Simple analytical models for flow over an erodable bed

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- applications in the Nature
- simple point of view
- simple models for the fluid
- simple models for the erodable bed
- stability analysis/ numerical computation

The coupled problem

- for a given soil $f(x,t)$
- ...

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- the flow erodes the soil.

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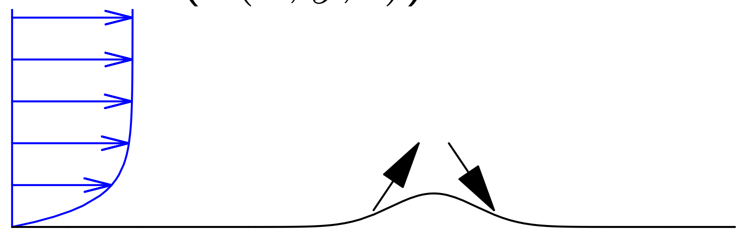

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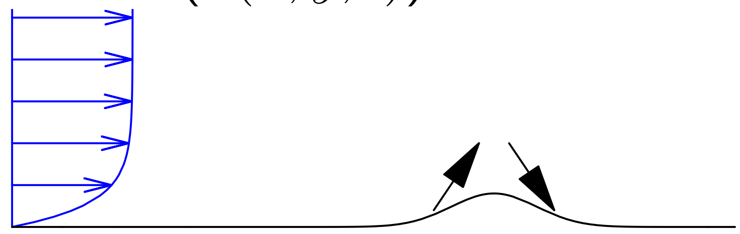

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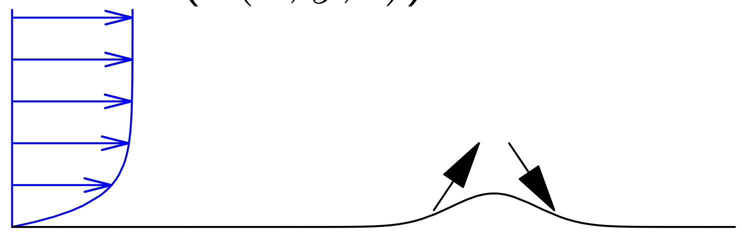

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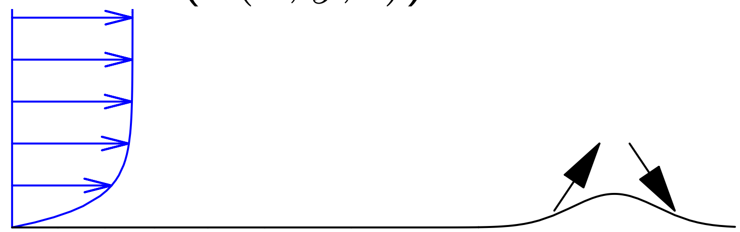

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we aim to present a simple description for the flow and obtain simple model equations to describe the interaction.

The fluid

Numerical resolution of Navier Stokes equations.
In real applications: viscosity changed... turbulence...

here we will present some simplifications:

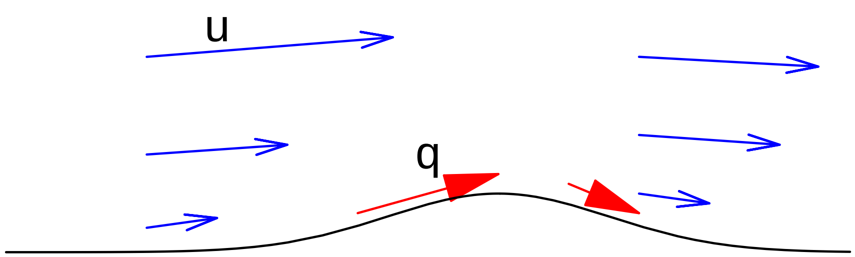
- Steady flow
- Linearized solutions
- ideal fluid at $Re = \infty$
- creeping flow at $Re = 0$
- asymptotic solution of N.S.: Triple Deck: laminar viscous theory at $Re = \infty$

The erodable bed

Mass conservation for the sediments:

$$\frac{\partial f}{\partial t} = -\frac{\partial q}{\partial x}.$$

Problem :
What is the relationship between q and the flow?
hint: the larger u the larger the erosion, the larger q
q seems to be proportional to the skin friction



The erodable bed: relations between q and u

$$\frac{\partial f}{\partial t} + \frac{\partial q}{\partial x} = 0$$

In the literature one founds Charru /Izumi & Parker / Yang / Blondeau

$$q = EH(\tau - \tau_s)((\tau - \tau_s)^a)$$

or with a slope correction for the threshold value:

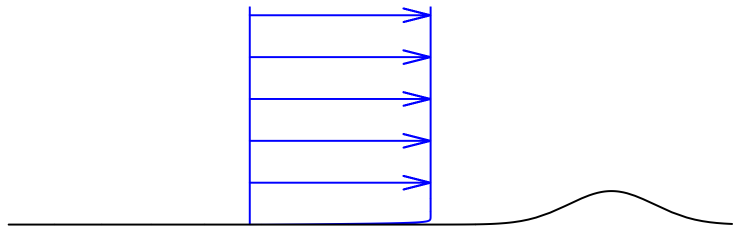
$$\tau_s + \lambda \frac{\partial f}{\partial x},$$

with H Heaviside function, a, E coefficients

Continuous theory

... leads to $q = \tau^3$

First example: Basic case, at $Re = \infty$



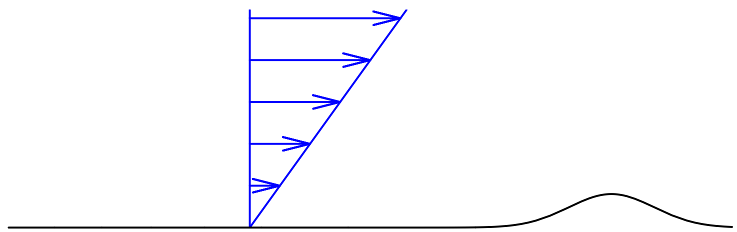
Uniform flow over a topography at large Reynolds number

Starting from an initial shape, the ideal fluid flow is computed (in the Small Perturbation Theory):

$$f(x,t) \text{ gives } u = (1 + \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{f'}{x-\xi} d\xi)$$

This is a very good approximation
But problems arise in the decelerated region (we will see this in the next section).

Second example: Basic case, at $Re = 0$

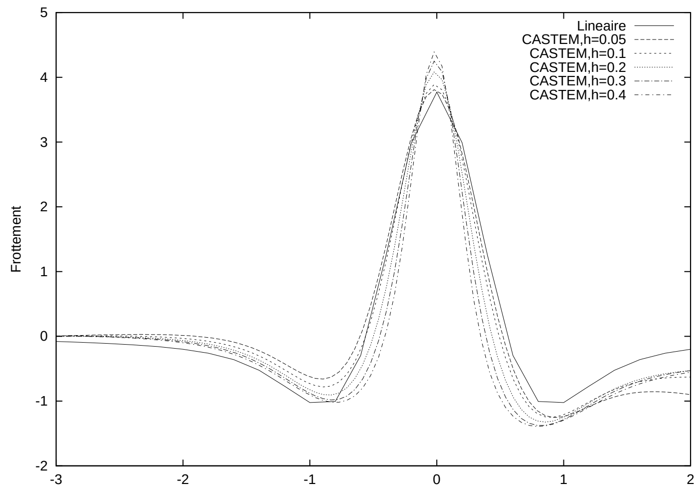


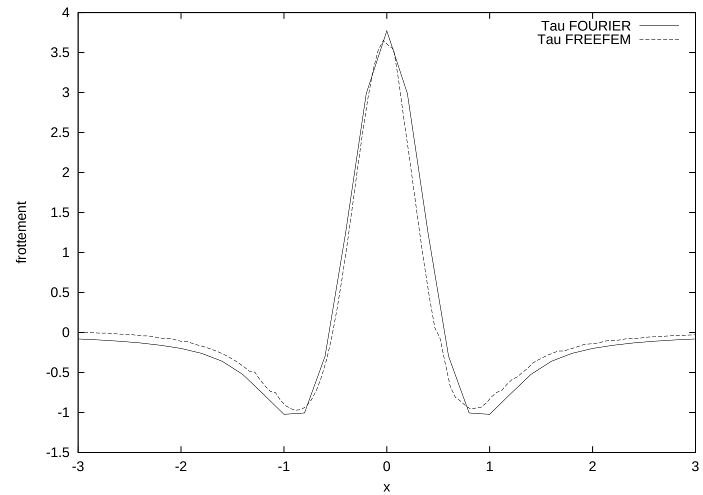
Shear flow over a topography $f(x,t)$ at small Reynolds number

Starting from an initial shape, the creeping flow is computed (in the Small Perturbation Theory), we obtain after some algebra:

$$\tau = 1 + \frac{2}{\pi}\int_{-\infty}^{\infty}\frac{f'}{x-\xi}d\xi$$

perturbation of a shear flow $Re = 0$





L: flow over a gaussian bump, comparisons linear theory/ computations
perturbation of skin friction computed with CASTEM $\frac{1}{h_0}\frac{\partial \overline{u}}{\partial \overline{y}}$ for $0.05 < h_0 < 0.4$ (bump size) and $Re = 1$
R: perturbation of skin friction computed with FreeFem.

