Stability of an Erodible Bed in a Shear Flow

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outline

- Boundary layer solutions of the basic flows
  - Steady shear flow
  - Decelerated flow
  - Oscillating flow

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- Perturbation of the basic flow
  - Equations near the wall
  - Linearised equations
  - Analytical law between the topography and the skin friction

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- linear stability analysis of the bed
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  - Decelerated flow case
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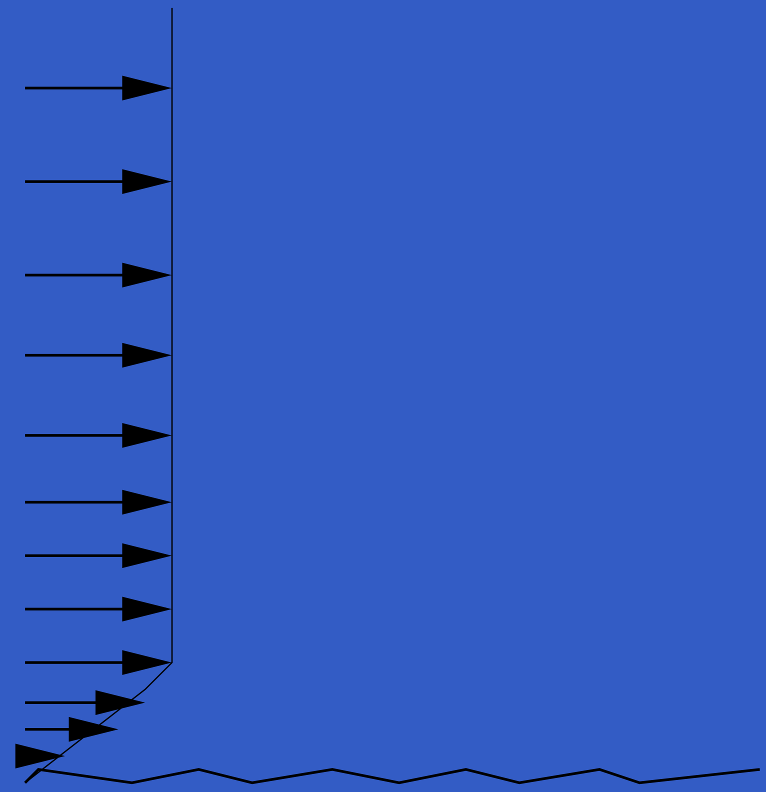
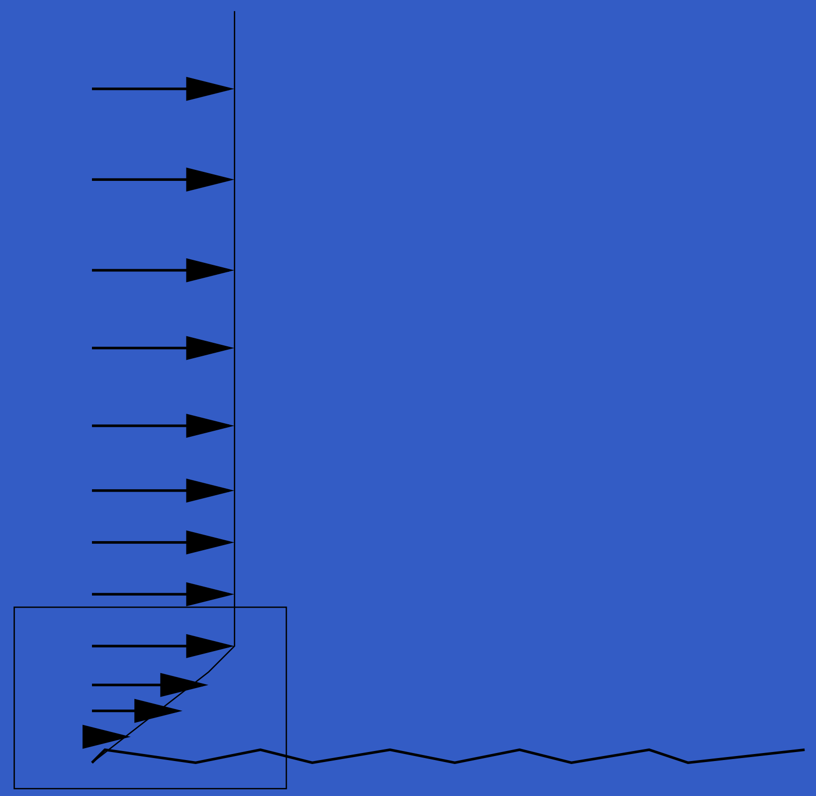
  - Oscillating flow case
- Examples of long time evolution of the bed

every profile is linear near the wall



every profile is linear near the wall



Boundary layer solutions of the basic flows

- **Steady basic flow**

$$u^* = U_0 \frac{y^*}{\delta} + \cdots$$

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$$u^* = U_0 \, Erf\left(\frac{y^*}{2\sqrt{\nu \, t^*}}\right) \quad = \frac{U_0 \, y^*}{\sqrt{\pi \, \nu \, t^*}} + \cdots$$

$$\bar{t} = \frac{\pi \, \nu}{\delta^2} \, t^* \quad and \quad \delta = \sqrt{\pi \, \nu \, T} \quad if \quad \mathcal{O}(t^*) = T$$

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- **Oscillating basic flow**

$$u^* = U_0 \, [\cos(\omega \, t^*) - e^{-\sqrt{\frac{\omega}{2\,\nu}} \, y^*} \, \cos(\omega \, t^* - \sqrt{\frac{\omega}{2\,\nu}} \, y^*)]$$

$$u^* = U_0 \, \sqrt{\frac{\omega}{\nu}} \, \cos(\omega \, t^* + \frac{\pi}{4}) \, y^* + \mathcal{O}(y^{*2}).$$

$$\bar{t} = \omega \, t^* + \frac{\pi}{4} \quad and \quad \delta = \sqrt{\frac{\nu}{\omega}}$$

Perturbation of the basic flow

with the scaling:

$$(x^*, y^*) = \delta\,(\overline{x}, \overline{y})$$

$$(u^*, v^*) = U_0\,(\overline{u}, \overline{v}) \qquad p^* = \rho\,(U_0^2\,\overline{p} - g\,\overline{y}\,\delta)$$

Navier Stokes equations:

$$\frac{\partial \overline{u}}{\partial \overline{x}} + \frac{\partial \overline{v}}{\partial \overline{y}} = 0$$

$$\frac{\partial(\overline{u}, \overline{v})}{\partial t} + Re_\delta\,([(\overline{u}, \overline{v}) \cdot \nabla](\overline{u}, \overline{v}) + \nabla\overline{p}) = \nabla^2(\overline{u}, \overline{v})$$

Perturbation of the basic flow

- **Rescaling**

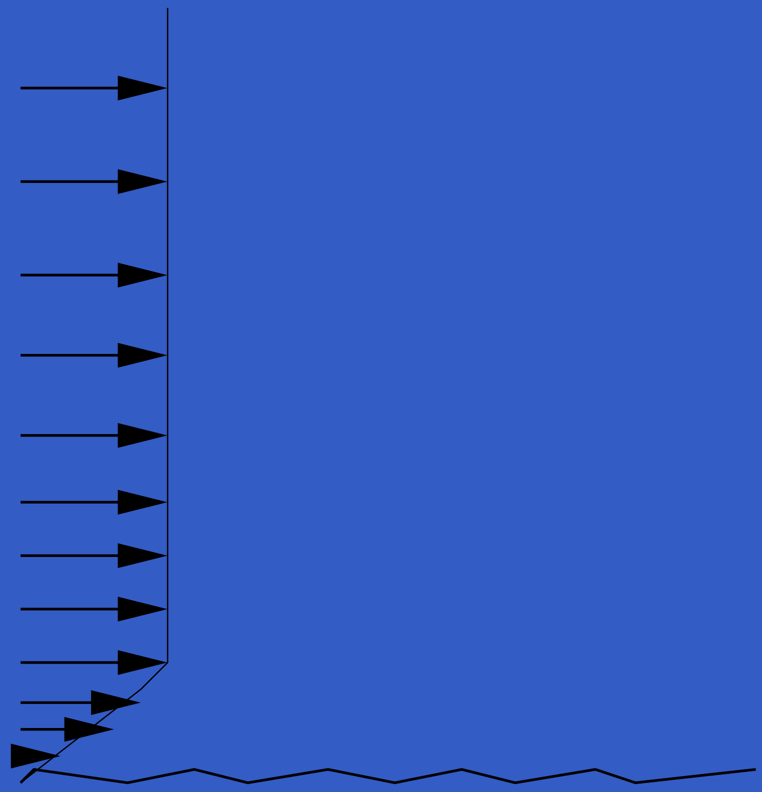
$$\overline{x} = \lambda\, x \quad and \quad \overline{y} = \varepsilon_b\, y$$

Restricting the field of study to dimensions of a bump of the disturbance of the bottom, one has:

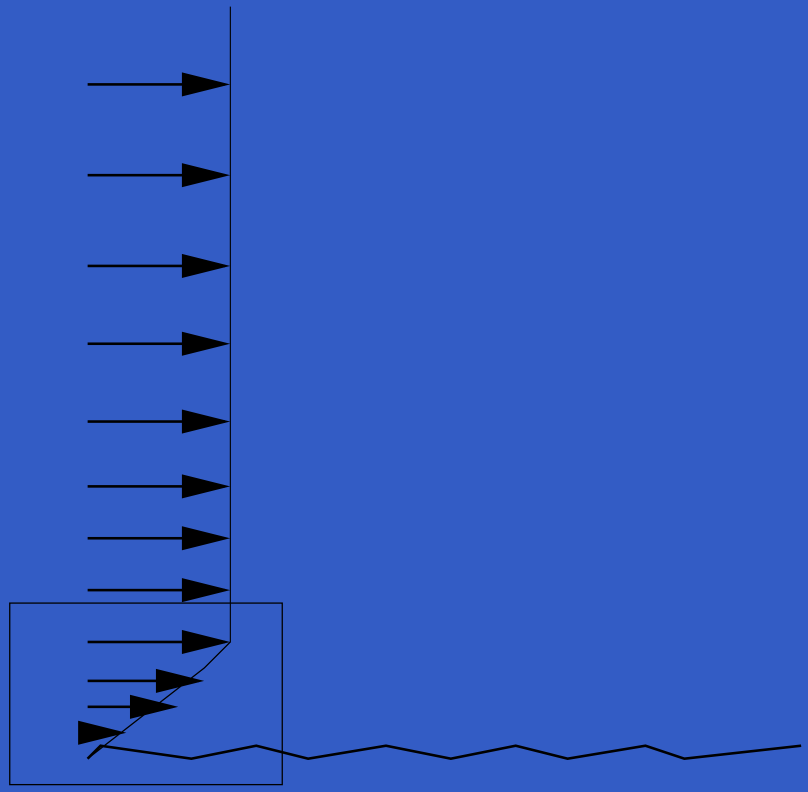
$$\overline{u} = U_s'(0)\,\overline{y} + \mathcal{O}(\overline{y}^2) = \varepsilon_b U_s'(0)\, y + \mathcal{O}(y^2)$$

where
$U_s'(0) = 1$ for the steady case:
$U_s'(0) = \frac{1}{\sqrt{\overline{t}}}$ for the decelerated flow and
$U_s'(0) = \cos(\overline{t})$ for the oscillating case
is a function of the alone variable $\overline{t}$, hydrodynamic time.

