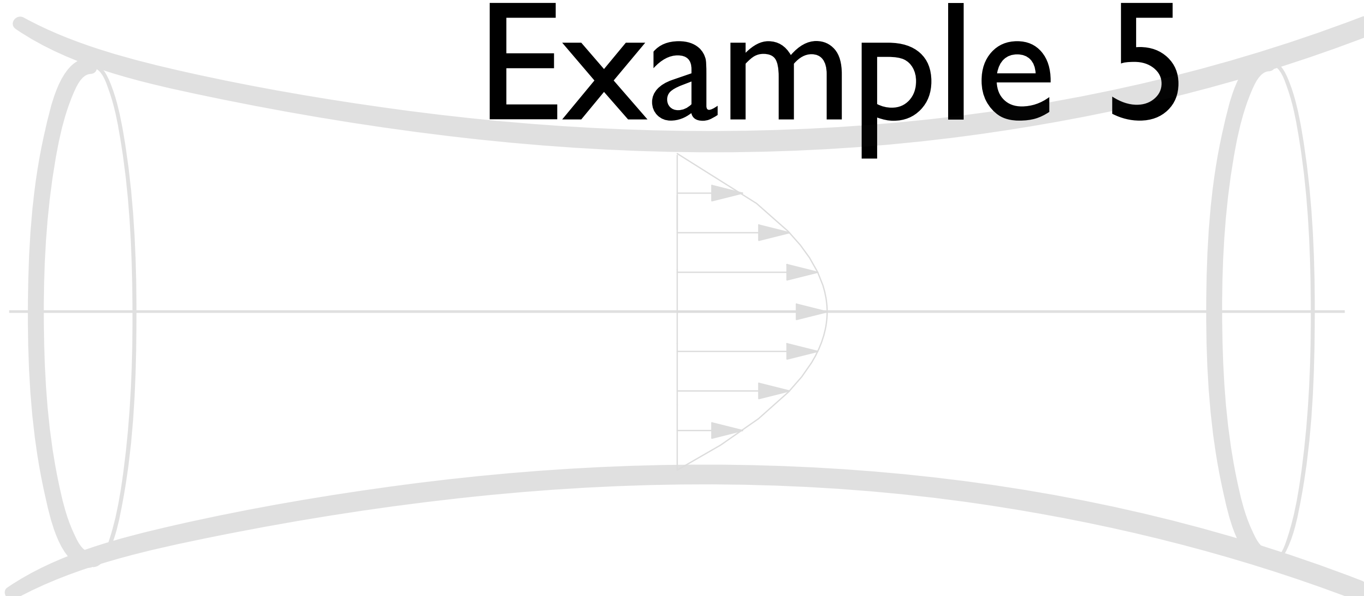




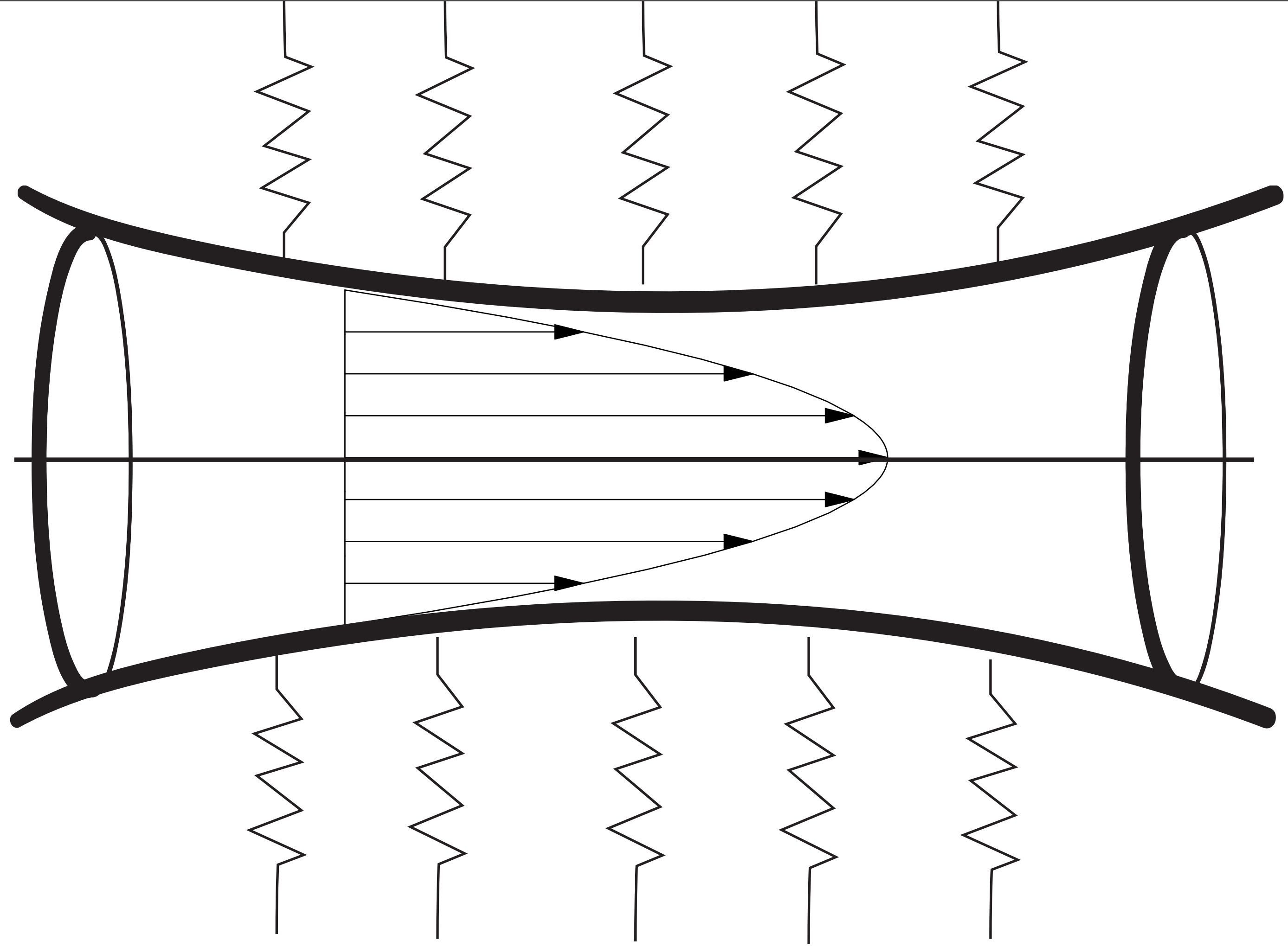




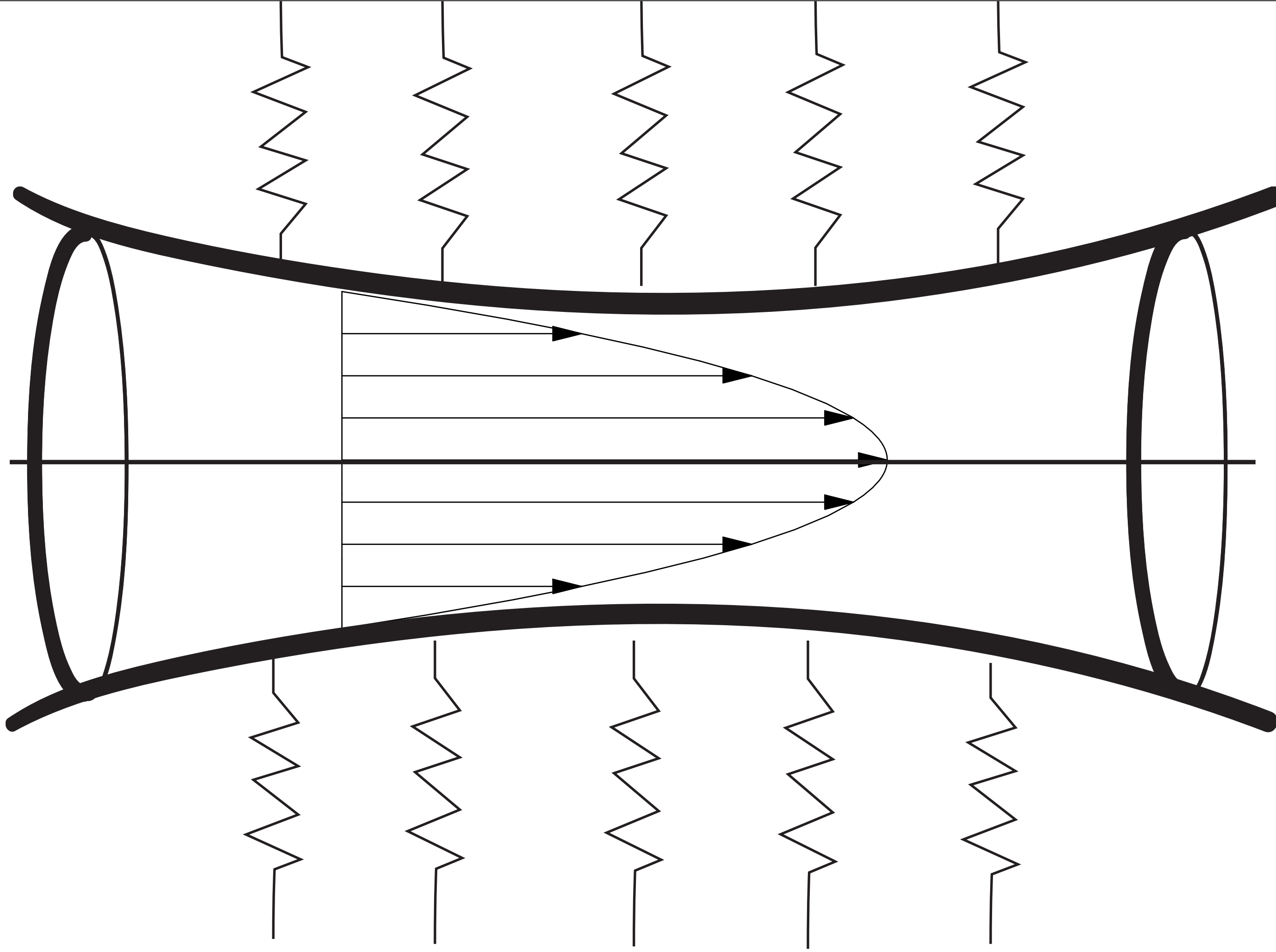