Linking q and u

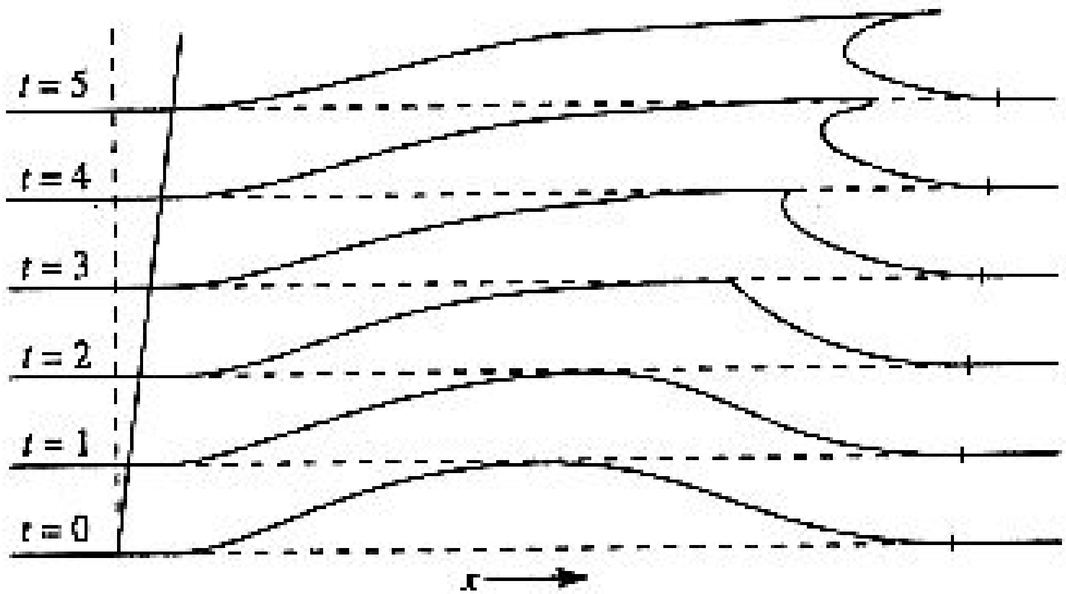
assuming that q is proportional to $u - 1$ or q proportional to $\tau - 1$ without threshold this gives the same relation in the two cases $(2!)$:

$$\frac{\partial f}{\partial t} = -\frac{1}{\pi}\frac{\partial}{\partial x}\int \frac{f'}{x - \xi} d\xi.$$

we recognize the linear Benjamin -Ono equation.

Supposed Evolution

The ideal fluid theory has been introduced by Exner.

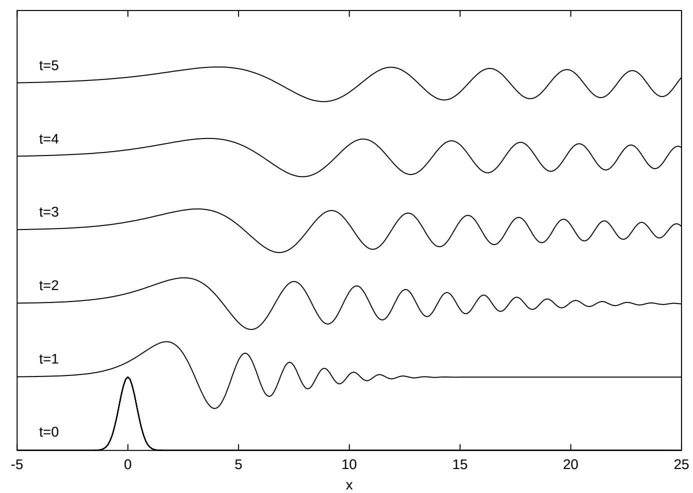


Issued from Yang (1995) reproduced from Exner (1925?).
"wave" inspiration in the dune evolution

Computed Evolution

Numerical resolution: finite differences, explicit
Tested on complete Benjamin - Ono: RHS$+\ 4f\partial f/\partial x$ gives the soliton $1/(1+x^2)$

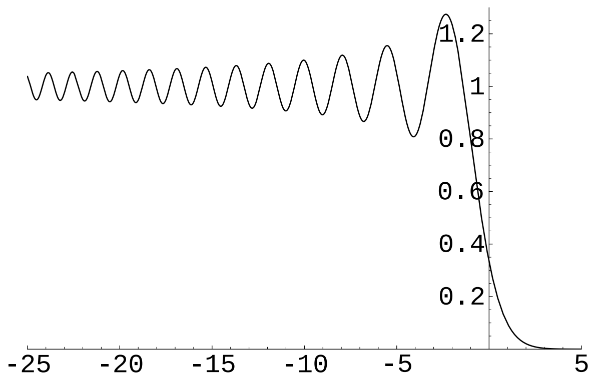
But here we observe the dispersion of the bump...

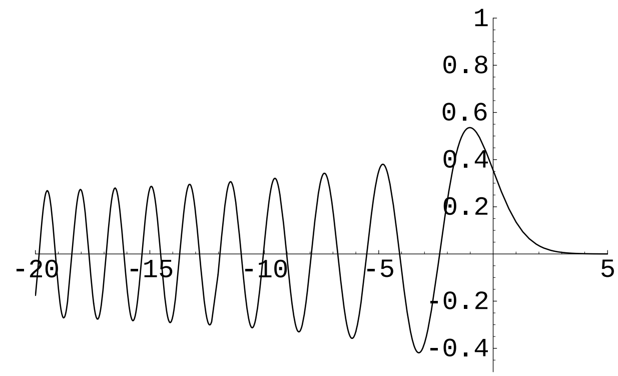


animation

Remark: linear KDV equation

The linear KDV equation reads $\frac{\partial f}{\partial t} = \frac{\partial^3 f}{\partial x^3}$, with selfsimilar solutions, $\eta = x t^{-1/3}$:





"Mascaret" solution: $f(x,t) = \int_{3^{-1/3}\eta}^{\infty} Ai(\xi) d\xi$; Airy solution: $f(x,t) = t^{-1/3} Ai(\frac{\eta}{3^{1/3}})$.

animation

asymptotic solution of L.B.O.

L.B.O. $$\frac{\partial f}{\partial t} = -\frac{1}{\pi}\frac{\partial}{\partial x}\int \frac{f'}{x-\xi}d\xi.$$

Selfsimilar variable $\eta = xt^{-1/2}$, self similar solution $f(x,t) = t^{-1/2}\phi(xt^{-1/2})$.