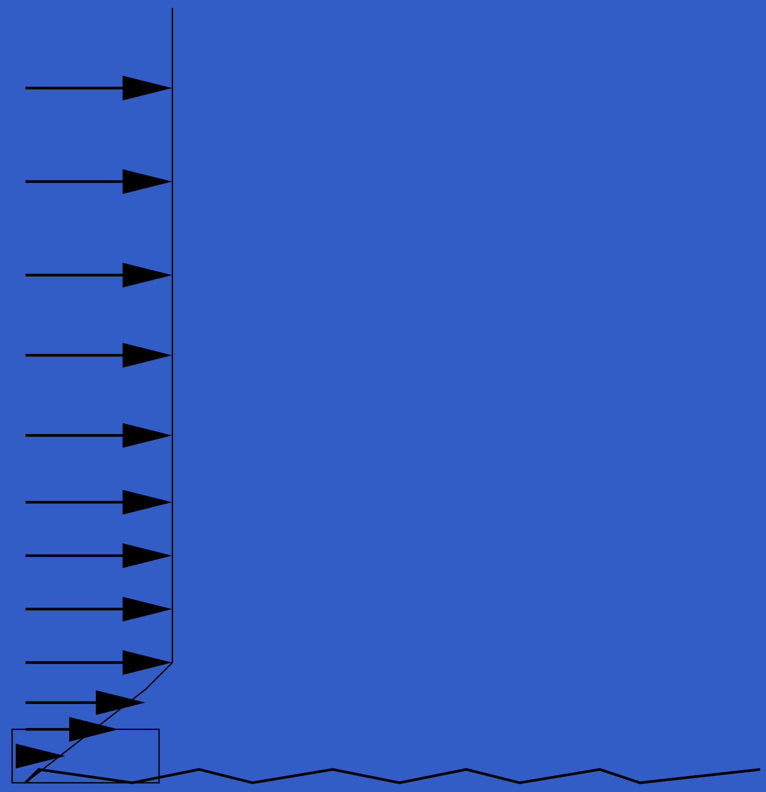
Perturbation of the basic flow



Perturbation of the basic flow

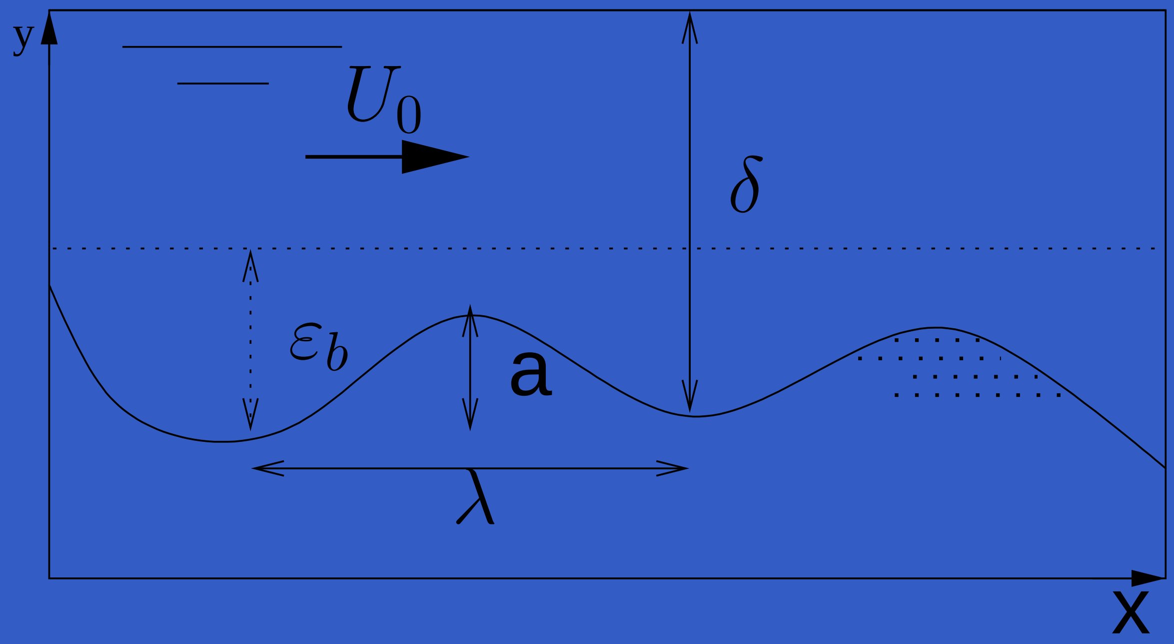


Perturbation of the basic flow



Perturbation of the basic flow

$$\frac{\varepsilon_b}{\delta} \ll 1 \qquad and \qquad \frac{\lambda}{\delta} \ll 1$$



Equations near the wall

- **As** $\varepsilon_b = \mathcal{O}(\lambda)$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$\varepsilon_b^2 \, Re_\delta \left\{ [(u,v) \cdot \nabla](u,v) + \nabla p \right\} = \Delta(u,v)$$

Equations near the wall

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$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$\varepsilon_b^2 \, Re_\delta \, \{[(u,v)\cdot\nabla](u,v) + \nabla p\} = \Delta(u,v)$$

- **As** $\varepsilon_b \ll \lambda$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$u\,\frac{\partial u}{\partial x} + v\,\frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x} + \frac{\lambda}{\varepsilon_b^3 \, Re_\delta}\,\frac{\partial^2 u}{\partial y^2}.$$

Equations near the wall

- Finally

$$x^* = \delta\, \varepsilon_b^3\, Re_\delta\, x, \qquad y^* = \delta\, \varepsilon_b\, y \qquad and \quad \varepsilon_b \ll 1$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$u\,\frac{\partial u}{\partial x} + v\,\frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x} + \frac{\partial^2 u}{\partial y^2}$$

$$0 = \frac{\partial p}{\partial y}.$$

Equations near the wall

The fact of having

$$\lambda \sim \varepsilon_b^3 \, Re_\delta = \frac{2\,A\,\varepsilon_b^3}{\delta} \qquad gives \qquad r = \frac{2\,A}{\lambda} \simeq \frac{\delta}{\varepsilon_b^3}$$

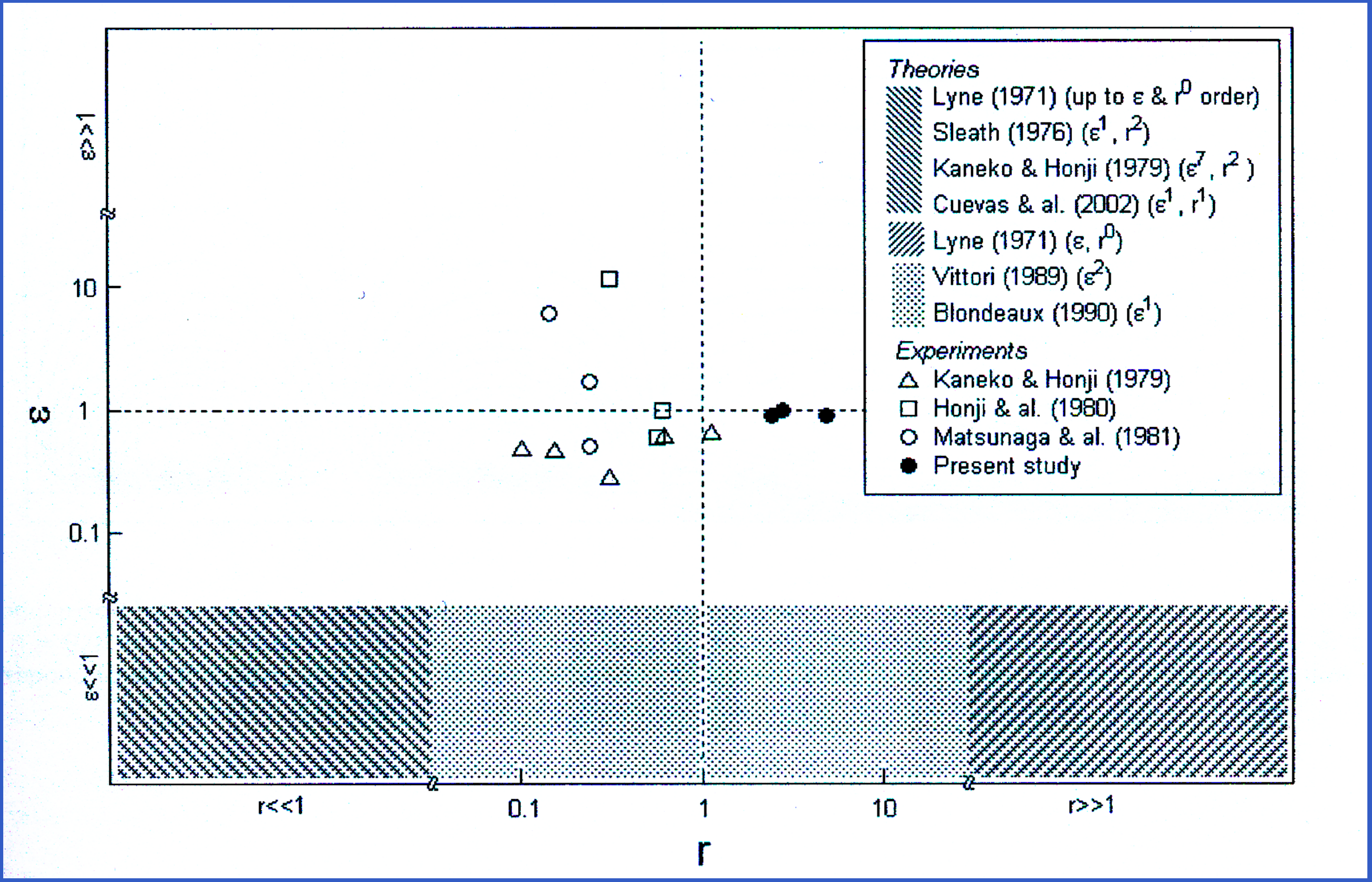
r is the aspect ratio between the characteristic scales uses in former studies. However

$$\frac{\delta}{\varepsilon_b} \gg 1 \qquad and \qquad \frac{1}{\varepsilon_b^2} \gg 1$$

one thus has well

$$r = \frac{2\,A}{\lambda} \simeq \frac{\delta}{\varepsilon_b} \, (\frac{1}{\varepsilon_b^2}) \gg 1$$

Equations near the wall



Rousseaux et al. (2003)