Example 5



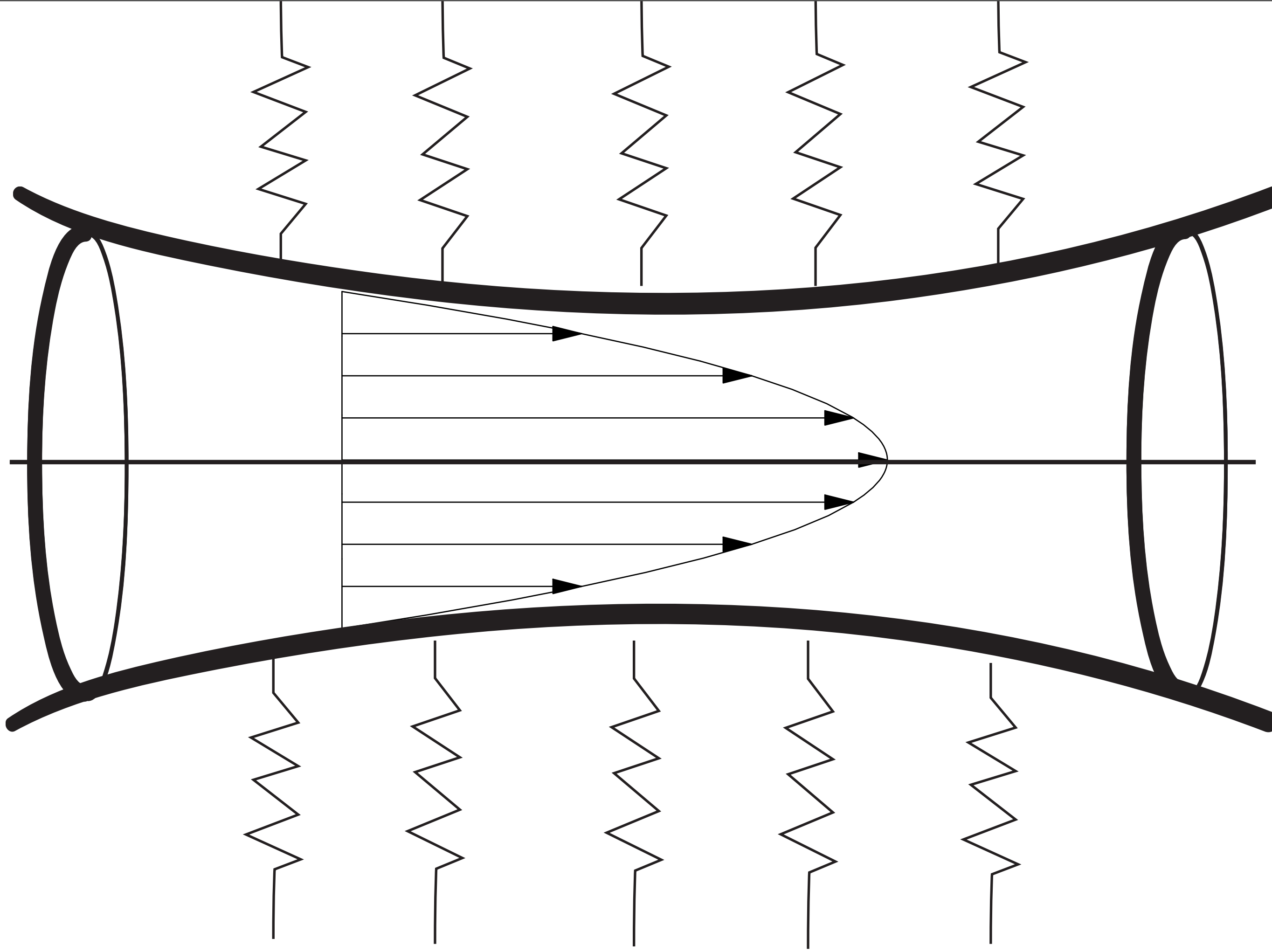
- Flow in a collapsible tube
- unsteady, elastic wall, no inertia