In the Fourier space $exp(-ikx)$ gives, in the RHS, $-i|k|kexp(-ikx)$, so:

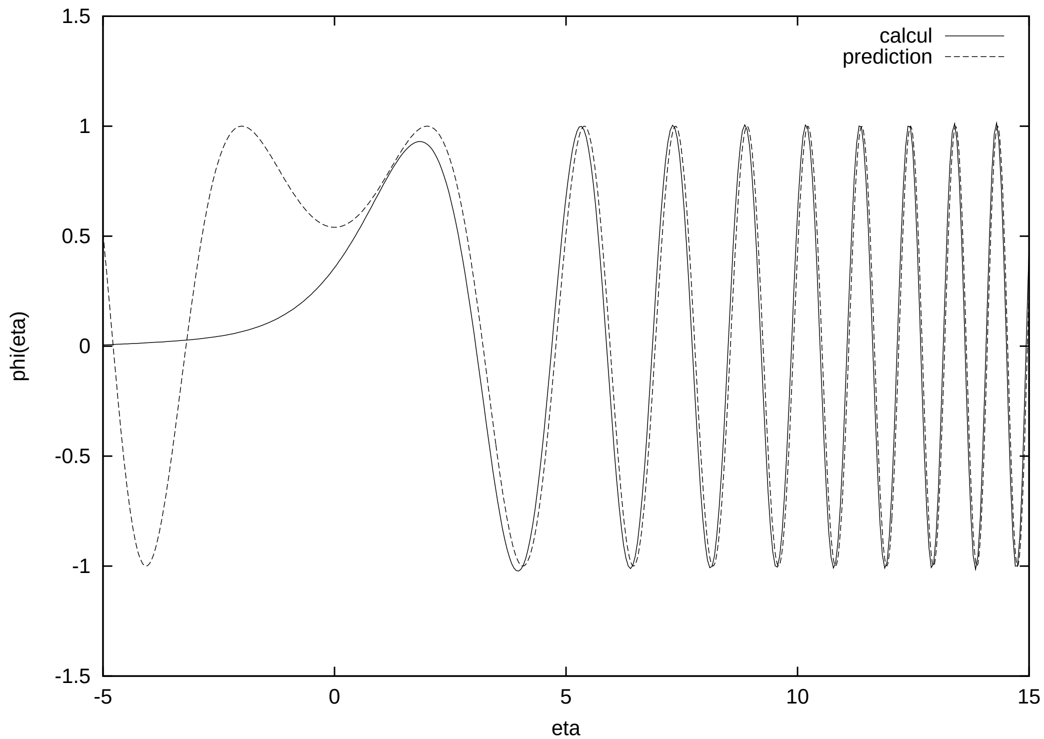
$$-\frac{1}{\pi}\frac{\partial}{\partial x}\int \frac{f'}{x-\xi}d\xi \simeq i\frac{\partial^2 f}{\partial x^2}$$

The self similar problem is approximated by:

$$\frac{-1}{2}(\phi(\eta) + \eta\phi'(\eta)) \simeq i\phi''(\eta).$$

whose exact solution is $\phi(\eta) = exp(i(\eta/2)^2)$

asymptotic solution of L.B.O.



Plot of the numerical solution $t^{1/2}f(x,t)$ function of $xt^{-1/2}$
the exact solution of the approximated problem $cos(1+(\eta/2)^2)$.

Conclusions for those two special cases L.B.O.

When $Re = 0$ or $Re = \infty$ (in a simple ideal flow theory), the topography is stable, there is a dispersive wave

There is an approximate solution (non linearities...?)

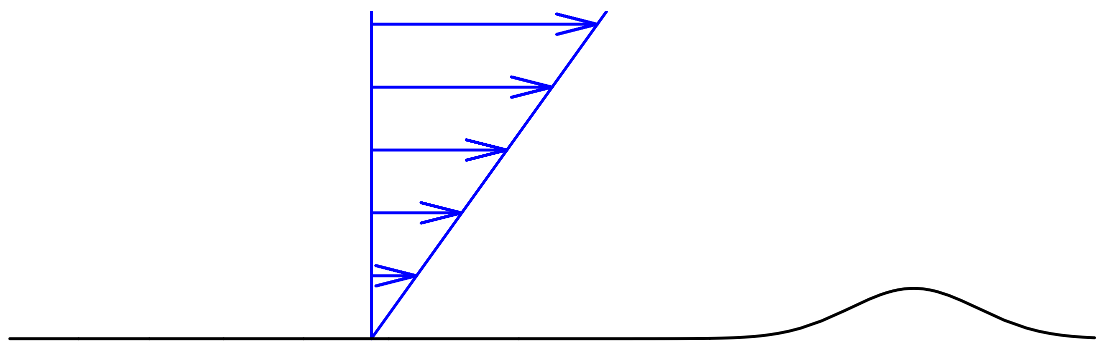
Stability arises because of the fact that skin friction is "in phase" with the bump crest

We have to introduce viscous effects, but small

It will change the the fluid response: skin friction will be "out of phase" (in advance) with the bump crest

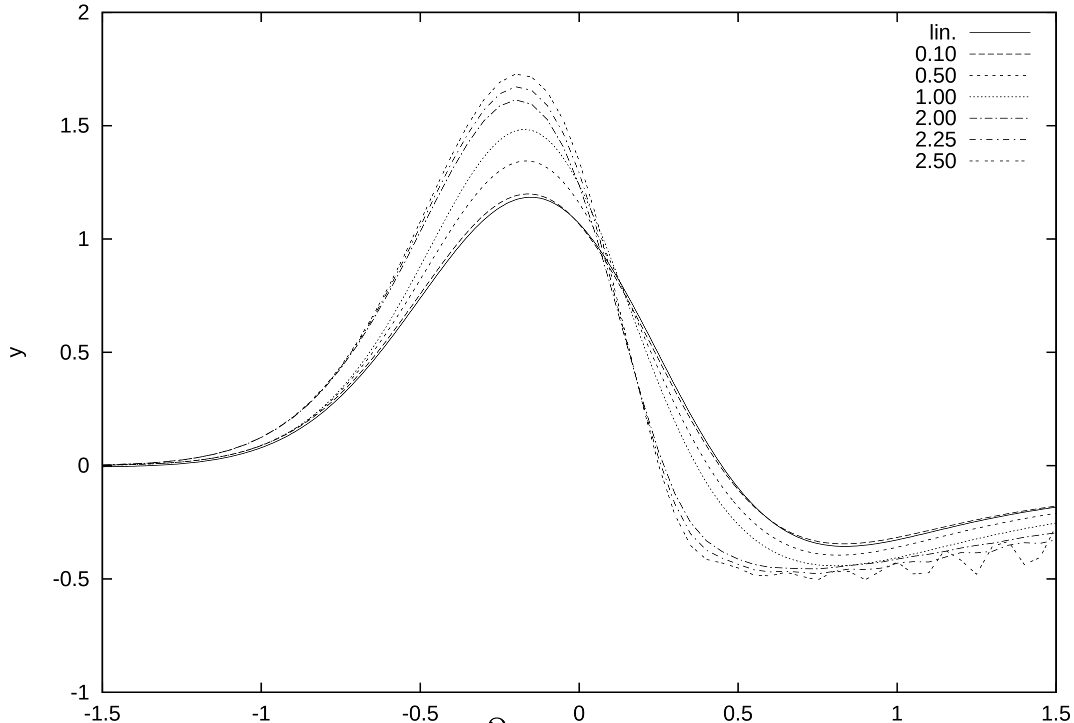
Asymptotic solution of the flow over a bump; double deck theory

We guess that viscous effects are important near the wall
Perturbation of a shear flow



$$\frac{\partial u}{\partial y}|_0 = 1 + \alpha FT^{-1}[(3Ai(0))(-ik)^{1/3}FT[f]] + O(\alpha^2).$$

Asymptotic solution of the flow over a bump; Linear/ Non Linear double deck theory



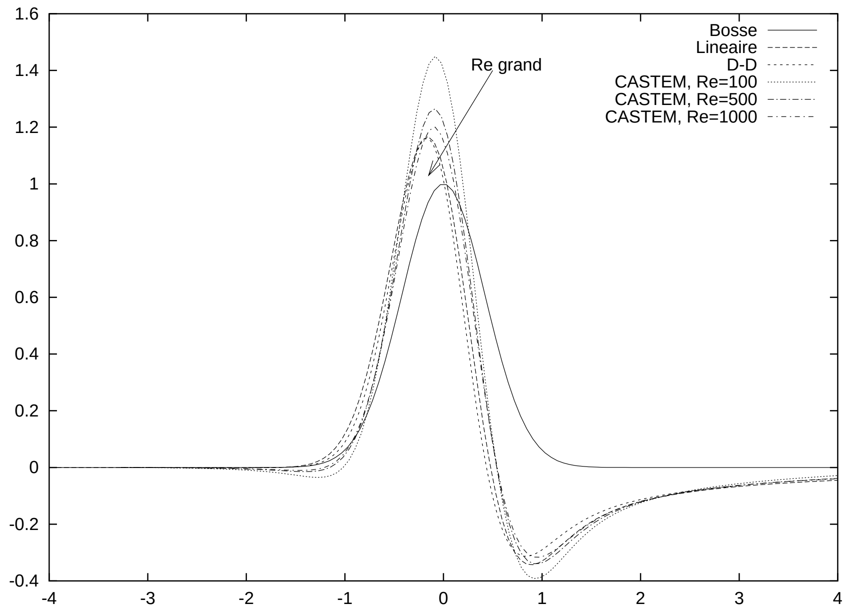
for the bump $\alpha e^{-\pi x^2}$
with $\alpha = 0.10, \alpha = 0.5, \alpha = 1.0,$
$\alpha = 2, \alpha = 2.25, \alpha = 2.50.$
The plain curve ("lin.") is the linear prediction , other curves come from the non linear numerical solution.