Linearised equations

$$f = a f_1$$

$$u = U_s'(0)\, y + \mathcal{O}(y^2)$$

that gives us the variables of the problem in the form

$$u = U_s'(0)\left[y + a u_1(x,y,t) + \cdots\right]$$

$$v = \qquad U_s'(0)\, a v_1(x,y,t) + \cdots$$

$$p = \qquad U_s'(0)\, a p_1(x,y,t) + \cdots$$

Linearised equations

we keep the equations with the 1^{st} order in a

$$\frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y} = 0$$

$$U_s'(0)\, y\, \frac{\partial u_1}{\partial x} + U_s'(0)\, v_1 = -\frac{\partial p_1}{\partial x} + \frac{\partial^2 u_1}{\partial y^2}$$

$$0 = \frac{\partial p_1}{\partial y}$$

Linearised equations

Decomposing in modes of Fourier, taking into account the continuity equation

$$f_1 = f_k \, e^{-i\,k\,x + \sigma\,t^L}$$

$$u_1 = \phi'(y)\, e^{-i\,k\,x + \sigma\,t^L}$$

$$v_1 = (i\,k)\phi(y)\, e^{-i\,k\,x + \sigma\,t^L}$$

$$p_1 = \psi(y)\, e^{-i\,k\,x + \sigma\,t^L}, \quad \psi_{,y} = 0$$

Perturbated solutions in Fourier space

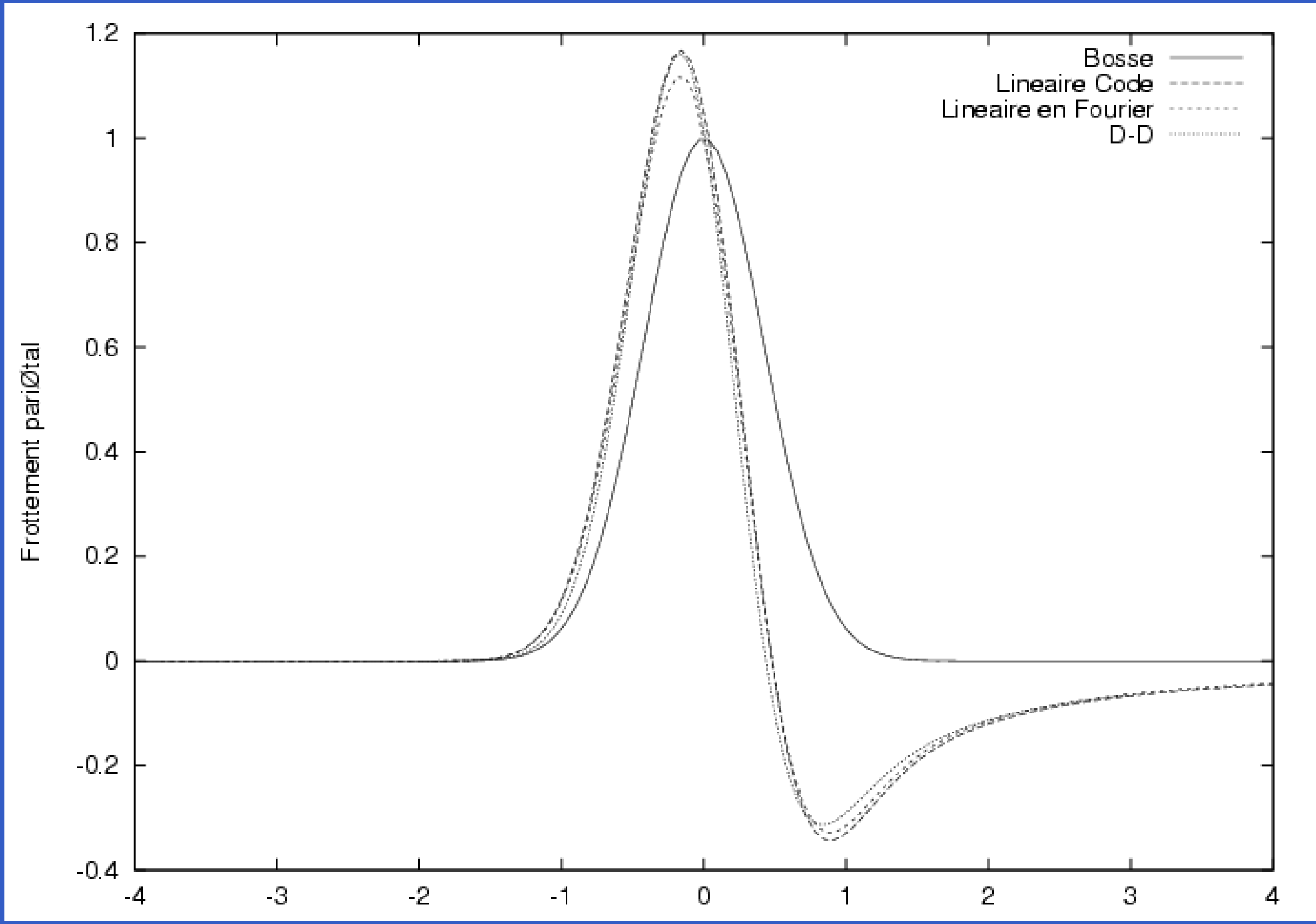
- Pression

$$p_1 = 3\,a\,Ai'(0)\,(U_s'(0))^{5/3}\,(-i\,k)^{-1/3}\,f_1$$

- friction

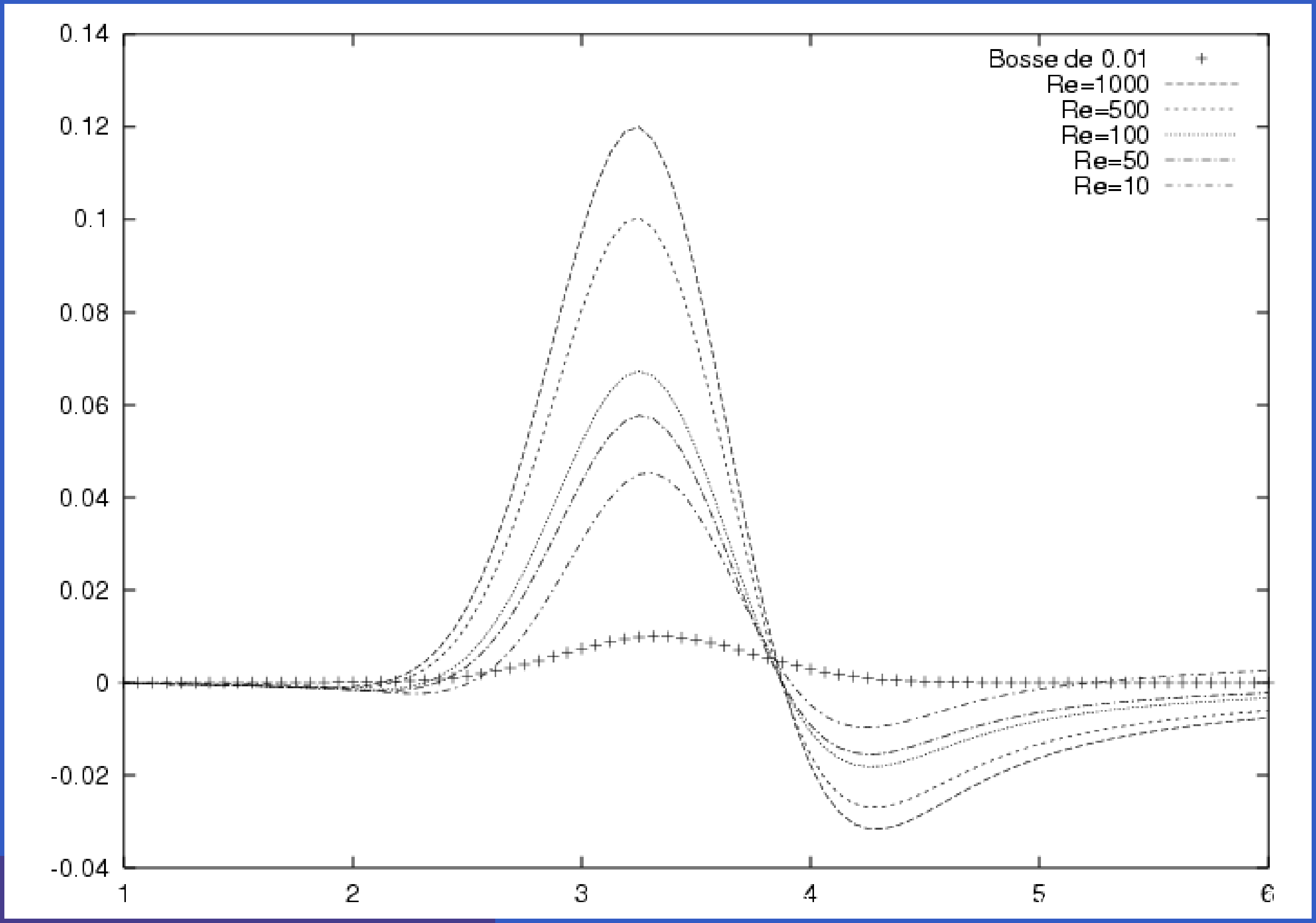
$$\tau_1 = \frac{\partial u_1}{\partial y} = 3\,a\,Ai(0)\,U_s'(0)\,(-i\,k\,U_s'(0))^{1/3}\,f_1$$

validation of linear friction



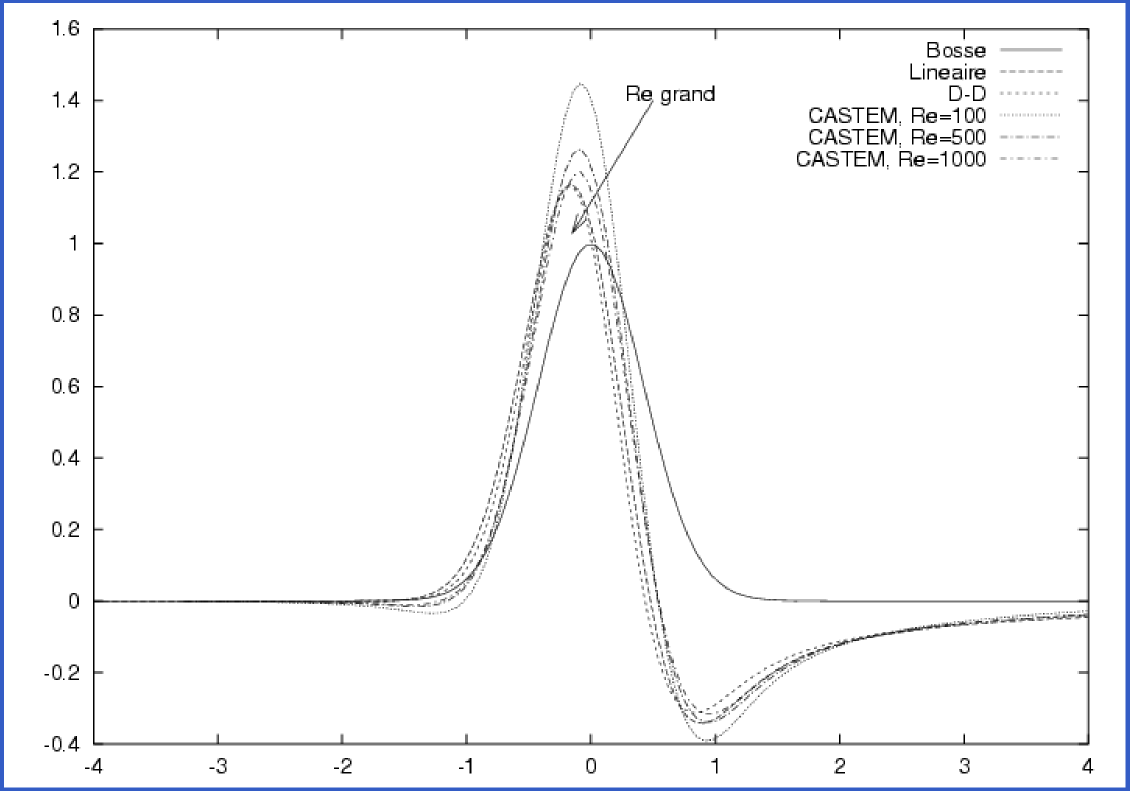
validation of linear friction

here, taking simply $U_s'(0) = 1$ (steady shear), the friction $(\tau - 1)$ calculated by CASTEM 2000 (Navier-Stokes)



validation of linear friction

here, taking simply $U_s'(0) = 1$ (steady shear), the friction $(\tau - 1)$ calculated by CASTEM 2000 (Navier-Stokes) and rescaled is compared to the linearised solution