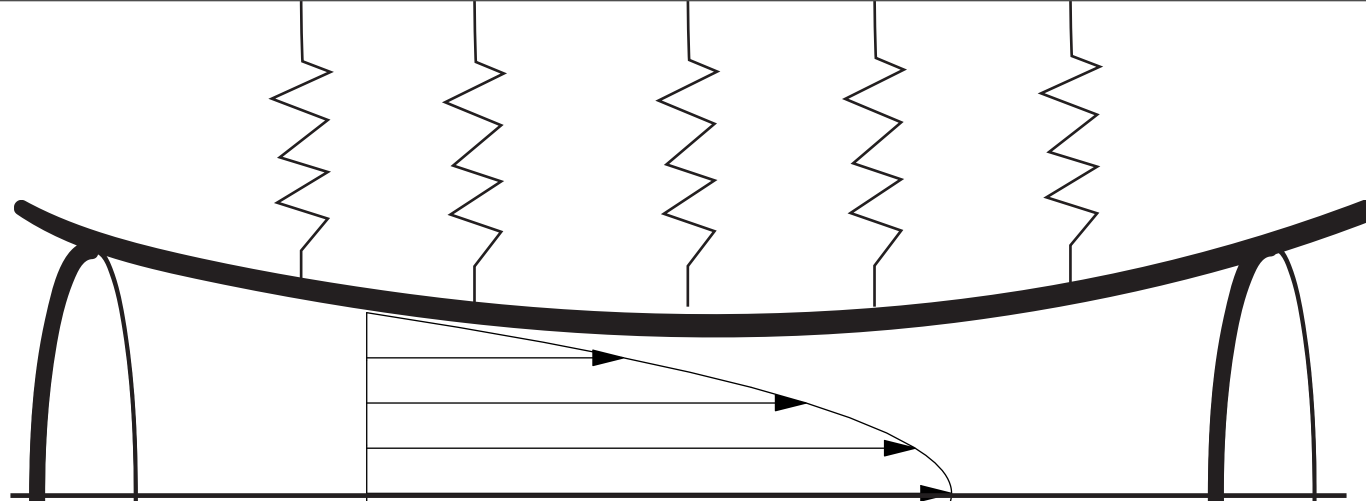
Collapsible tube

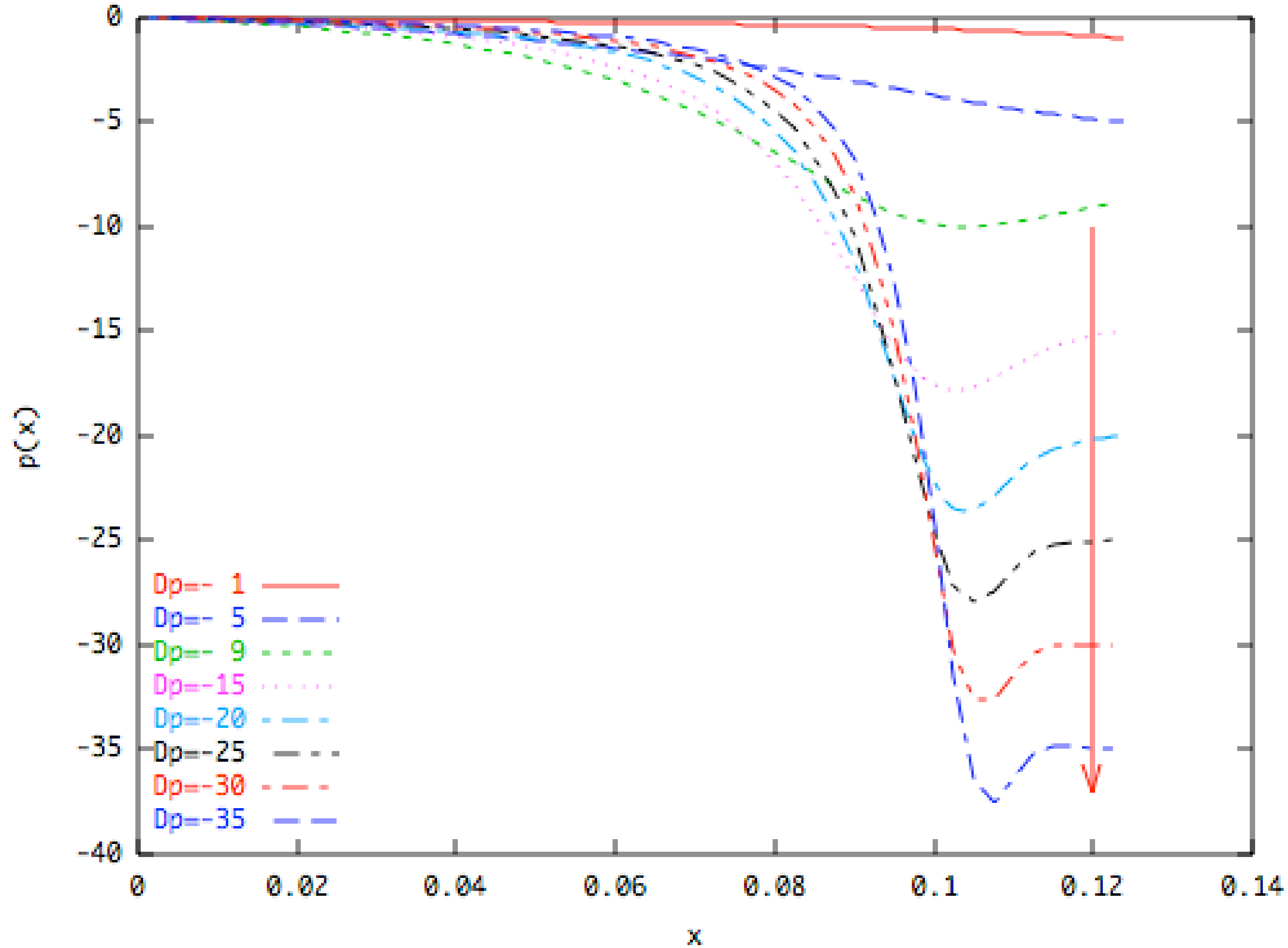


R^n gives p^{n+1}

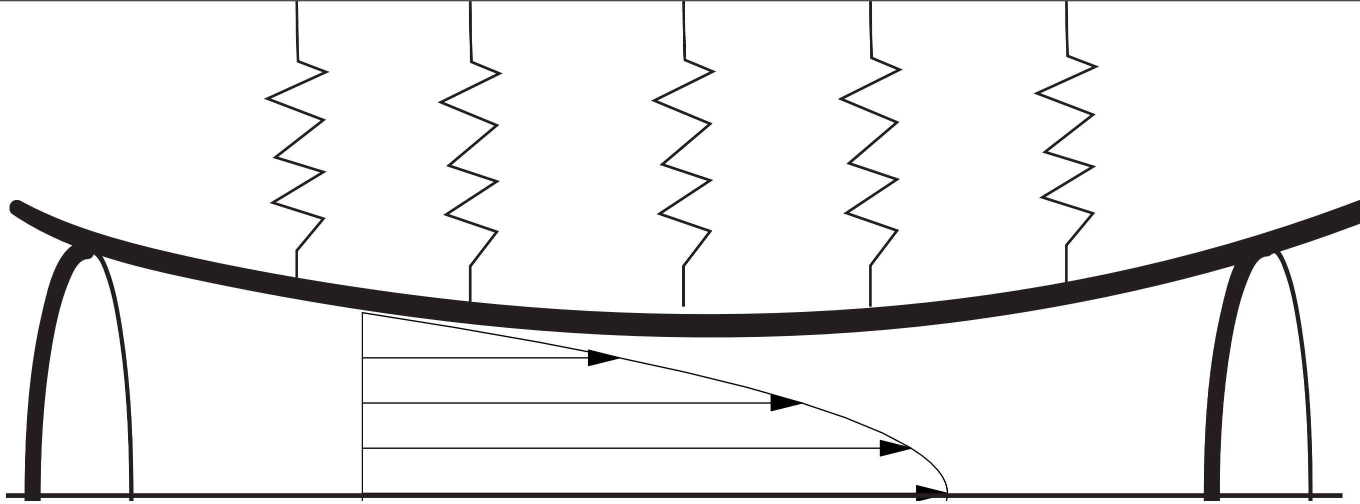


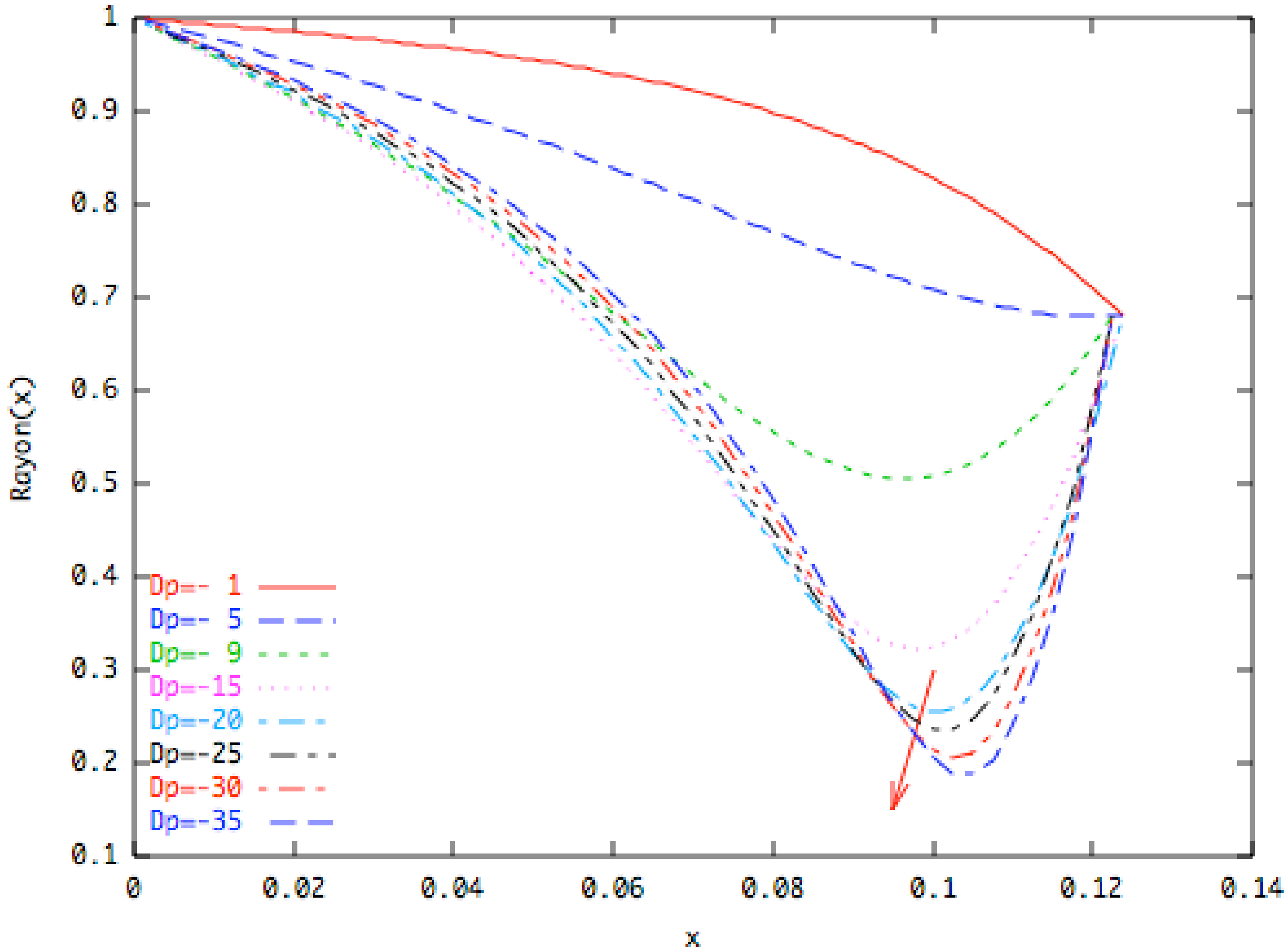
R^n gives p^{n+1} $\longrightarrow$ $p^{n+1} = k(R^{n+1} - 1)$