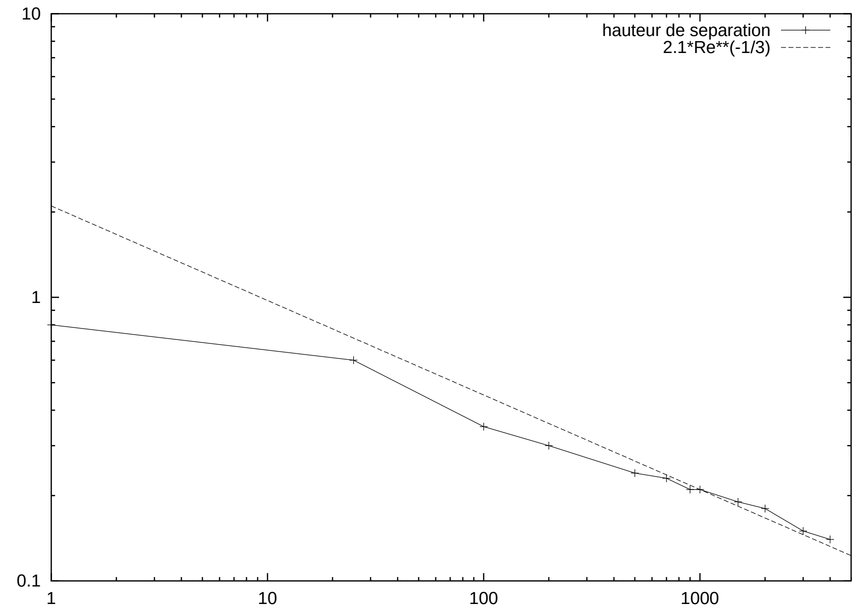
Notice the numerical oscillations in the case of separated

flow (separation is for $\alpha > 2.1$)

Reduced wall shear $(\frac{\partial u}{\partial y}_0 - 1)/\alpha$
function of x

Comparison with Navier Stokes

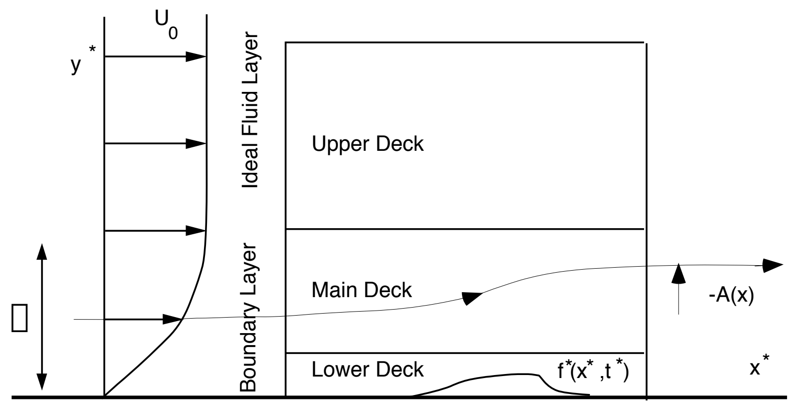




good!

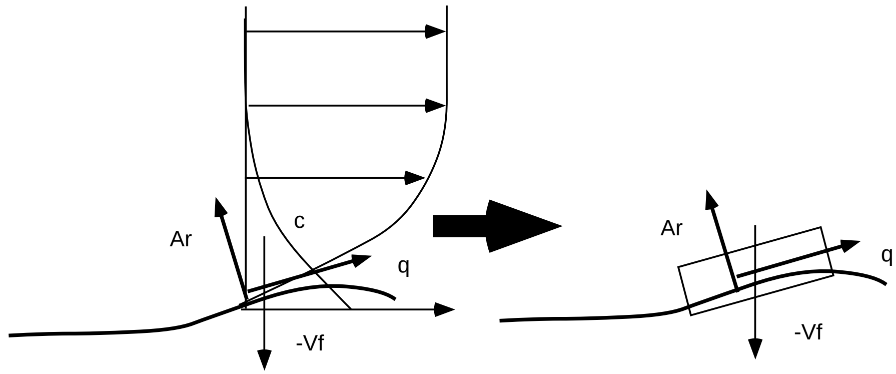
conclusion: Perturbation of shear flow is in advance compared to the bump crest.

Other asymptotic solution of the flow over a bump; triple deck theory



$FT[\tau] = \frac{(-ik)^{2/3}}{Ai'(0)} Ai(0) \frac{FT[f]}{\beta^* - \beta_{pf}}$, with $\beta^* = (3Ai'(0))^{-1}(-ik)^{1/3}$

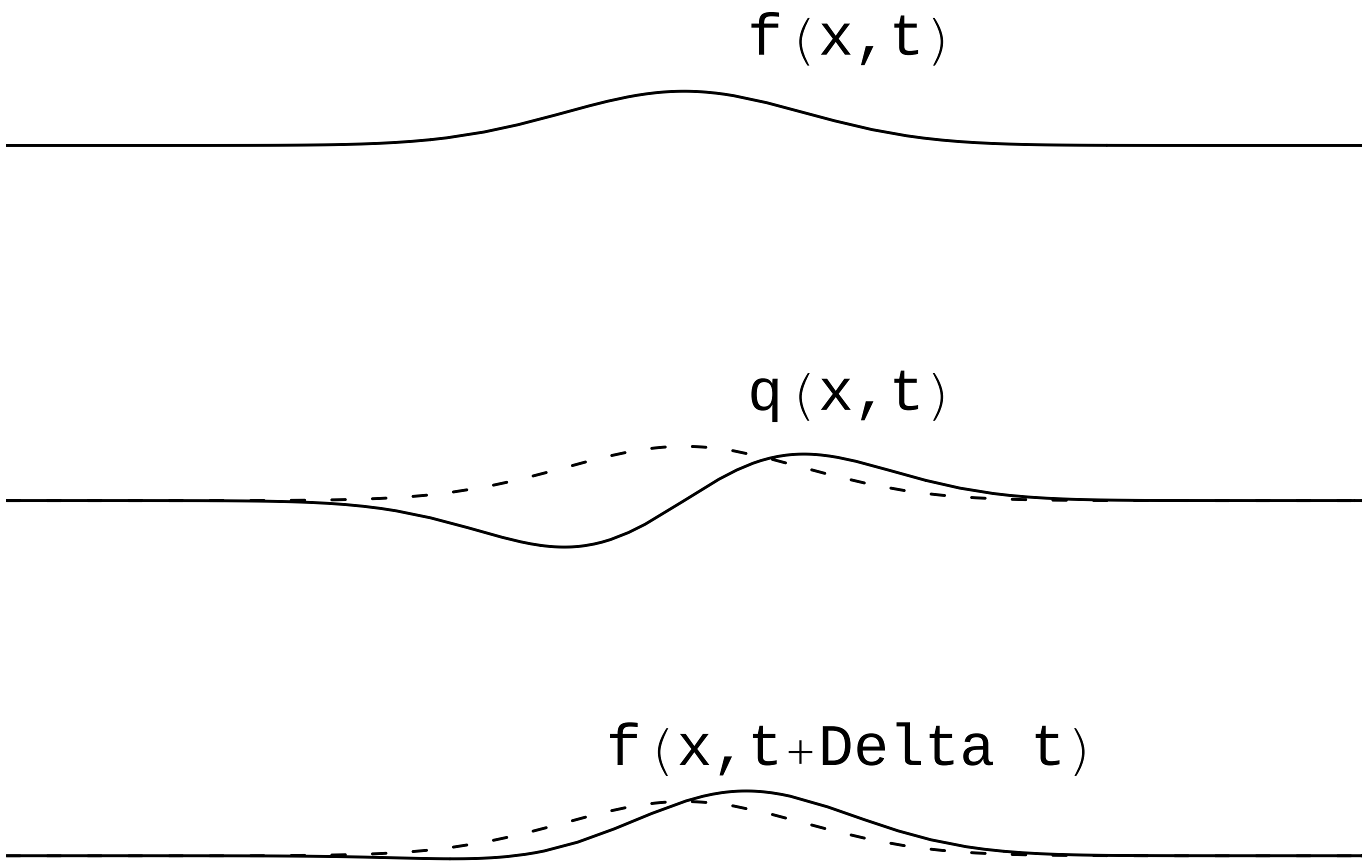
Simplification of mass transport



$$\frac{\partial}{\partial x} q + V q = \beta(H(\tau - \tau_s - \lambda \frac{\partial f}{\partial x})(\tau - \tau_s - \lambda \frac{\partial f}{\partial x}))^{\gamma}.$$

- total flux of convected sediments q (left figure).
- threshold effect τ_s
- "slope" effect $\lambda \frac{\partial f}{\partial x}$
- H: Heaviside function, γ, V, β...