Decelerated basic flow

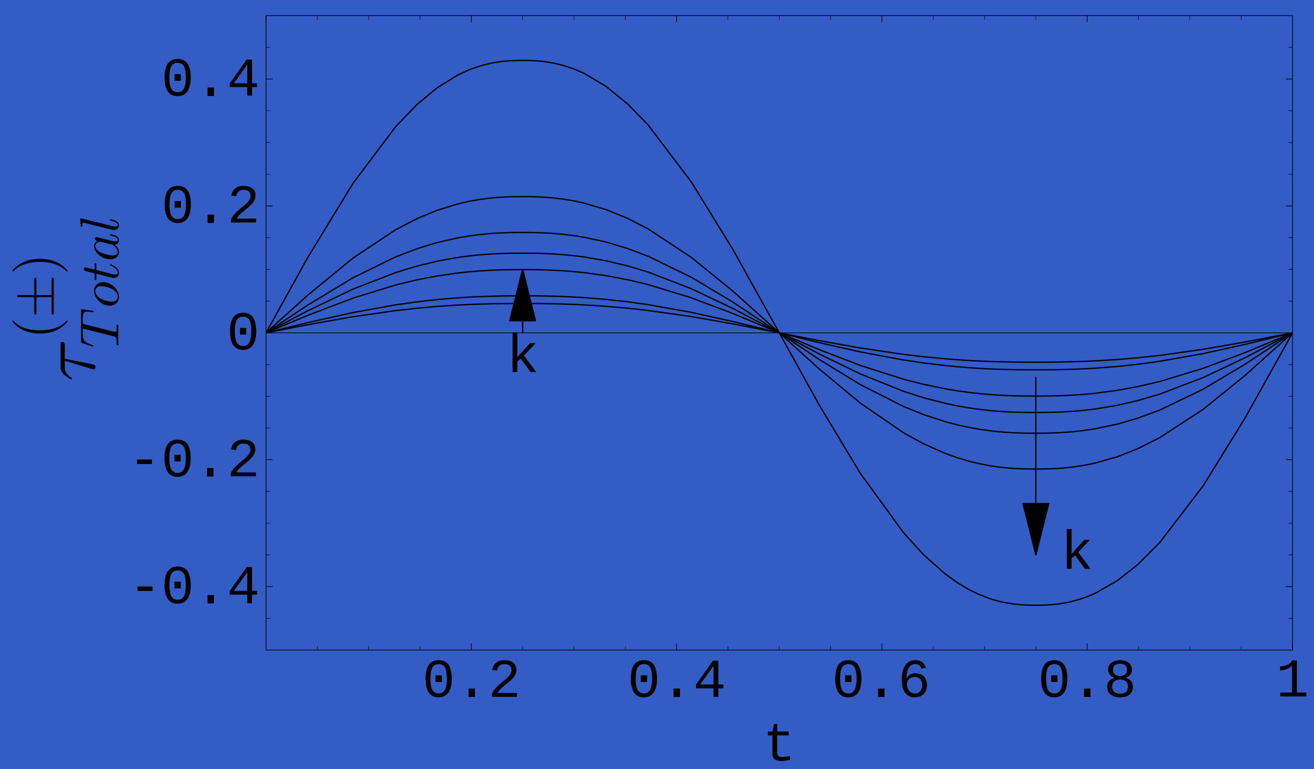
$$U_s'(0) = \frac{1}{\sqrt{t}},$$

the bottom friction is

$$\tau_{Total} = \frac{1}{\sqrt{t}} + TF^{-1}\{3\,Ai(0)\,(-i\,k)^{1/3}\,[t]^{-2/3}\,e^{-i\,k\,x+\sigma\,t^L}\}(x,t)$$

Oscillating basic flow

For one period of oscillation



Oscillating basic flow

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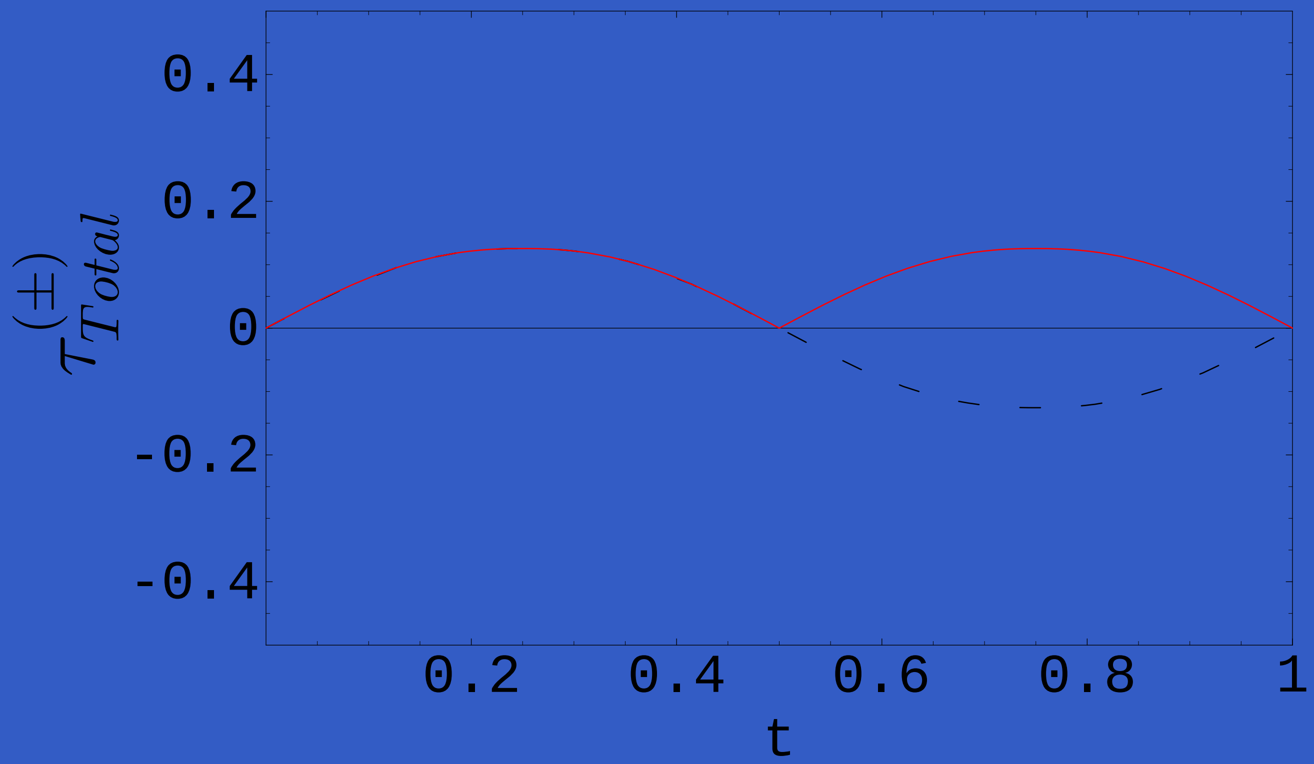

Oscillating basic flow

during one cycle the topography does not change

we take the mean value of all the quantities

Multiscale analysis...

$$U_s'(0) = cos(t),$$

$$\tau_{Total}^{(+)} = cos(t) + TF^{-1}\{3\,Ai(0)\,(-i\,k)^{1/3}\,[cos(t)]^{4/3}\,e^{-i\,k\,x+\sigma\,t^L}\}(x,t)$$

$$\tau_{Total}^{(-)} = -cos(t) - TF^{-1}\{3\,Ai(0)\,(-i\,k)^{1/3}\,[cos(t)]^{4/3}\,e^{-i\,k\,x+\sigma\,t^L}\}(x,t)$$

Oscillating basic flow

$$< \tau >_{Total} = \frac{1}{T} \, [\int_0^{t_p} \tau_{Total}^{(+)} \, dt + \int_{t_p}^{T} \tau_{Total}^{(-)} \, dt \,]$$

$$< \tau >_{Total} = \frac{9 \, Ai[0] \, [(-i \, k)^{1/3} - (i \, k)^{1/3}] \, \Gamma(\frac{7}{6})}{4 \, \sqrt{\pi} \, \Gamma(\frac{2}{3})}$$

Fluid

Up to now, we have for any initial profile, the skin friction,

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need for a law of matter flux.

Laws of matter flux

In the majority of their work, B. Sumer (1984), P. Blondeaux (1990), G. Parker (1995), K. Richards (1999), F. Charru (2002), K. Kroy, Hermann Sauermann (2002) , established that

$$q \propto \tau^{\frac{3}{2}}.$$

As

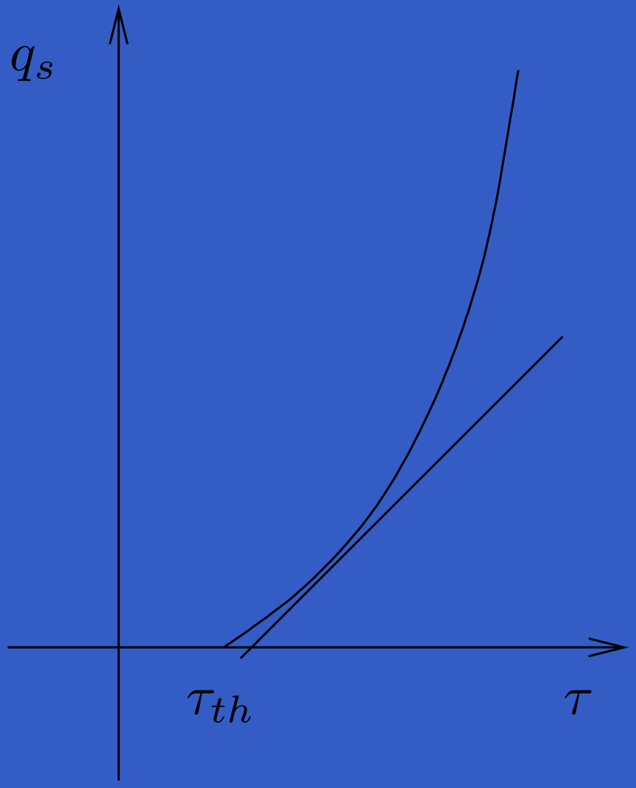
$$u = U_s'(0)\,[y + u_1(x,y,t) + \cdots],$$
$$\tau = U_s'(0)\,[1 + \tau_1(x,y,t) + \cdots] \quad with \quad |\tau_1| \ll 1$$

so

$$q \propto (1+\tau_1)^{\frac{3}{2}} \approx 1 + \frac{3}{2}\tau_1$$

Laws of matter flux

$$\begin{cases} if \quad \tau > \tau_{th} & q = \tau - \tau_{th} \\ \qquad\qquad else & q = 0 \end{cases}$$



Laws of matter flux

- Linear form (Yang (1995), Fredsøe and Deigaard (1992))

$$q = \tau - \tau_{th} - \Lambda \frac{\partial f}{\partial x}$$

- An another form (Andreotti and al. (2002) simplified Kroy and al (2002) Sauermann and al (2001))

$$l_K \frac{\partial q}{\partial x} + q = \tau - \tau_{th}$$

with l_K proportional to $\frac{1}{U_s'(0)}$.