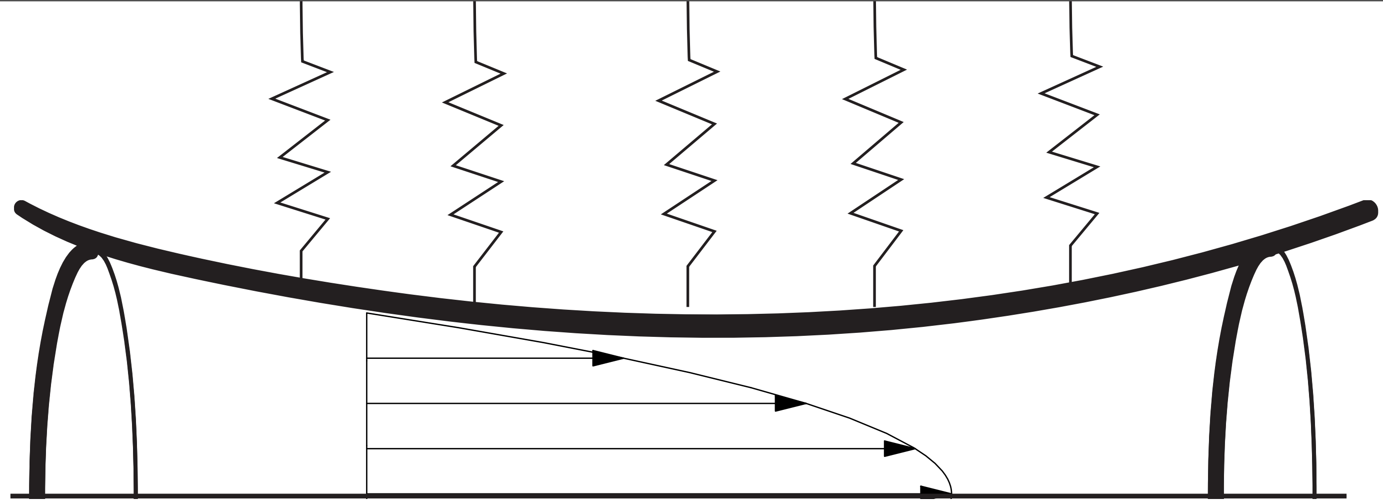


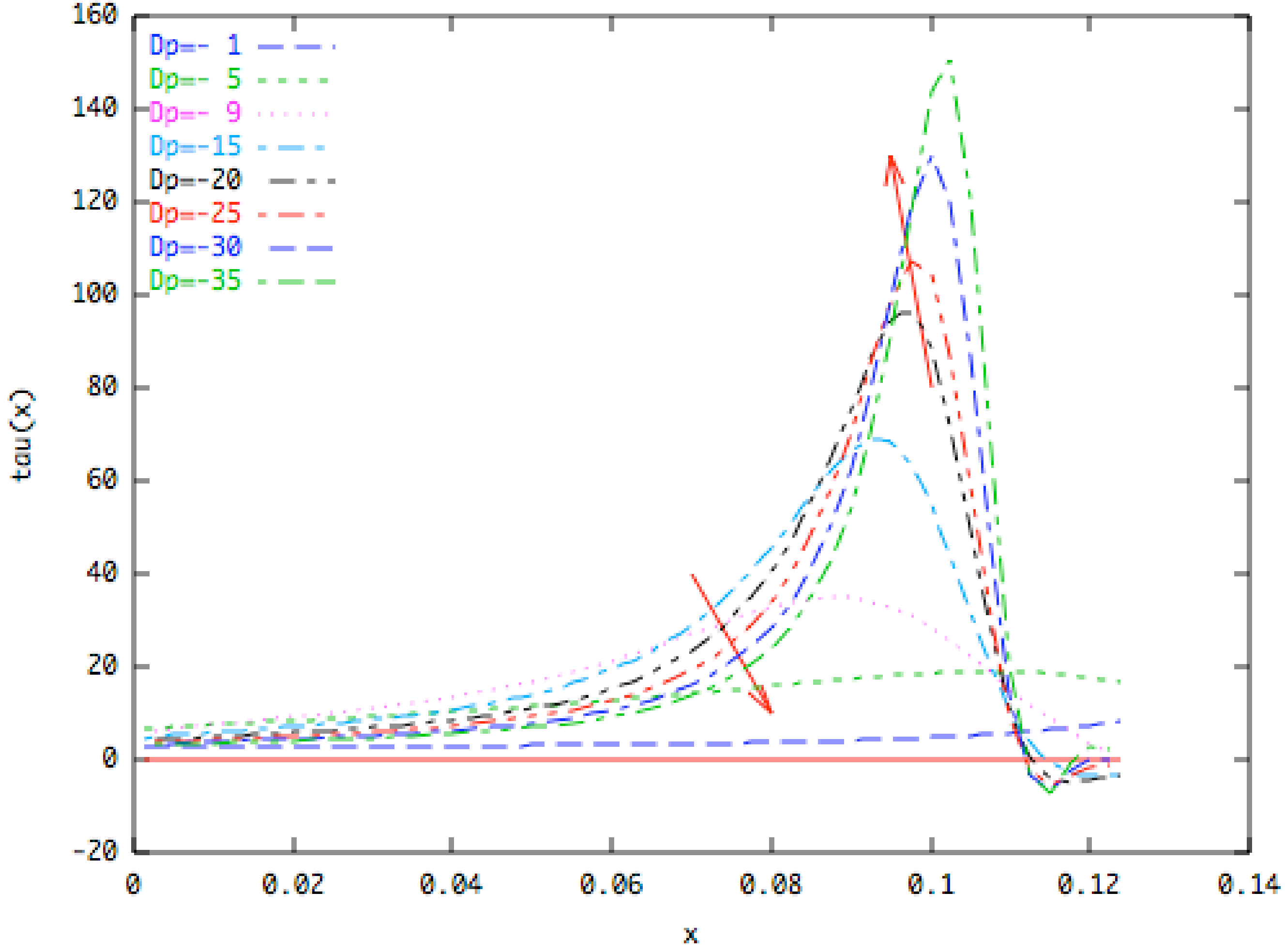
p(x)
x
0
-5
-10
-15
-20
-25
-30
-35
-40
0
0.02
0.04
0.06
0.08
0.1
0.12
0.14
Dp=- 1
Dp=- 5
Dp=- 9
Dp=-15
Dp=-20
Dp=-25
Dp=-30
Dp=-35