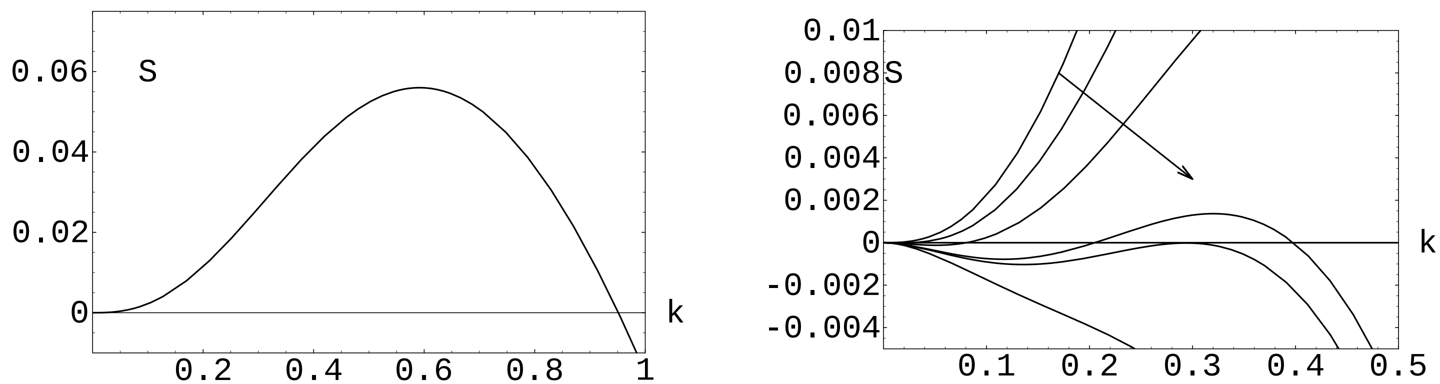
Simple instability mechanism



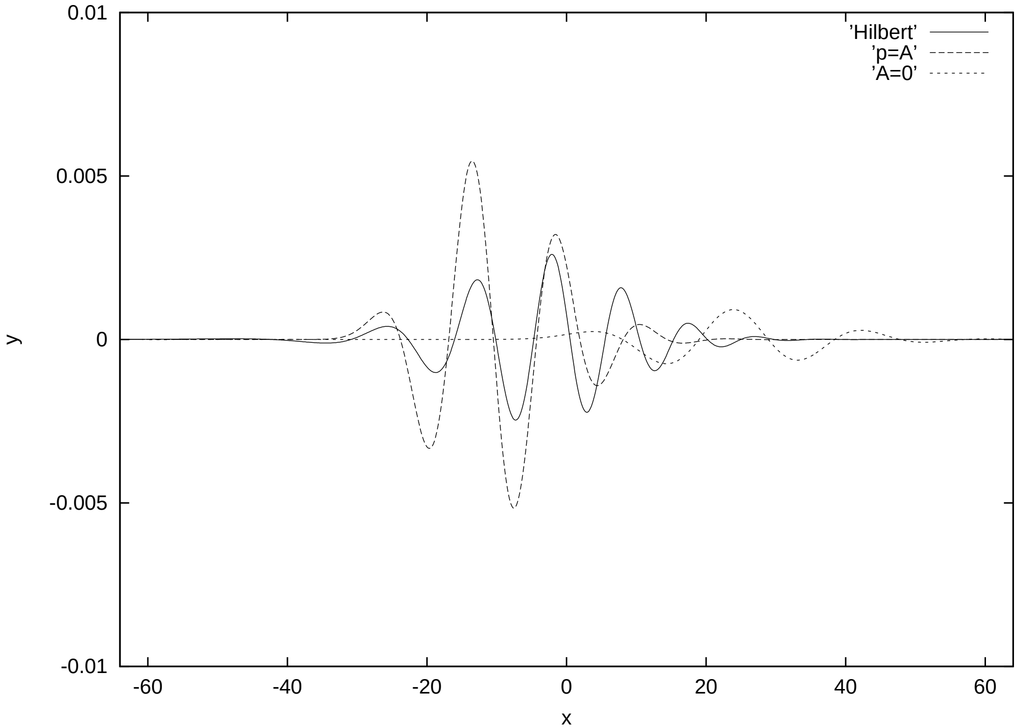
Stability analysis

- If pure shear flow $\lambda = 0$, $V = 0$, $Re(\sigma) >$ for any k
  $\lambda \neq 0$ or $V \neq 0$ stabilises high frequencies

- Infinite depth case (Hilbert case). The real part of σ for $\beta = V = \gamma = 1$ as function of the wave length k:
  - on the left figure $\lambda = 0$: there is no slope effect
  - on the right figure, we focus on the small k which are amplified when $\lambda = 0$, but are damped for $\lambda > 0$ (following the arrow, from up to down $\lambda = 0$, $\lambda = 0.1$, $\lambda = 0.2$, $\lambda = 0.3$, $\lambda = 0.316$ and $\lambda = 0.4$).