Fluid/ bed coupling

Up to now, we have for any initial profile, the skin friction, and then the flux of matter

$$q \leftarrow \tau \leftrightarrow f$$

Fluid/ bed coupling

Up to now, we have for any initial profile, the skin friction, and then the flux of matter

$$q \leftarrow \tau \leftrightarrow f$$

$$\frac{\partial f}{\partial t} = -\frac{\partial q}{\partial x}$$

Linear stability analysis

- steady shear case

$$FT[\tau] = FT[f](3Ai(0))(-(ik))^{1/3} \qquad \frac{\partial f}{\partial t} = -\frac{\partial q}{\partial x}$$

So, for a mode k, looking to $f = e^{\sigma t + i\omega t} e^{-ikx}$,

$$\sigma + i\omega = \frac{3^{\frac{1}{3}}}{\Gamma(\frac{2}{3})}(1/2 + i\sqrt{3}/2)(k)^{4/3} - \Lambda\, k^2$$

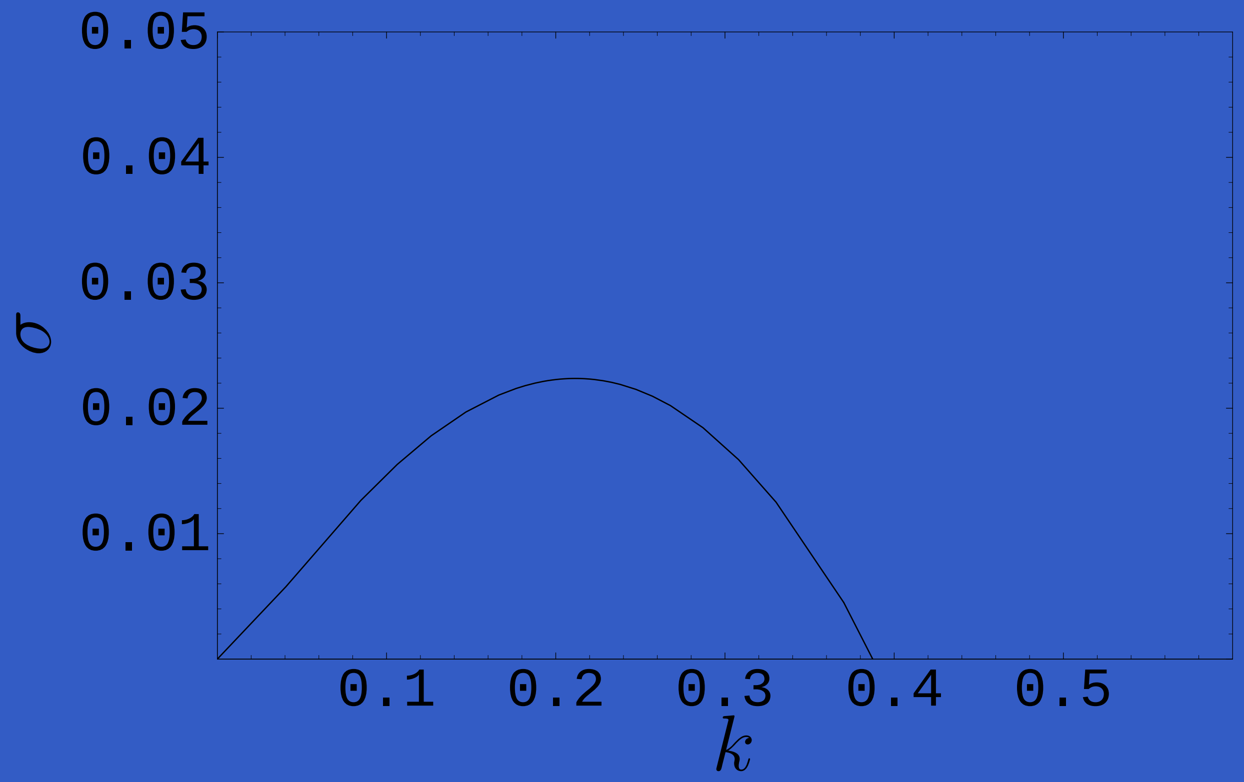
With $\Lambda = 0$ all waves are always instable

slope effect $\Lambda \neq 0$ give an amplification for long waves; short waves always instable.

Or length of saturation effect give an amplification for long waves which are always stable; short waves always instable.

Linear stability analysis

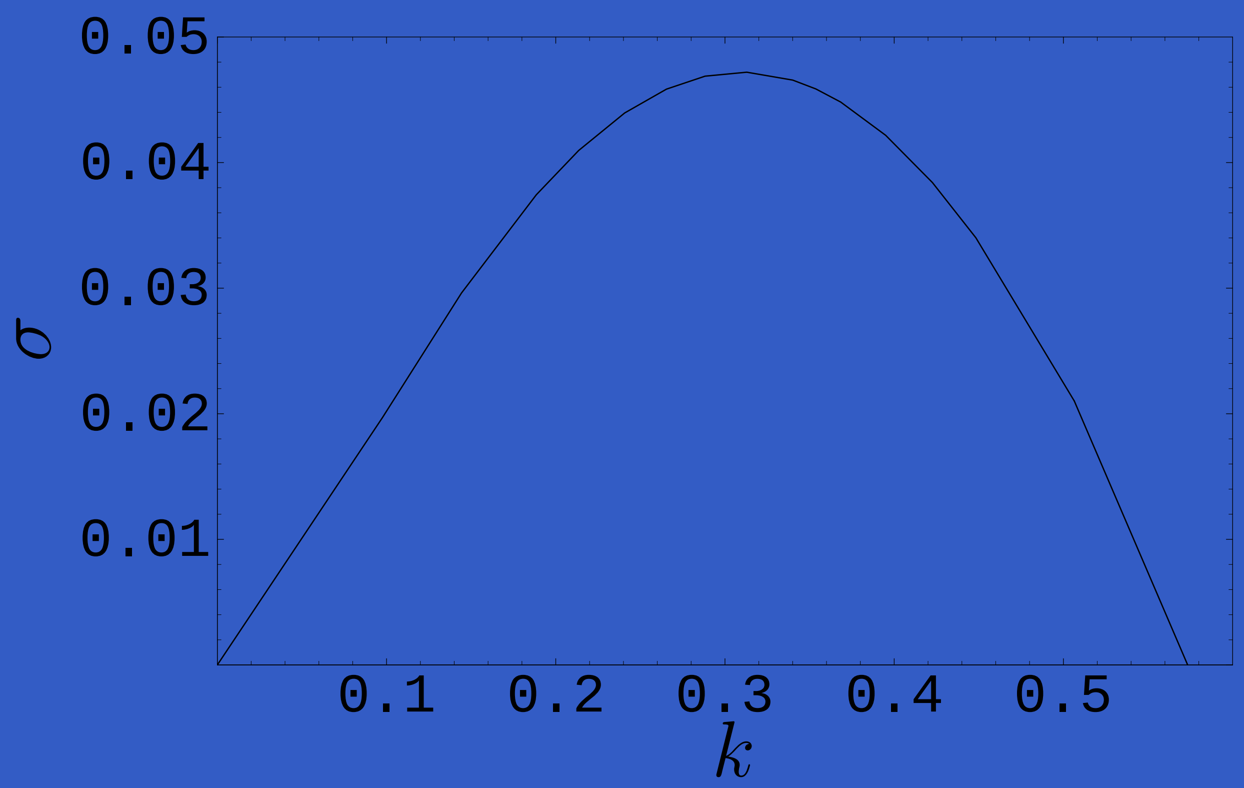
- steady shear case



Constant shear, $U_s'(0) = 1$, amplification factor σ as function of number k (here $q = \tau - \tau_{th} - \Lambda \frac{\partial f}{\partial x}$ with $\Lambda = 1$), decreasing Λ increases the cut off value of k.

Linear stability analysis

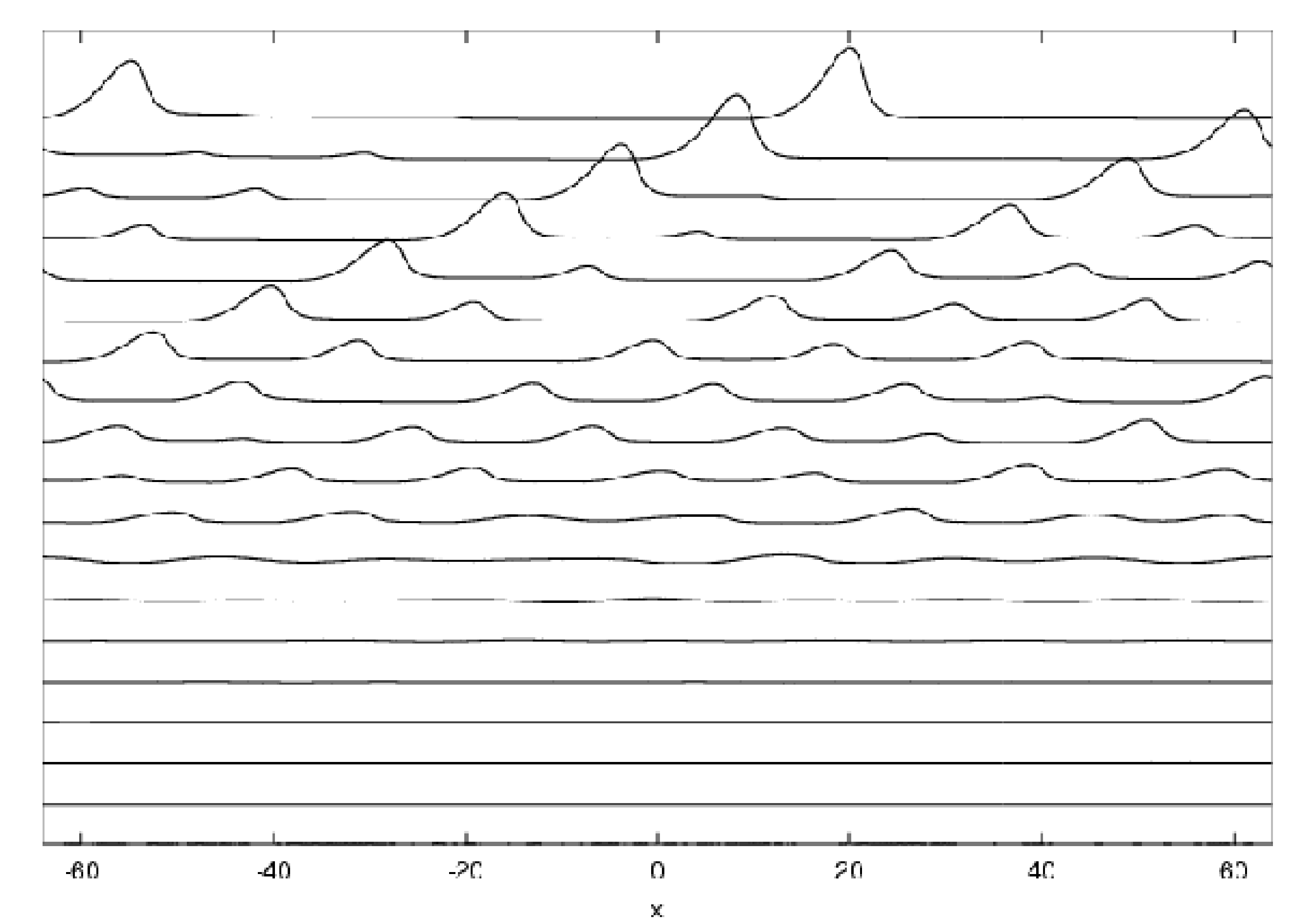
- steady shear case



Constant shear, $U_s'(0) = 1$, amplification factor σ as function of number k (here $l_K \frac{\partial q}{\partial x} + q = \tau - \tau_{th}$ with $l_K = 1$), decreasing l_K increases the cut off value of k.