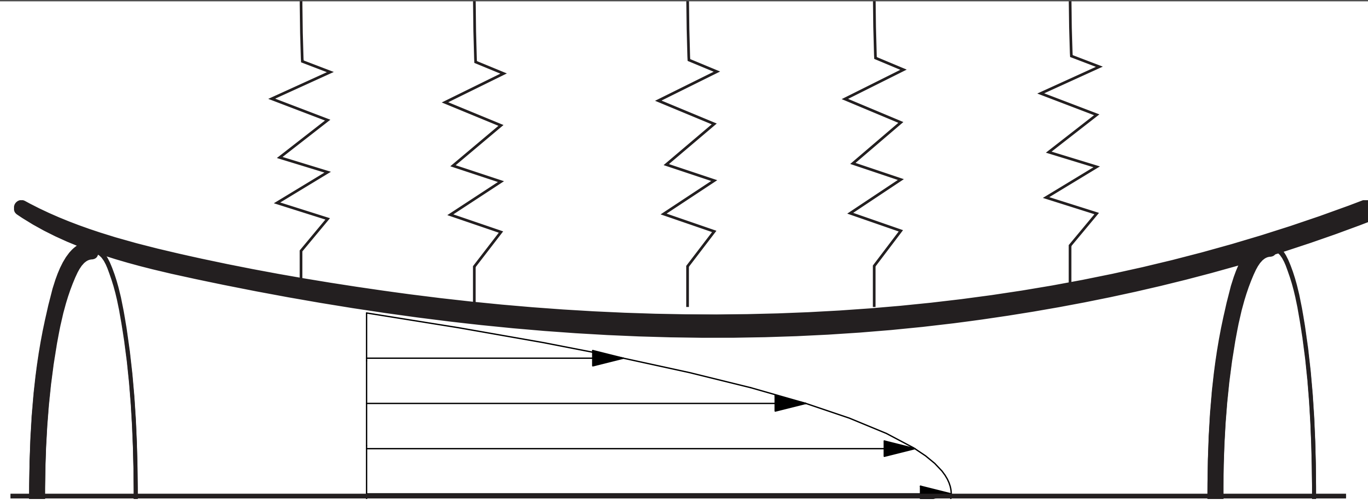


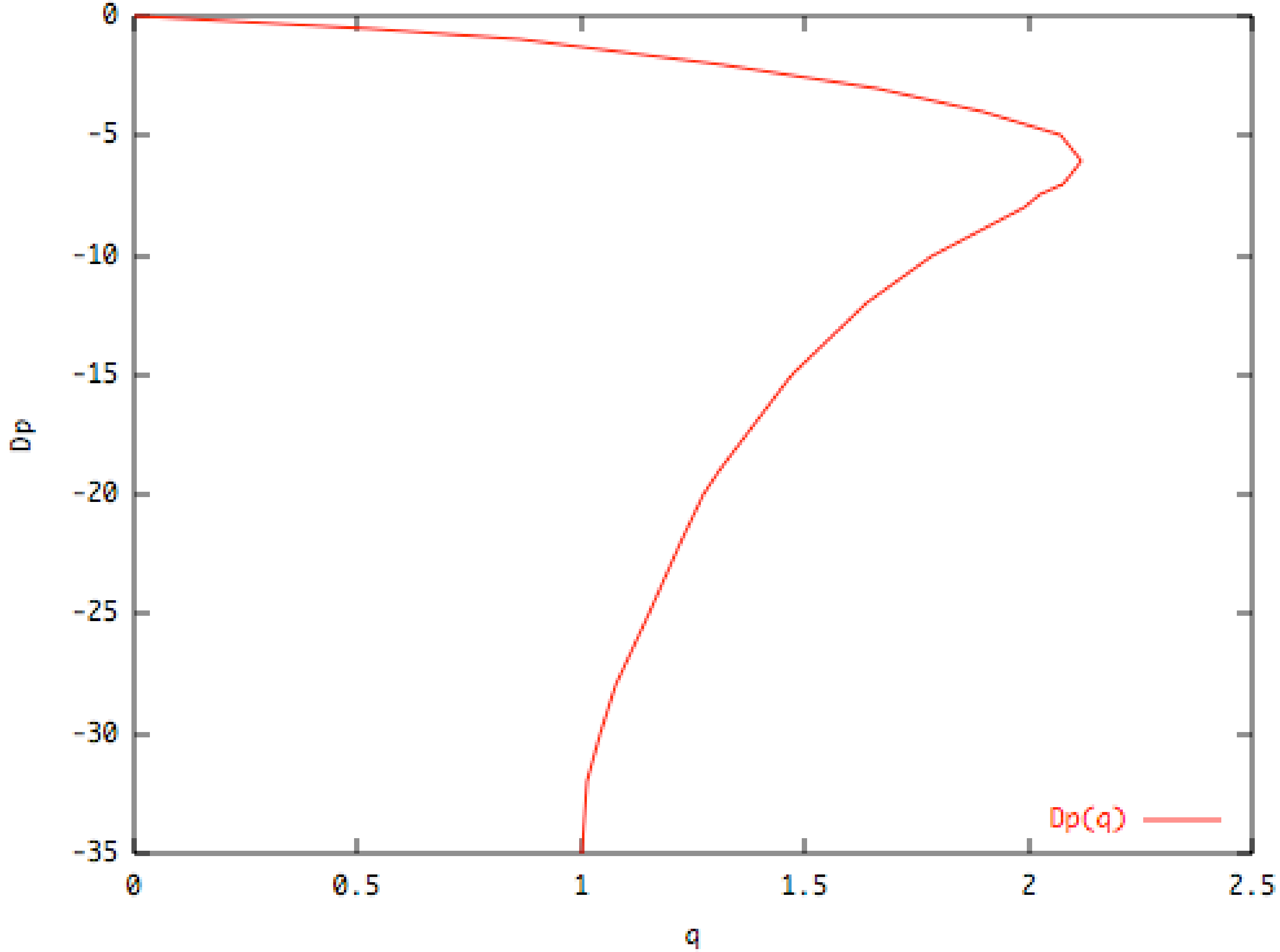
Rayon(x)
x
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0.02
0.04
0.06
0.08
0.1
0.12
0.14
Dp=- 1
Dp=- 5
Dp=- 9
Dp=-15
Dp=-20
Dp=-25
Dp=-30
Dp=-35