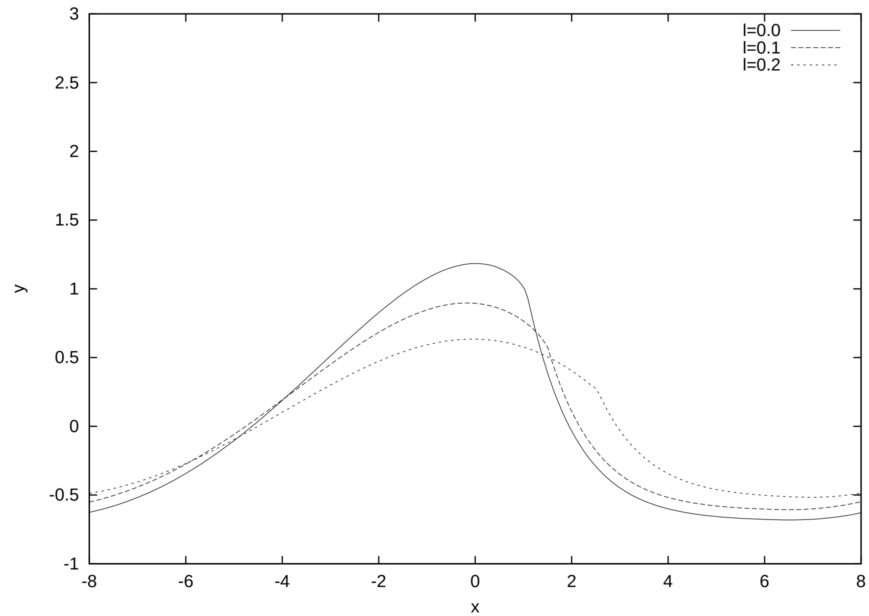


Time Evolution



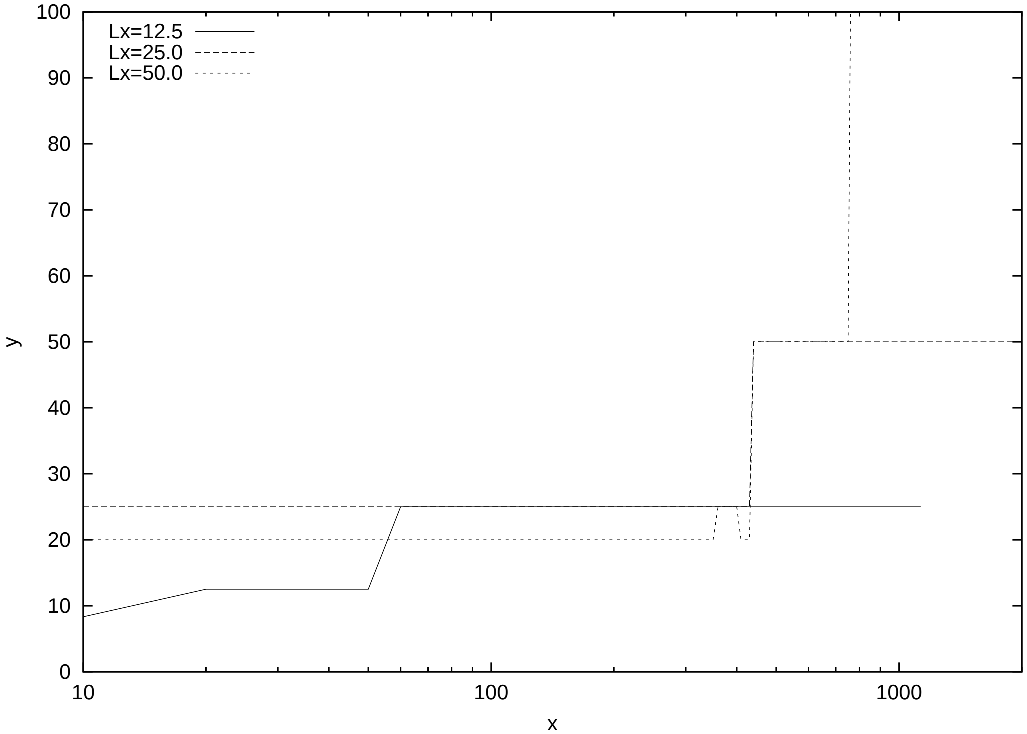
The topography at time $t = 100$ *for the three unstable régimes (*$\beta = V = \gamma = 1$*,* $\lambda = 0$*,* $L_x = 64$ $\tau_s = -0.1$*). At time* $t = 0$*, a random noise of level* 0.001 *was introduced. The spatial frequency* k_M *giving the larger* $Re(\sigma)$ *in the band* $0 < k_M < k_m$ *has been selected.*

Slope effect: influence of λ



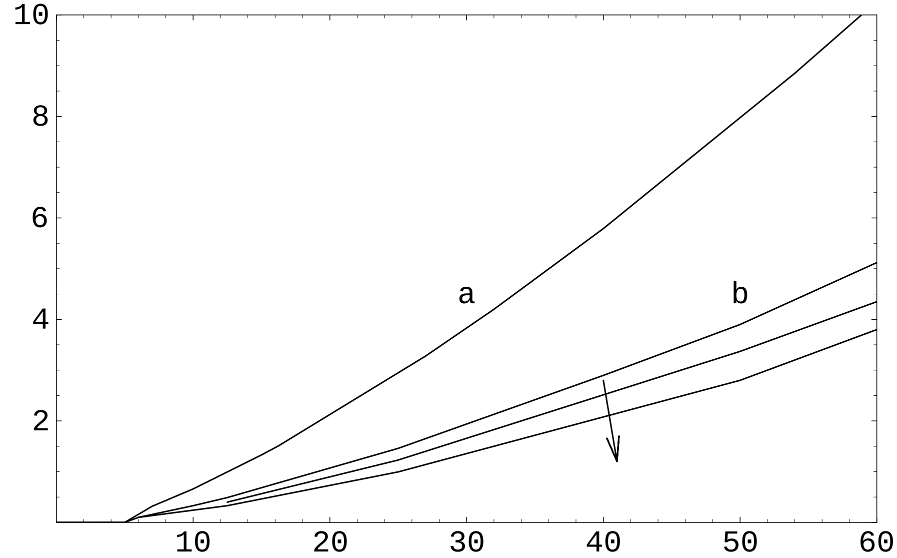
Bump shape $t = 500$, (4 bumps coexist with $\beta = 1$, $\gamma = 1$, $V = 1$, $\tau_s = -0.05$), $\lambda = 0$, $\lambda = 0.1$ and $\lambda = 0.2$ (the curves are shifted to place the maximum at the origin)

Coarsening process, pure shear flow case



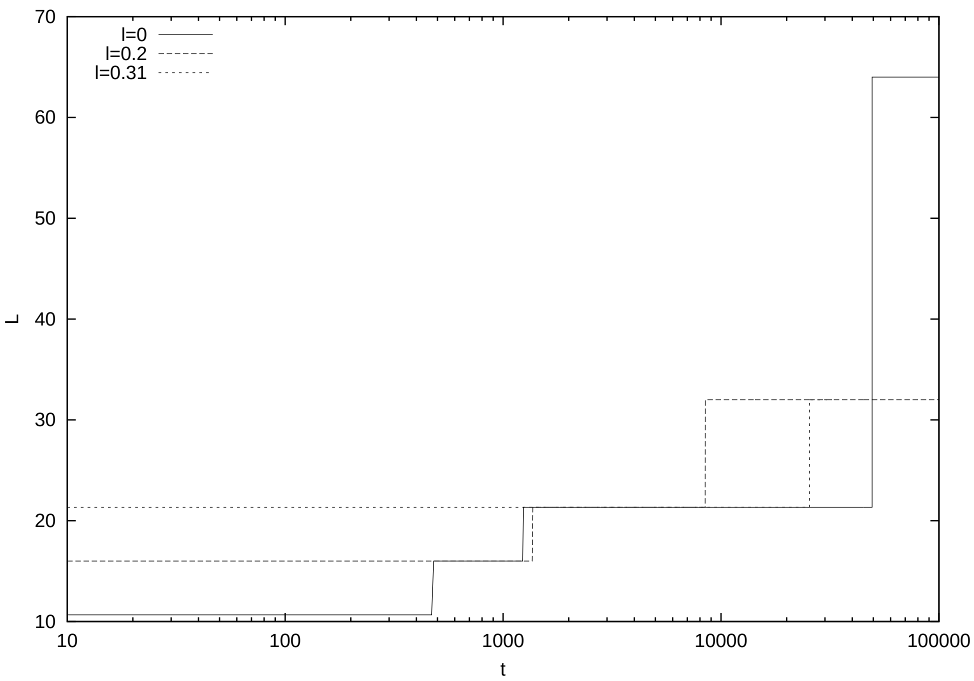
The wave length $2\pi/k$ of the maximum of the bump spectrum versus time, corresponding mostly to the number of bumps present in the domain, is plotted as function of time (log scale), here in the case $A = 0$. As time increases, there is less and less bumps present in the domain, finally a single bump fills it: $2\pi/k_{final} = 2L_x$. Here, $L_x = 12.5$, 25 and 50. The long wave are unstable in such a way that the final length of the bump is the size of the computational domain.

Coarsening process, pure shear flow case



The maximum of the final bump height h_{max} plotted as a function of half the domain size L_x in the case $A = 0$. The case $\tau_s = -0.1$, $V = 1$, $\lambda = 0$ is the upper curve. The lower curves correspond to $\tau_s = -0.05$, $V = 1$, the arrow is directed to the increasing values of λ ($\lambda = 0,\ 0.1$ and 0.2). The subcritical case gives qualitatively the same results.

Coarsening process,Hilbert case



Examples of long time evolution of $2\pi/k$ the wave length value maximizing the bump spectrum (corresponding mostly to the number of bumps present in the domain). This is an infinite depth case for a domain of length $2L_x$. If $\lambda = 0$, there is finally only one bump of size $2L_x$ (the largest possible). If $\lambda < 0.316$, two bumps (of size L_x) are present, the larger are damped. If λ is increased, there is no dune anymore as predicted by the linearized theory. Here $V = \beta = 1$, $L_x = 32$, $\tau_s = -0.25$. Notice that several bumps may live during a very long time: here in the case $\lambda = 0.31$, during a very long time $(10 < t < 25000)$ three bumps are present.

Conclusion

avalanches…

Equations sans dimension

$$q = \frac{R^2}{2} \qquad E = \frac{R^3}{3}$$

$$h(x,t) = Z(x,t) + R(x,t)$$

$$\frac{\partial h}{\partial t} + R\frac{\partial R}{\partial x} = 0$$

$$\frac{\partial R}{\partial t} + R\frac{\partial R}{\partial x} = -\frac{1}{1 + (\frac{\partial Z}{\partial x})^2}(\frac{\partial Z}{\partial x} + \mu_I) + \nu\frac{\partial^2 R}{\partial x^2}$$

Cas BRdG 98

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$\underline{ActeI}$

$\underline{ActeIetII}$

$\underline{ActeIetIIetIII}$

Cas DAD 99 modifié

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relation $\mu_I(R, Z', ...)$

Exemples

- $\underline{sans \quad frein}$:
  si $|\frac{\partial Z}{\partial x}| > \mu_s$ alors $\mu_I = \mu_s$