Examples of long time evolution : steady shear flow

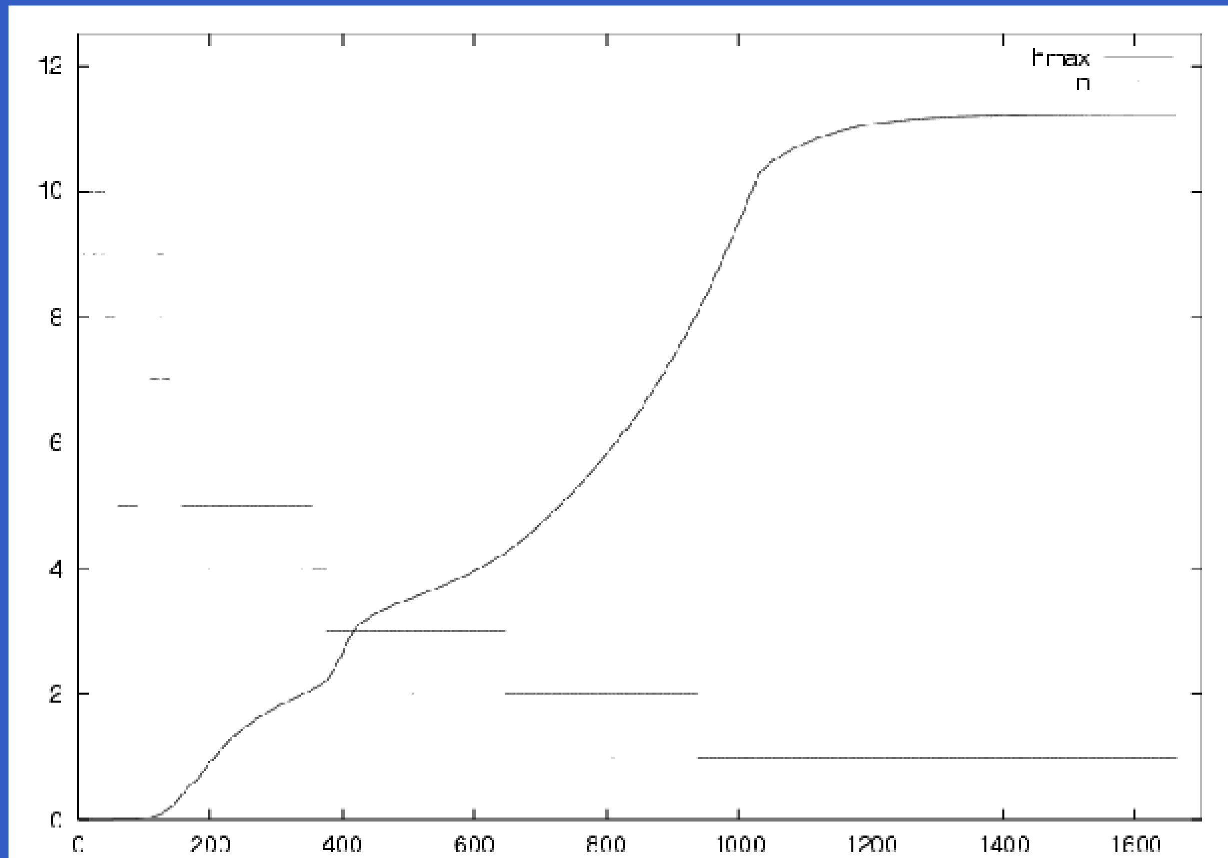
coarsening



$l_K \frac{\partial q}{\partial x} + q = \tau - \tau_{th}$ with $l_K = \frac{1}{U_s'(0)} = 1$

animation

Examples of long time evolution : steady shear flow



Number of dunes and maximal height versus time,

$$l_K \frac{\partial q}{\partial x} + q = \tau - \tau_{th} \text{ with } l_K = \frac{1}{U_s'(0)}$$

Linear stability analysis

- Decelerated shear case

$$\frac{\partial f}{\partial t^L} = -\frac{\partial q}{\partial x}$$

while

$t^L \simeq \mathcal{O}(t)$

$$f = f_k(t)\, e^{-i\,k\,x}, \qquad u_1 = u_k(t)\, e^{-i\,k\,x} \qquad \cdots$$

$$\frac{\partial f_k(t)}{\partial t} = -\,3\,Ai(0)\,(-i\,k)\,(-i\,k)^{1/3}\,t^{-2/3}\; - \Lambda\,k^2\,f_k(t).$$

The logarithm of each mode of Fourier of f

$$\log(f_k(t)) = -\,9\,Ai(0)\,(-i\,k\,)\,(-i\,k\,)^{1/3}\,t^{-1/3}\; - \Lambda\,k^2\,t$$

Linear stability analysis

- Decelerated shear case

With $\Lambda = 0$ all waves are always instable

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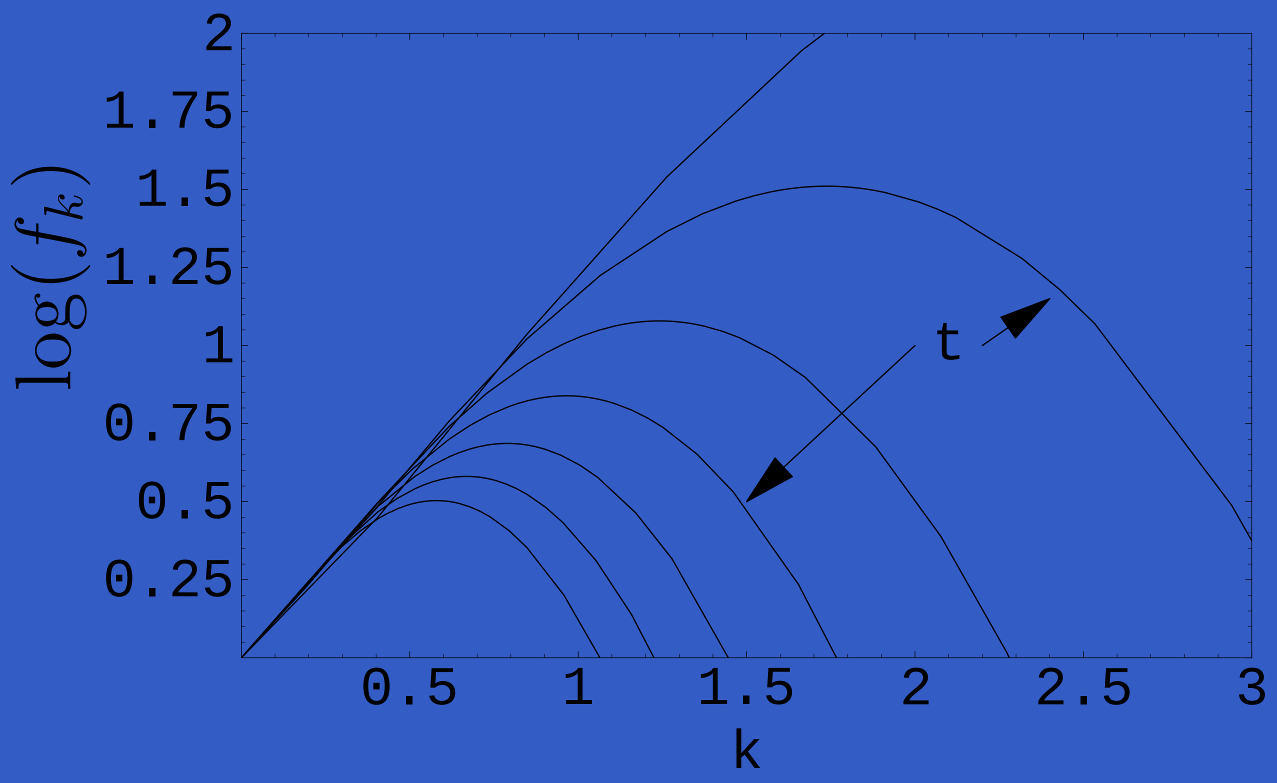
Or length of saturation effect give an amplification for long waves which are always stable; short waves always instable.

Linear stability analysis

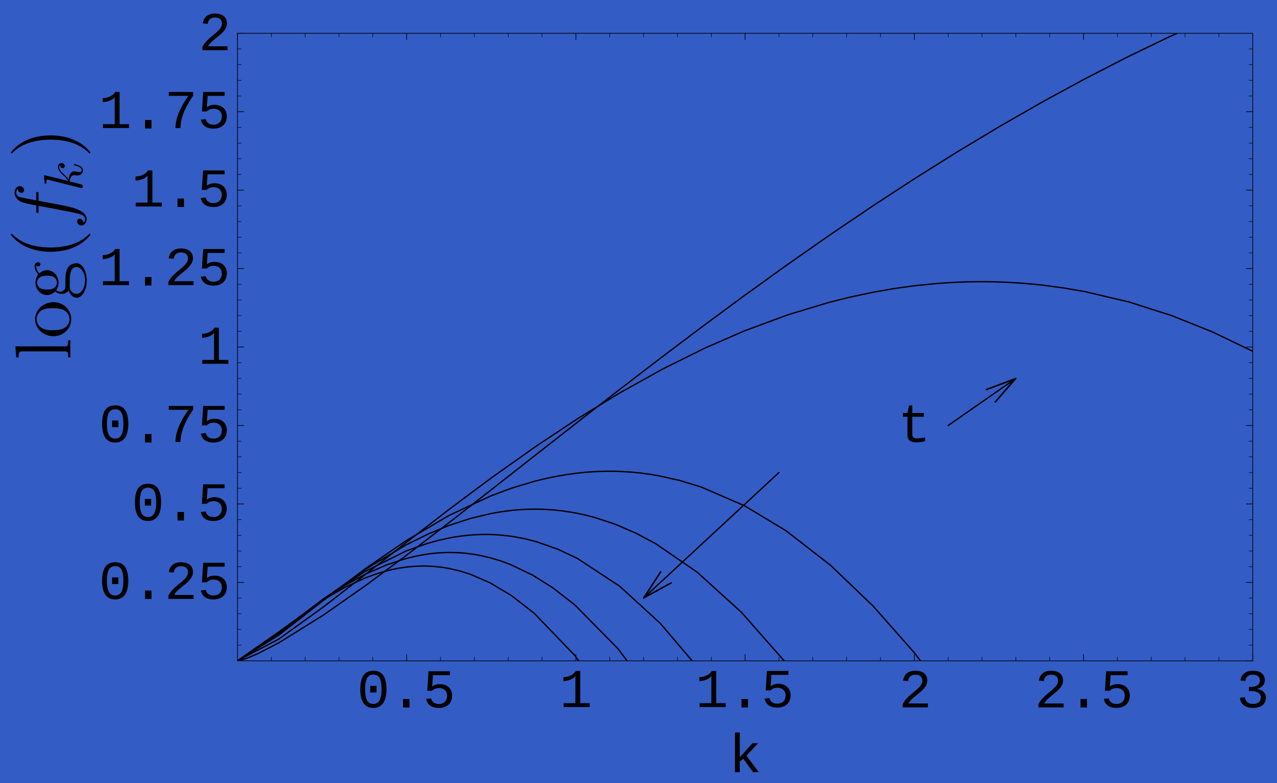
- decelerated shear case, law of q with saturation effect



No saturation effect ($q = \tau - \Lambda \frac{\partial f}{\partial x}$) with $\Lambda = 0.4$