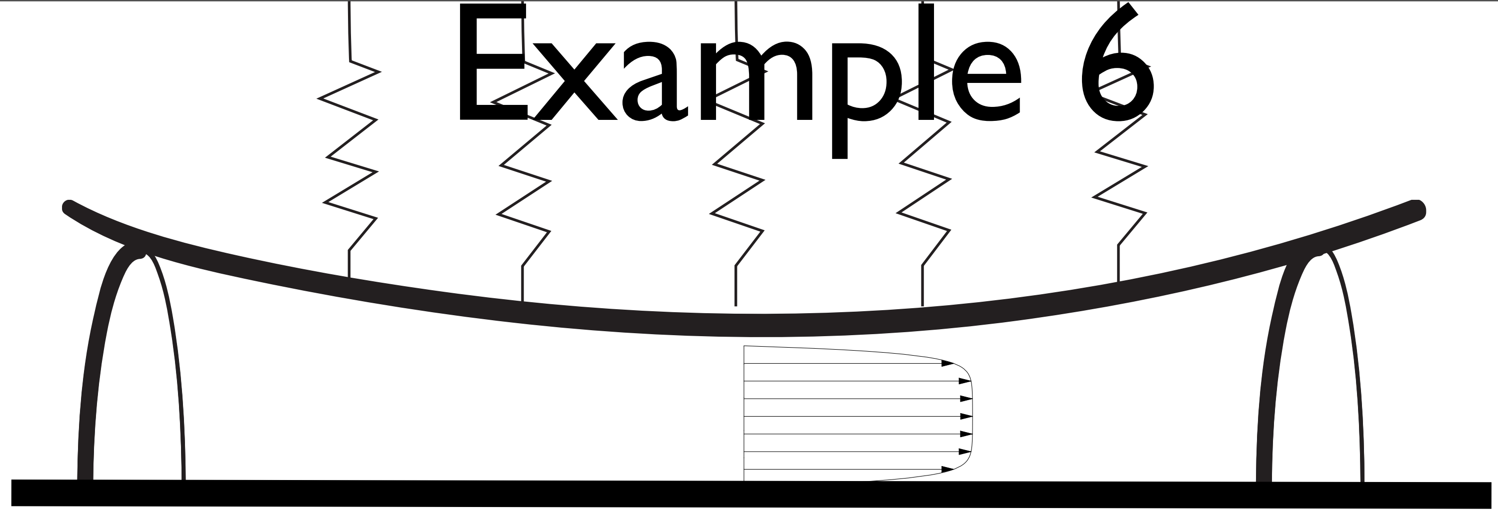


Dp=- 1
Dp=- 5
Dp=- 9
Dp=-15
Dp=-20
Dp=-25
Dp=-30
Dp=-35
tau(x)
x
160
140
120
100
80
60
40
20
0
-20
0
0.02
0.04
0.06
0.08
0.1
0.12
0.14




0
-5
-10
-15
-20
-25
-30
-35
Dp
0
0.5
1
1.5
2
2.5
q
Dp(q)

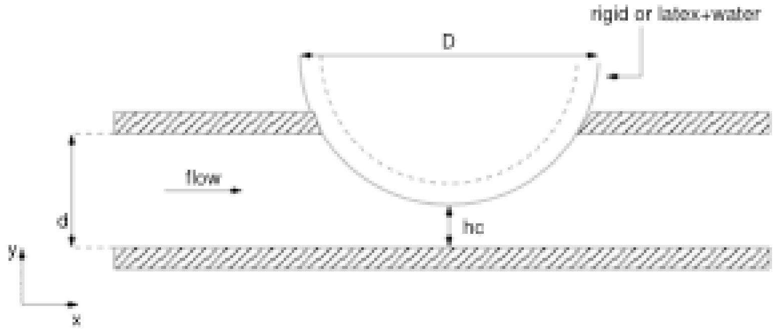
Example 6



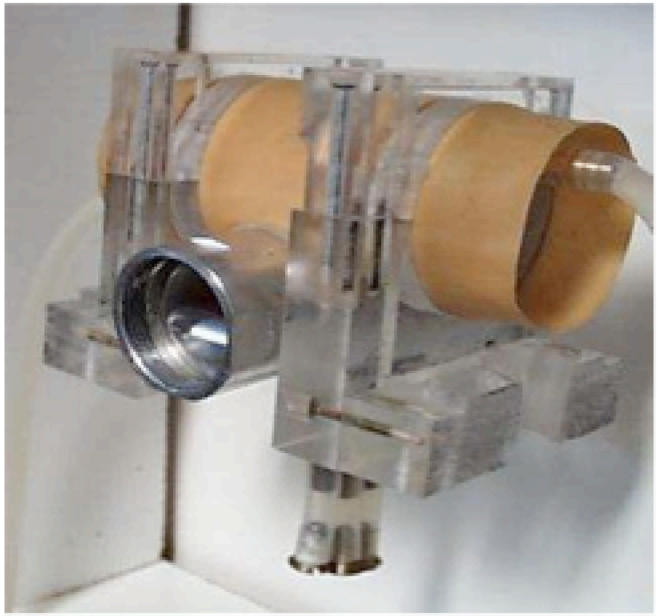
Franz Chouly   Yohan Payan (TIMC Grenoble)

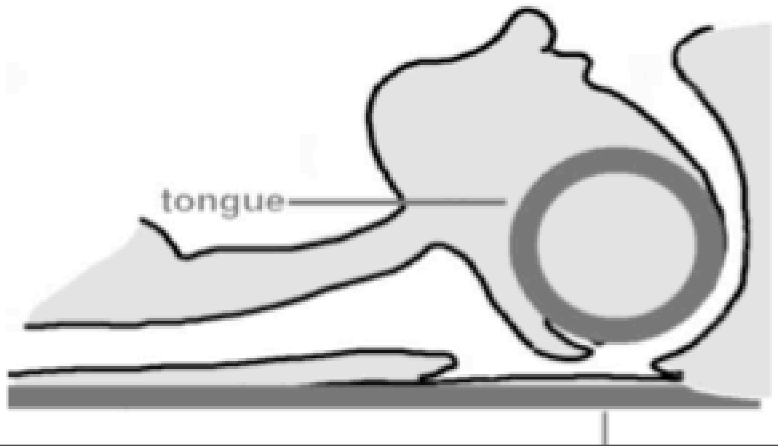
Xavier Pelorson & Annemie van Hirtum (ICP Grenoble)

Obstructive Sleep Apnea Syndrome

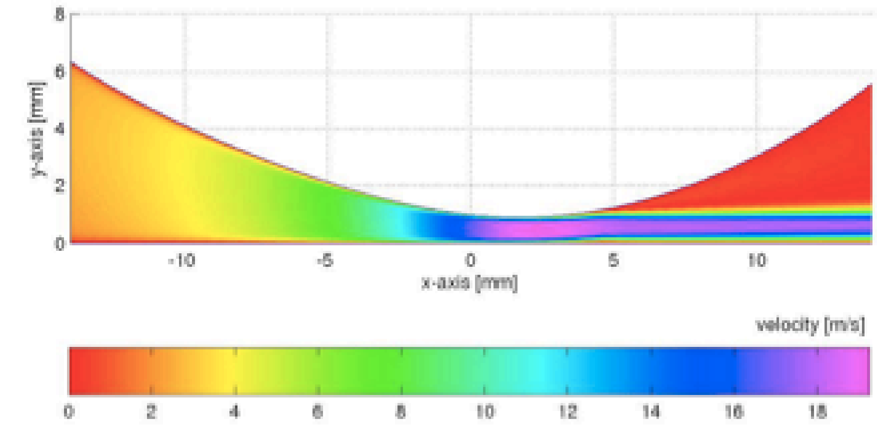


(a)

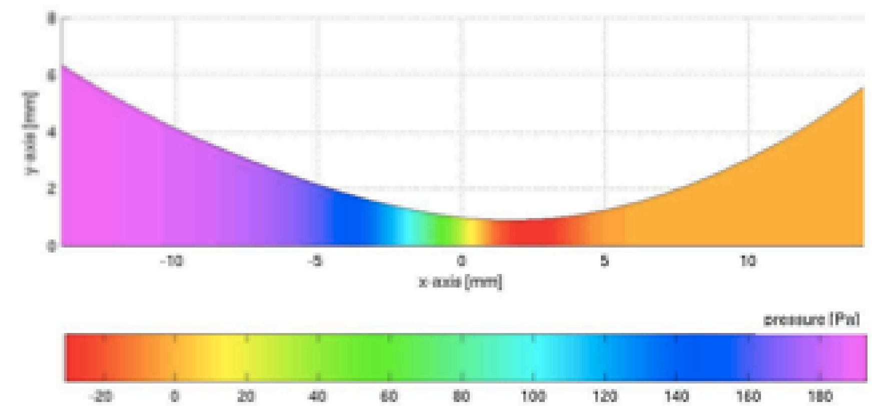

(b)