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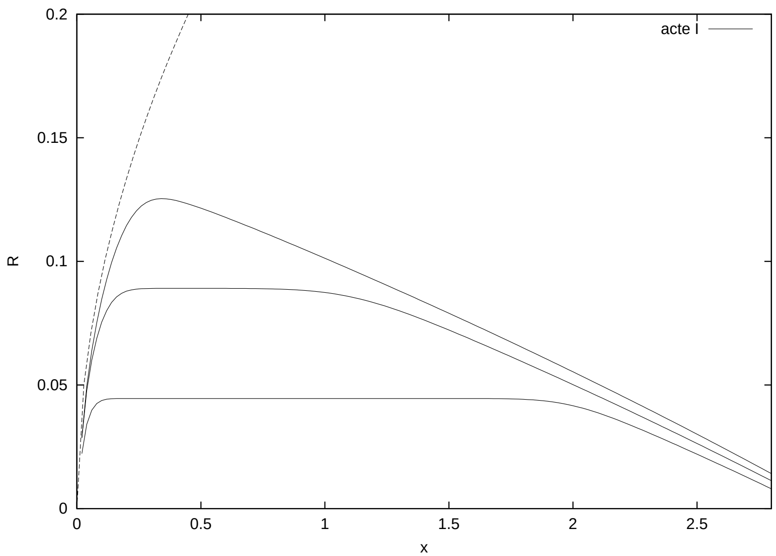
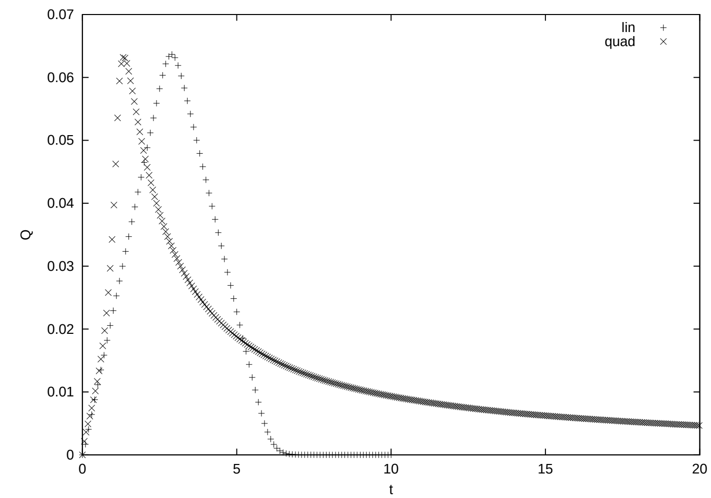
```
un cas lin\'eaire
dt=.0001
tmax=10
dx=0.01
nx=1000
nu=0.05
mu=0.3
modelmu=4
tas=-1
theta=0.35
K=0
Nlin=0
v0=1



dt=.001
```

```
tmax=350
dx=0.01
nx=1000
nu=0.05
mu=0.3
modelmu=4
tas=-1
theta=0.35
K=0
Nlin=1
v0=0
```

Non Linear velocity: The Acts



acte I
R
x
0
0.05
0.1
0.15
0.2
0.5
1
1.5
2
2.5



lin
quad
Q
t
0
0.01
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0.03
0.04
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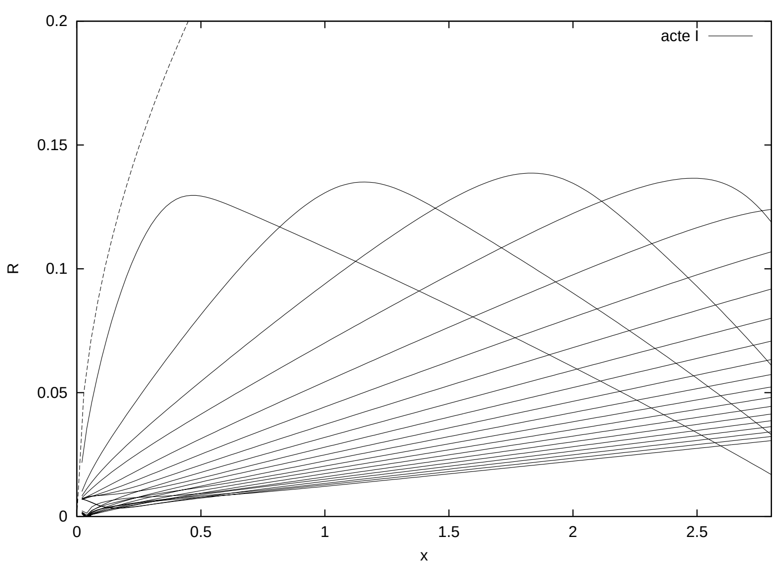
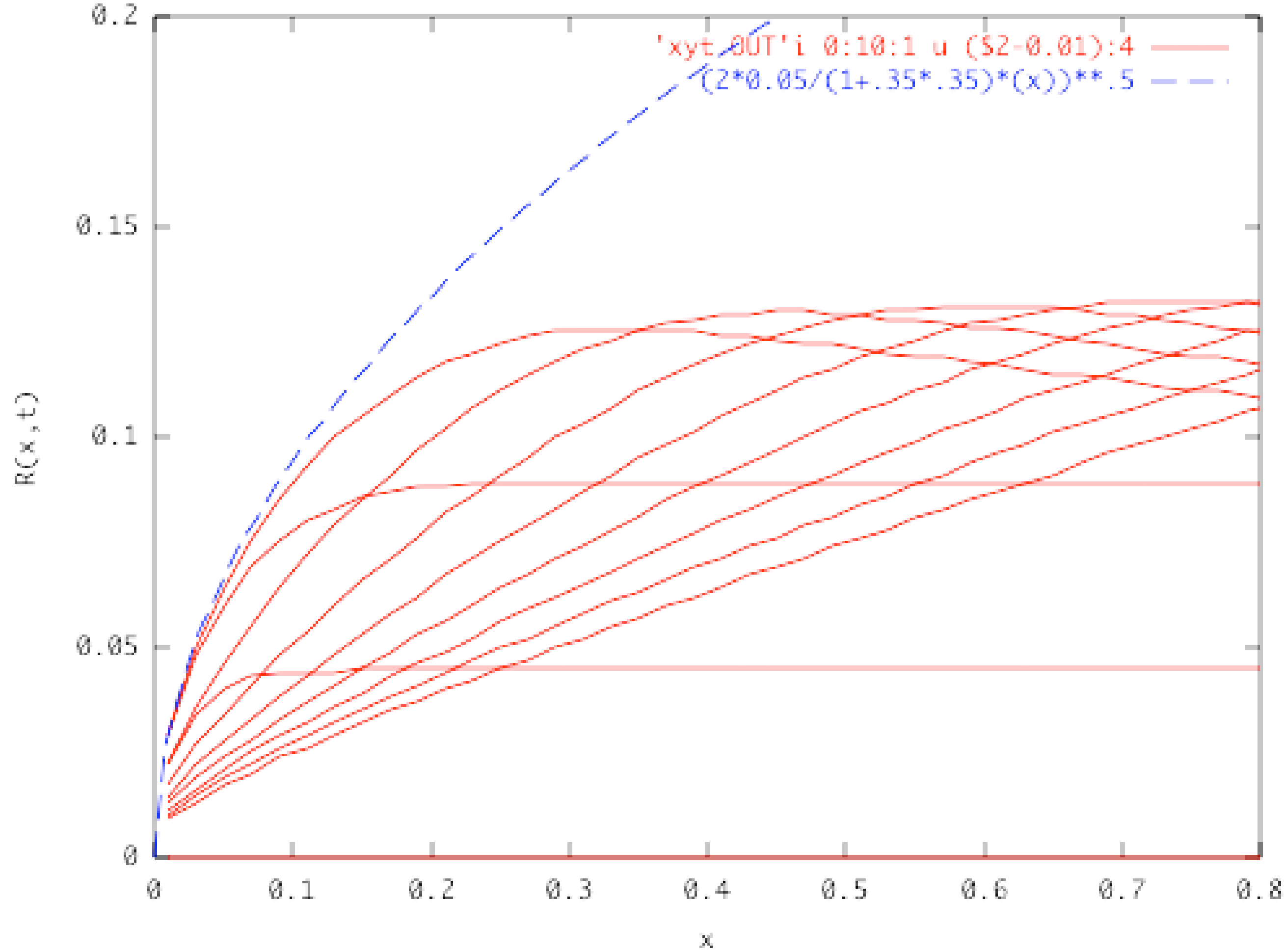
0.07
5
10
15
20