Linear stability analysis

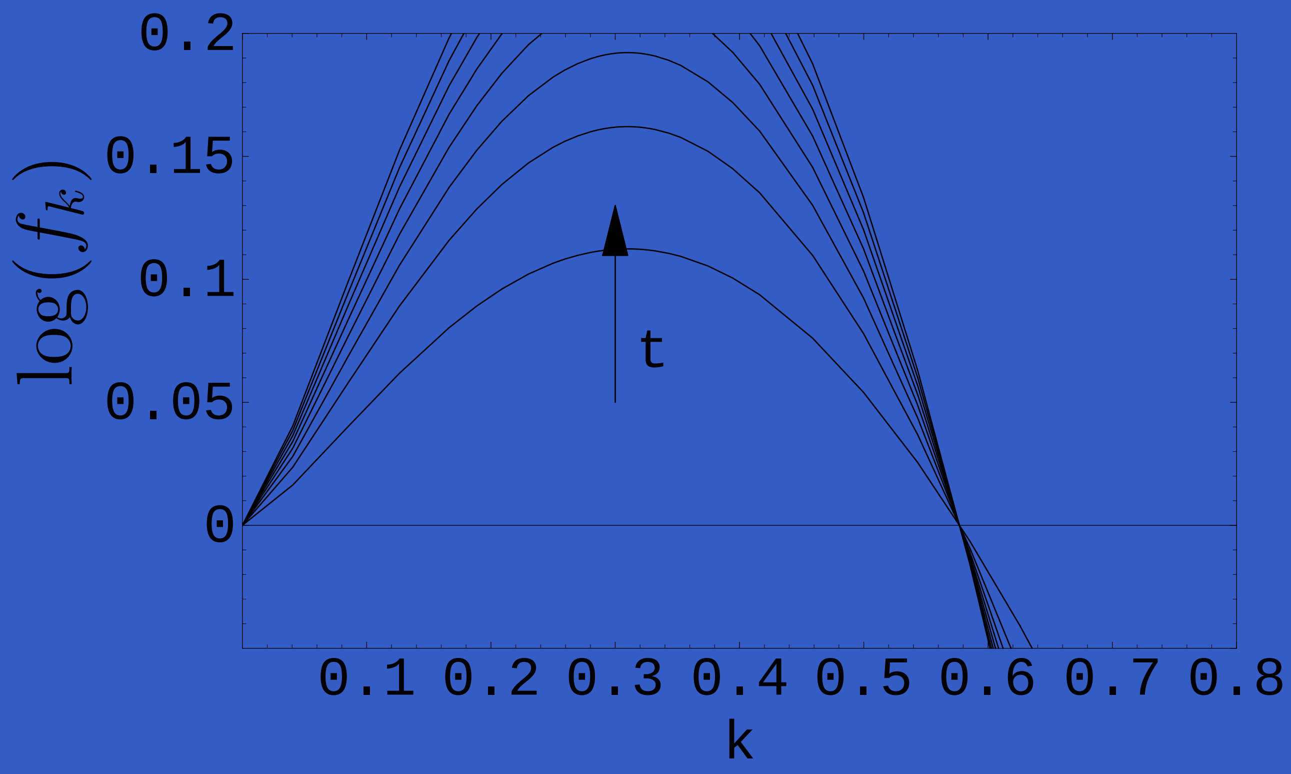
- decelerated shear case, law of q with saturation effect



Saturation effect $l_K \frac{\partial q}{\partial x} + q = \tau - \tau_{th}$ with $l_K = \frac{1}{U_s'(0)}$

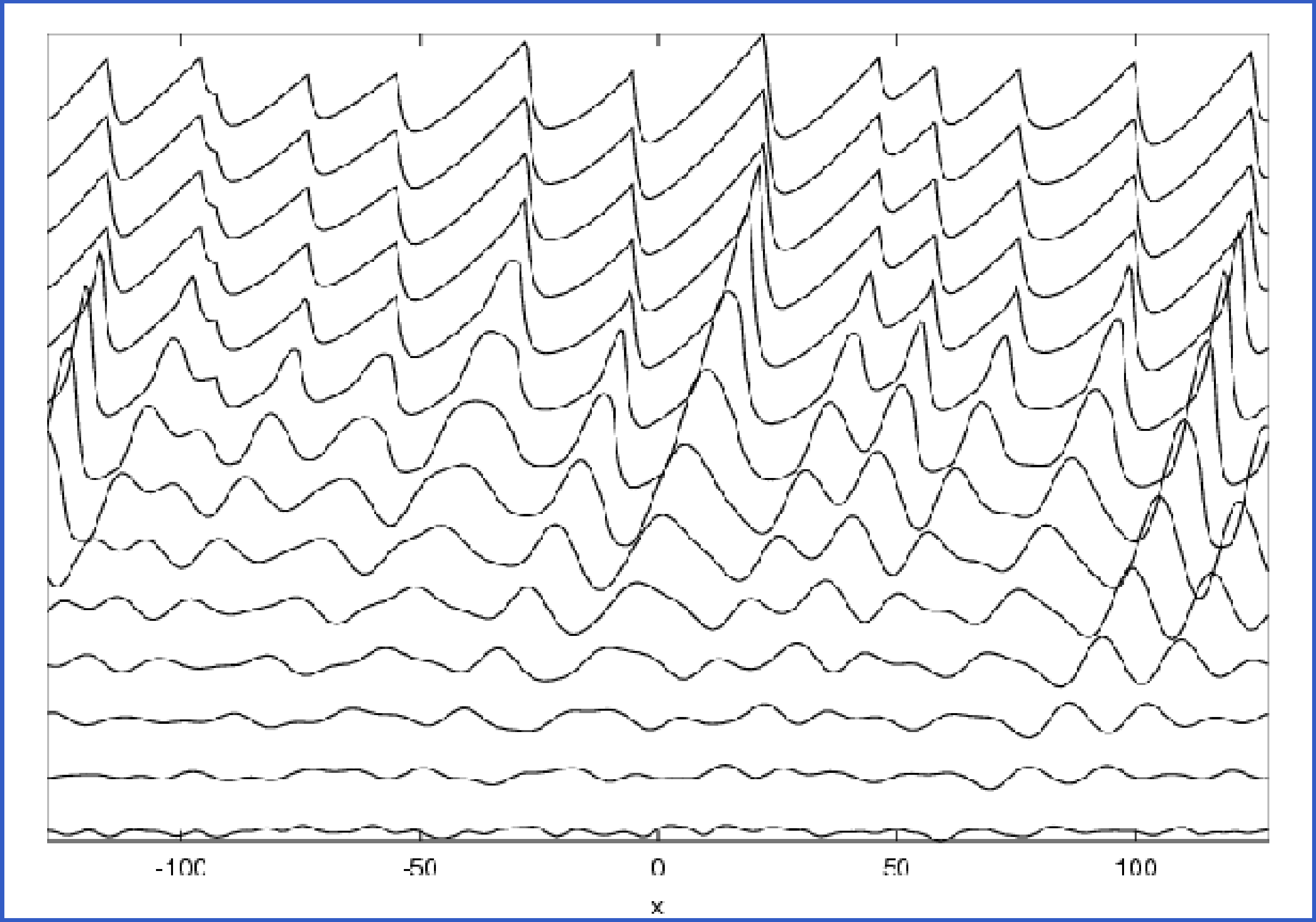
Linear stability analysis

- decelerated shear case, law of q with saturation effect



Saturation effect with $l_K \frac{\partial q}{\partial x} + q = \tau - \tau_{th}$ with $l_K = 1$

Examples of long time evolution : decelerated shear case, with saturation effect



Saturation effect with $l_K \frac{\partial q}{\partial x} + q = \tau - \tau_{th}$ with $l_K = 1$

animation

Linear stability analysis: Oscillating Flow

- Multiscale analysis for the flux relation

$$\frac{\partial f}{\partial t} = -\theta \frac{\partial q}{\partial x}$$

with $\theta << 1$, $t_0 = t$, and $t_1 = \theta t$ the long time.

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial t_0} + \theta \frac{\partial}{\partial t_1}$$

Let $f = f_0(t_0, t_1) + \theta f_1(t_0, t_1)$
and $q = q_0(t_0, t_1) + \theta q_1(t_0, t_1)$

$$\frac{\partial f_0}{\partial t_0} = 0$$

ie the topology is quasisteady

Linear stability analysis: Oscillating Flow

- Multiscale analysis for the flux relation

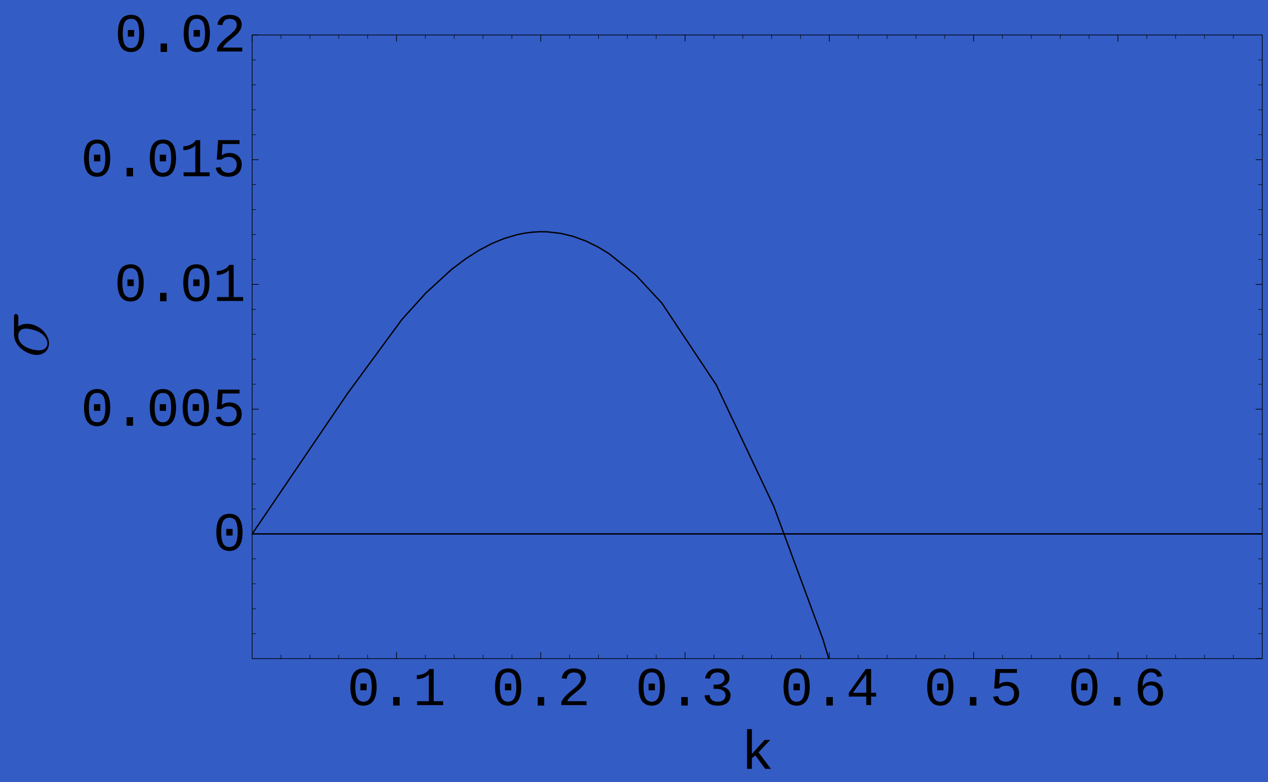
$$\frac{\partial f_0}{\partial t_1} + \frac{\partial f_1}{\partial t_0} = -\frac{\partial q}{\partial x}$$

so $f_0(t_0, t_1) = F_0(t_1)$, decomposition: q is $Q + q'$ where $Q = < q >$ and $< q' > = 0$ so

$$\frac{\partial f_1}{\partial t_0} = (-\frac{\partial q'}{\partial x}) + (-\frac{\partial Q}{\partial x} - \frac{\partial F_0}{\partial t_1})$$