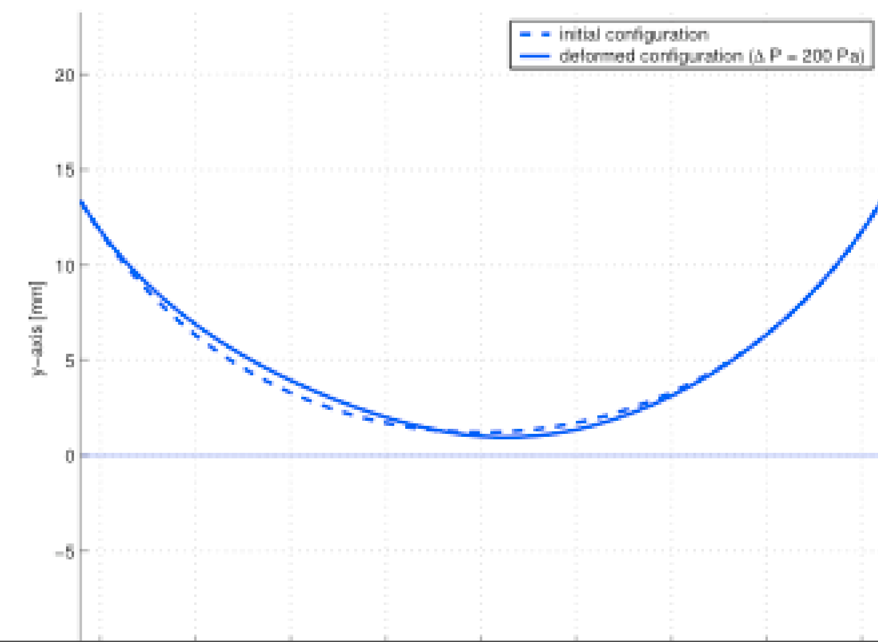
RNSP + Ansys

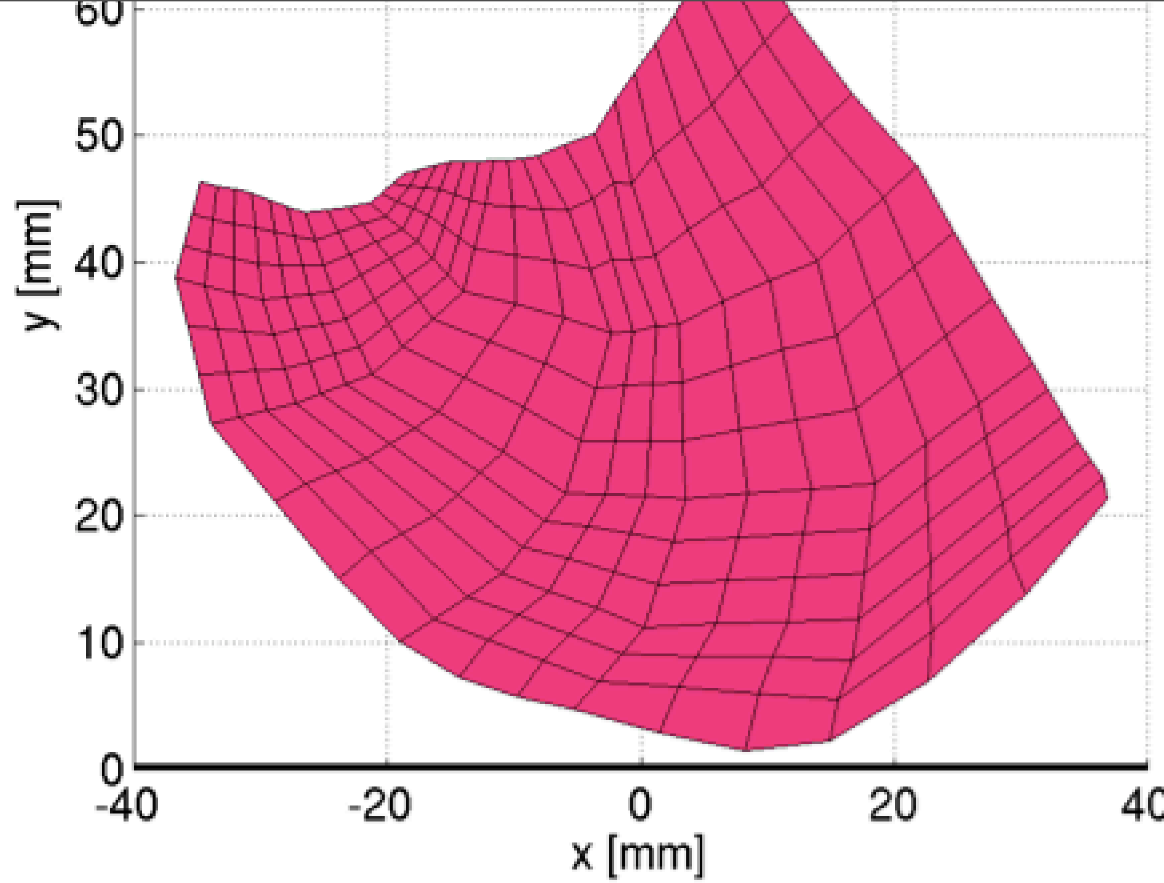


(a)

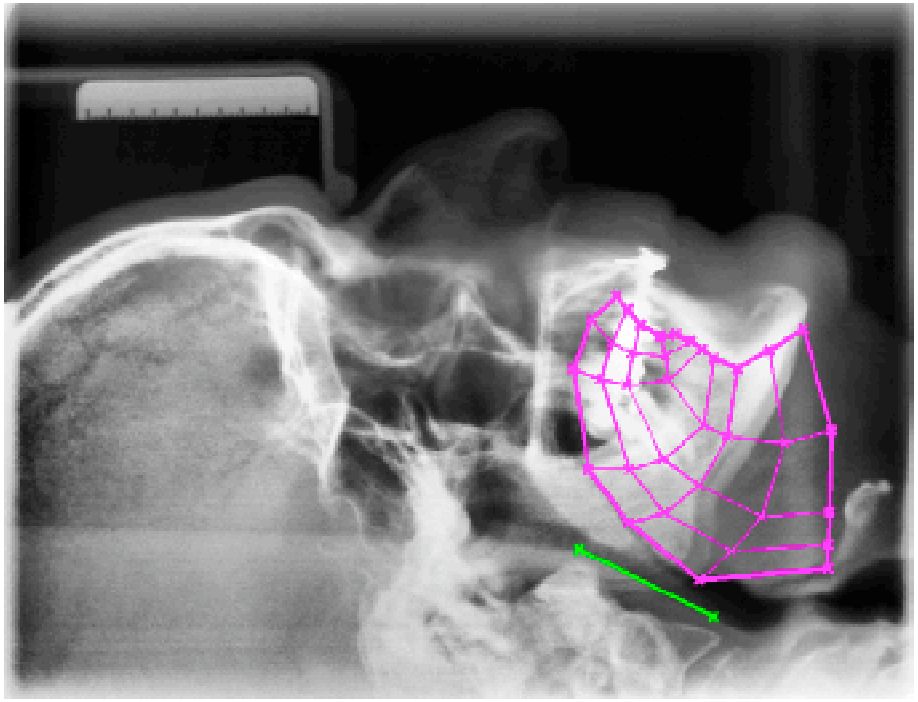


(b)