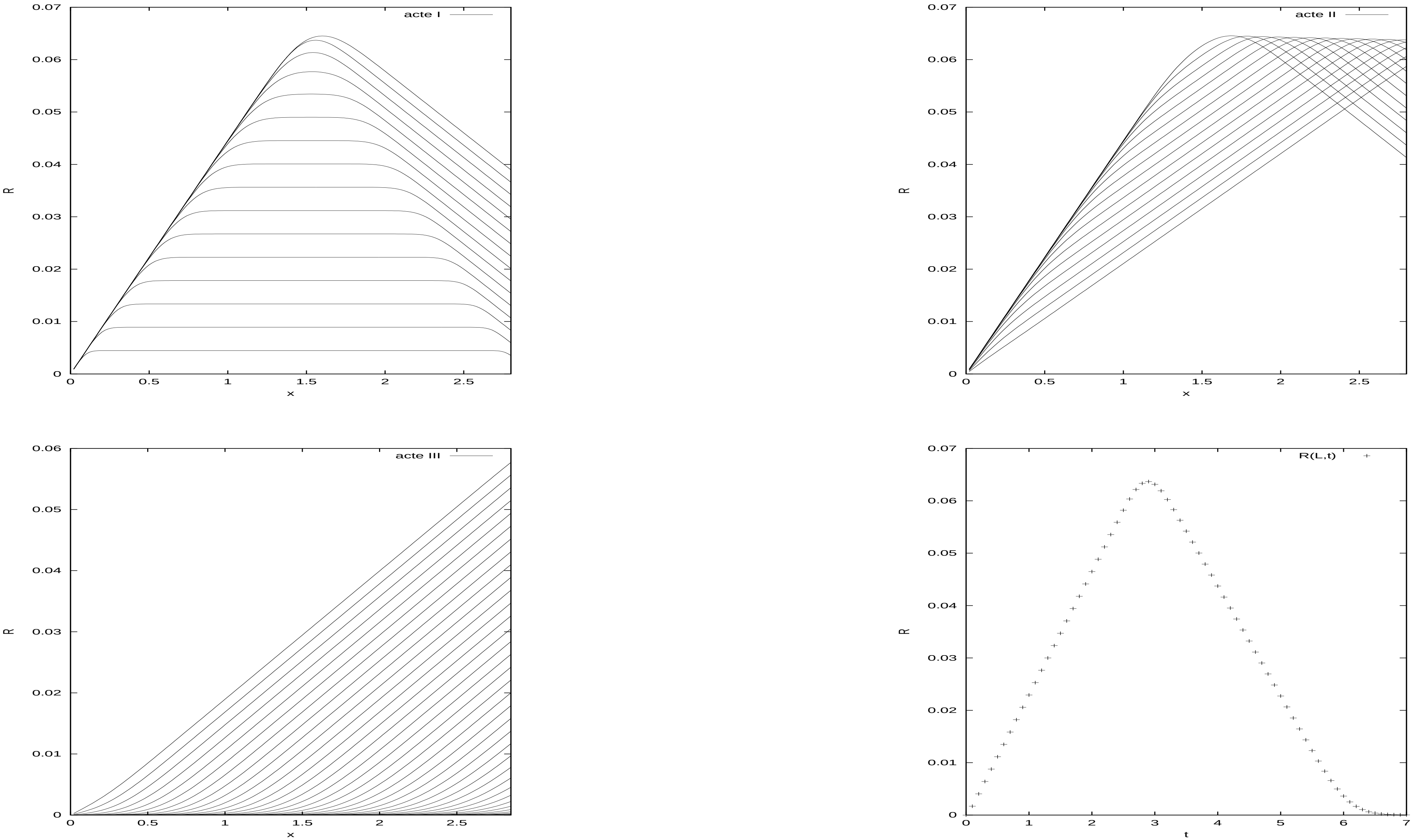


acte I
R
x
0
0.05
0.1
0.15
0.2
0.5
1
1.5
2
2.5



'xyt.OUT'i 0:10:1 u ($2-0.01):4
(2*0.05/(1+.35*.35)*(x))**.5
R(x,t)
x
0
0.05
0.1
0.15
0.2
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8

Linear velocity: The Acts


acte I
acte II
acte III
R(L,t)
R
x
t
0.07
0.06
0.05
0.04
0.03
0.02
0.01
0
0.5
1
1.5
2
2.5
3
4
5
6
7

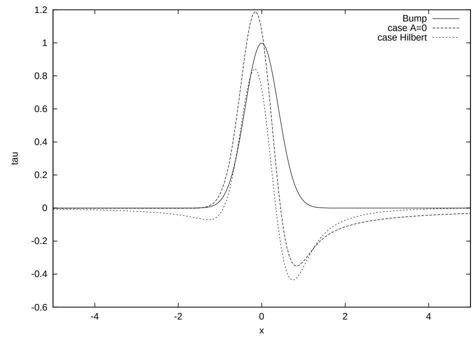


Figure 1: *The linear solution (in the Triple Deck scales) for the perturbation of the wall shear function of* x*, in the* $A = 0$ *case, and in the Hilbert case* $p = -\pi^{-1}\int(x-\xi)^{-1}(-A')d\xi$*. The bump perturbation is here* $e^{-\pi x^2}$*. The case* $A = 0$ *leads to no upstream influence, the Hilbert case leads to a small upstream influence: the skin friction anticipates the bump. The skin friction is extreme before the maximum of the bump. Skin friction s larger in the wind side than in the lee side.*

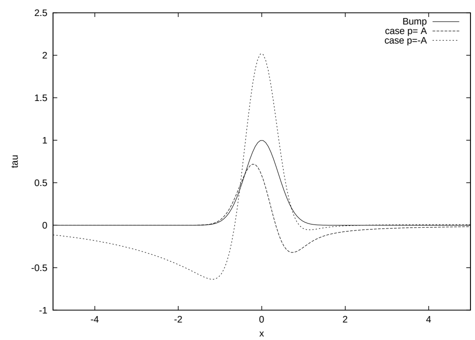


Figure 2: *The linear solution (in the Triple Deck scales) for the perturbation of the wall shear function of* x*, in the* $p = -A$ *case, and in the* $p = A$*. The bump perturbation is here* $e^{-\pi x^2}$*. The case* $p = A$ *(subcritical) leads to no upstream influence, the case* $p = -A$ *(supercritical) leads to a strong upstream influence: the skin friction anticipates the bump. The skin friction is extreme before the maximum of the bump only in the fluvial case.*

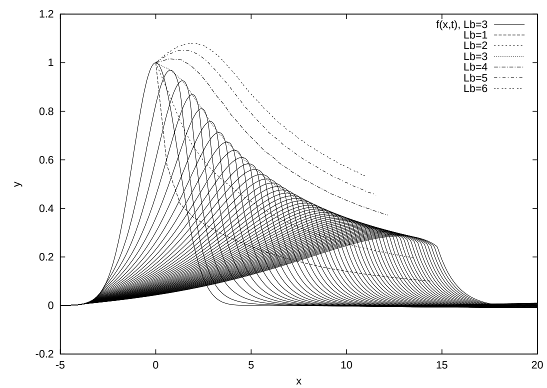


Figure 3: *Influence of the initial width L_b of a bump $exp(-\pi(x/L_b)^2)$, the maximum of the bump is plotted for $L_b = 1, 2, 3, 4$ and 5 for $t < 100$; $f(x, t)$ is plotted as well (for $t = 0, 2, 4, 6, \ldots, 100$ with $L_b = 3$). The larger the bump, the smaller its velocity; $\beta = 1$, $\gamma = 1$, $V = 1$, $\lambda = 0$ and $\tau_s = 0$.*

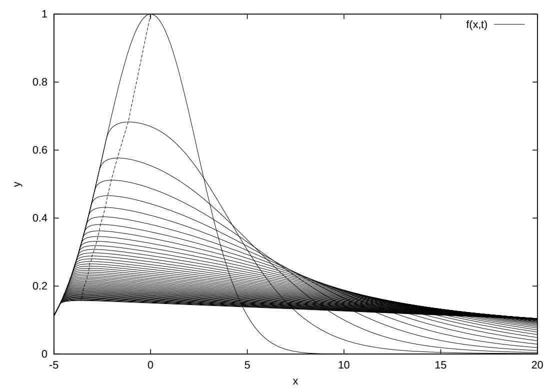


Figure 4: *Destruction of a bump* $exp(-\pi(x/6)^2)$ *in the supercritical régime, with* $\beta = 1$*,* $\gamma = 1$*,* $V = 1$*,* $\lambda = 0$ *and* $\tau_s = 0$*; the maximum of the bump is plotted for* $t < 100$*, it is moving upstream;* $f(x,t)$ *is plotted as well (for* $t = 0, 2, 4, 6, \ldots, 100$*).*

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