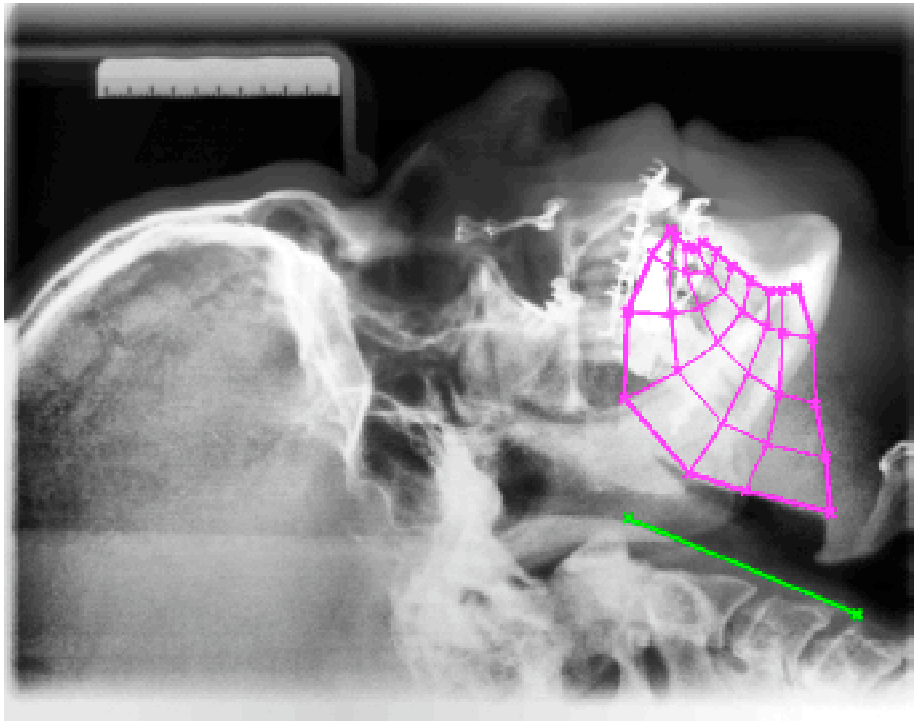




[redacted]



(PreOp)



(PostOp)

Conclusion

- starting from Navier Stokes
- set of simple equations RNSP
- set of more simple equations Integral
-

Conclusion

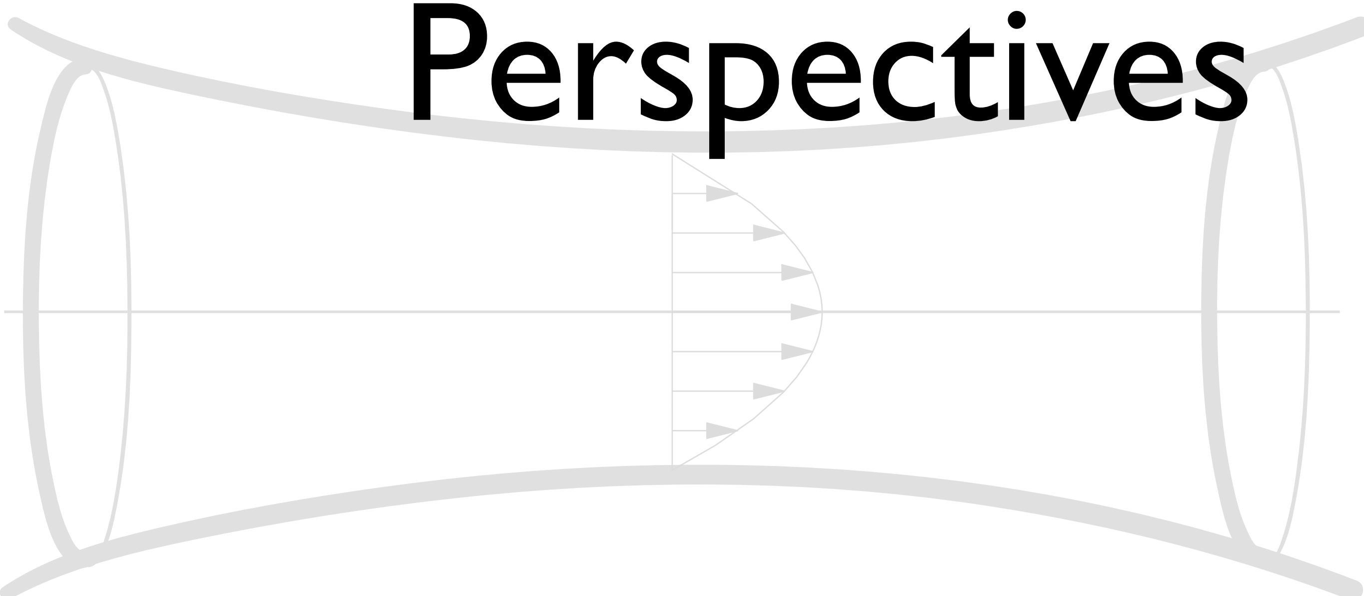
- starting from Navier Stokes
- set of simple equations RNSP
- set of more simple equations Integral
- Good agreement with full Navier Stokes

-

Conclusion

- starting from Navier Stokes
- set of simple equations RNSP
- set of more simple equations Integral
- Good agreement with full Navier Stokes
- “explain” the features of the flow
- boundary conditions for full NS
- real time simulation

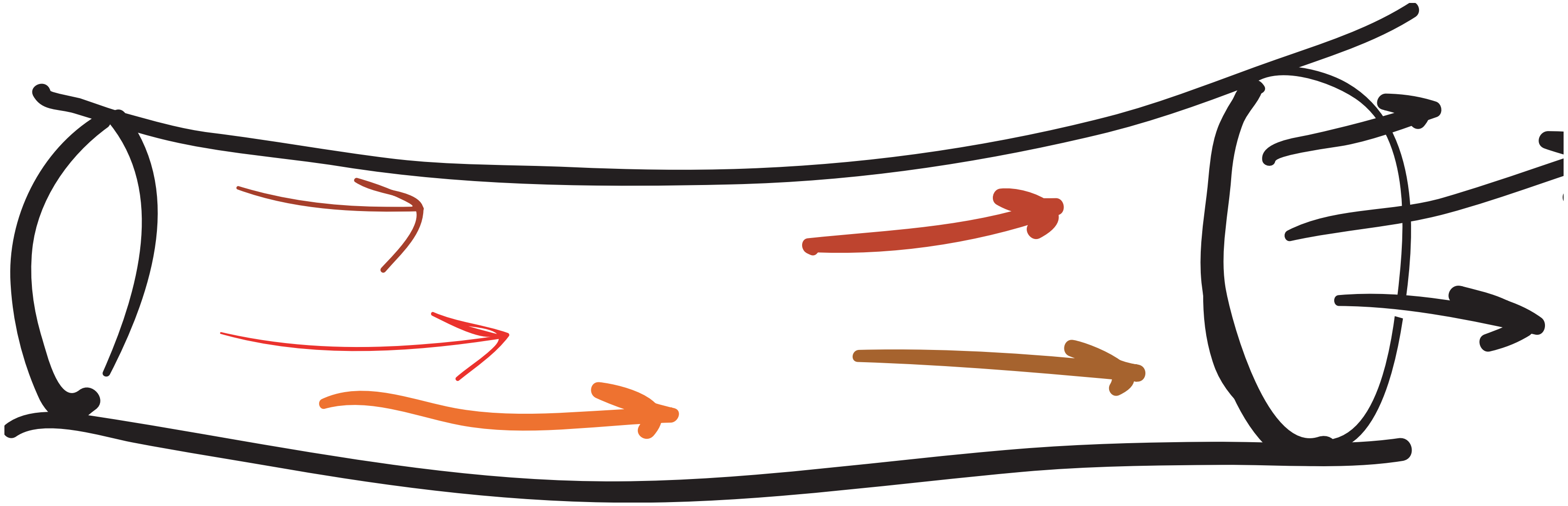
Perspectives



Aneurysm:

OSA:

- Use Acrobat Reader 7.05 to see animations
- Updated version may be found here.



- Use Acrobat Reader 7.05 to see animations
- Updated version may be found here.