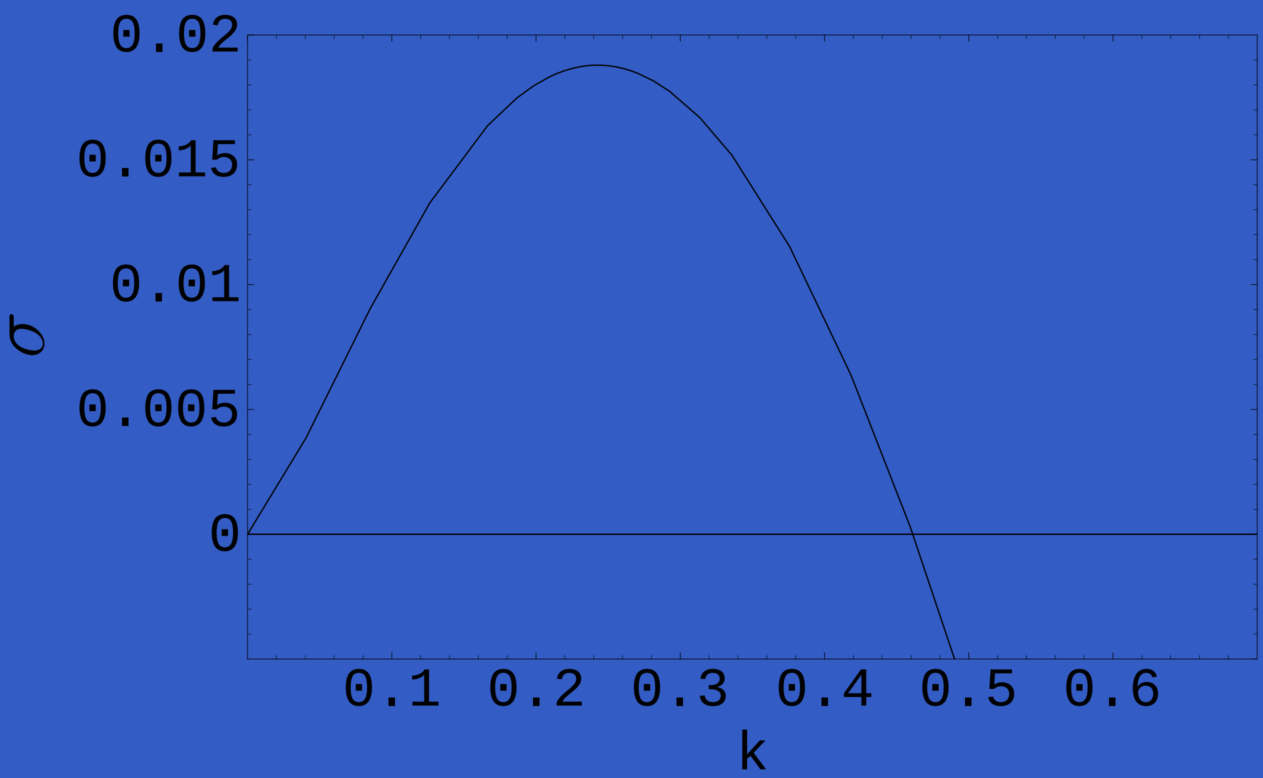
secular term: $(-\frac{\partial Q}{\partial x} - \frac{\partial F_0}{\partial t_1})$ must be 0, q' must be borned.

Linear stability analysis: Oscillating Flow



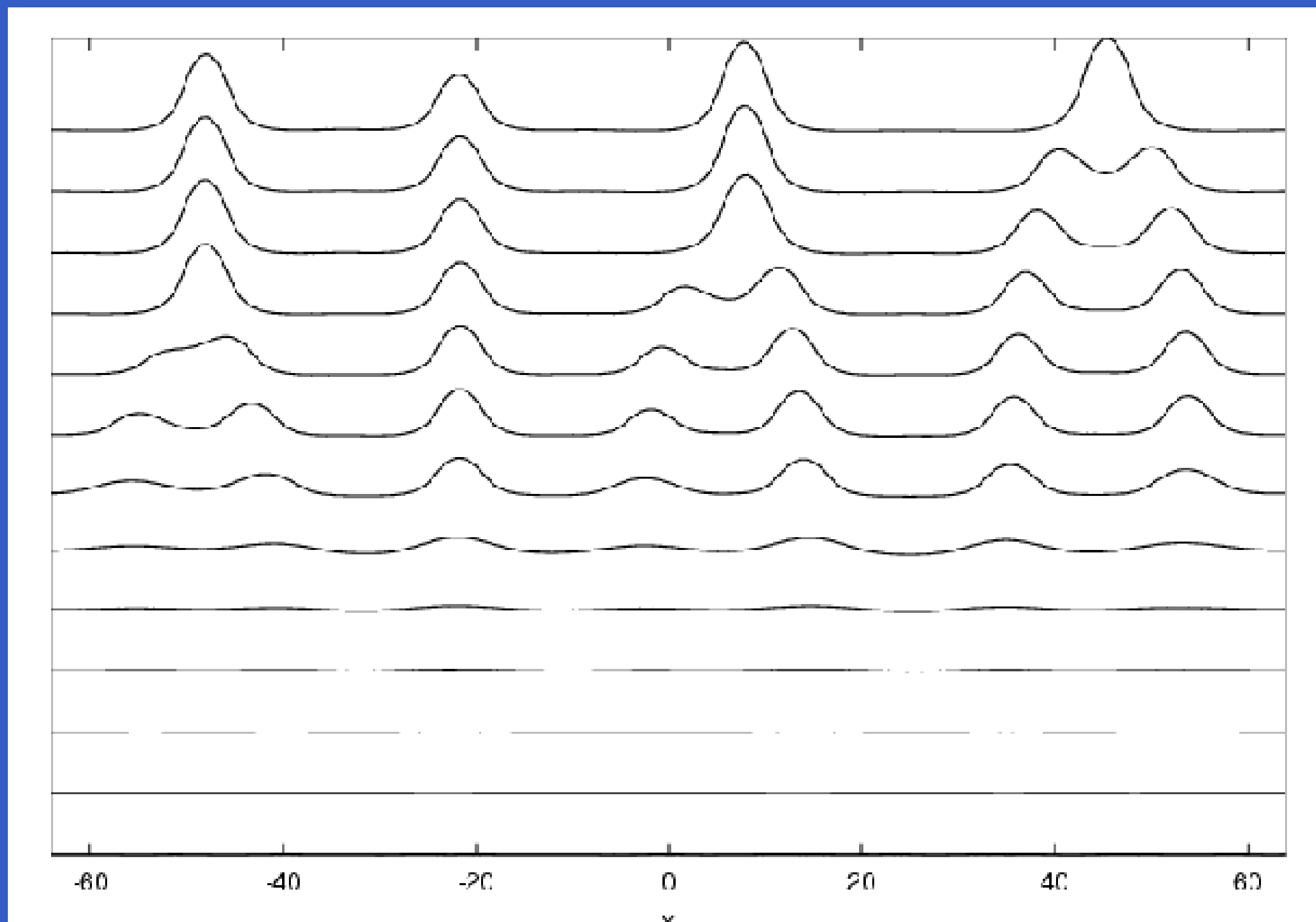
no saturation effect $q = \tau - \Lambda \frac{\partial f}{\partial x}$ with $\Lambda = 0.01$

Linear stability analysis: Oscillating Flow



with saturation effect and $l_K \frac{\partial q}{\partial x} + q = \tau - \tau_{th}$ with $l_K = \frac{1}{U_s'(0)}$

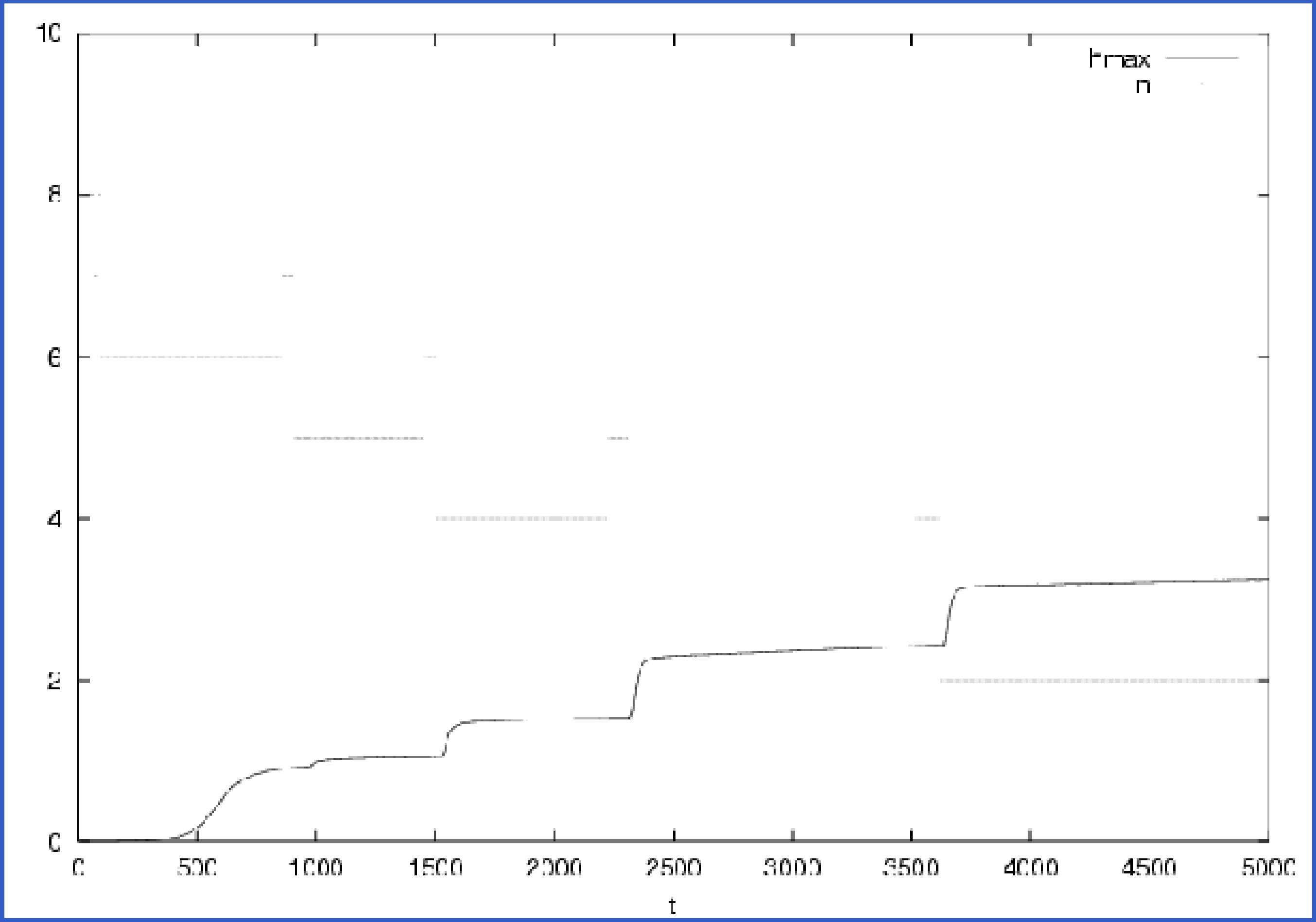
Examples of long time evolution : Oscillating Flow



$l_K \frac{\partial q}{\partial x} + q = \tau - \tau_{th}$ with $l_K = 1/U_s'(0)$

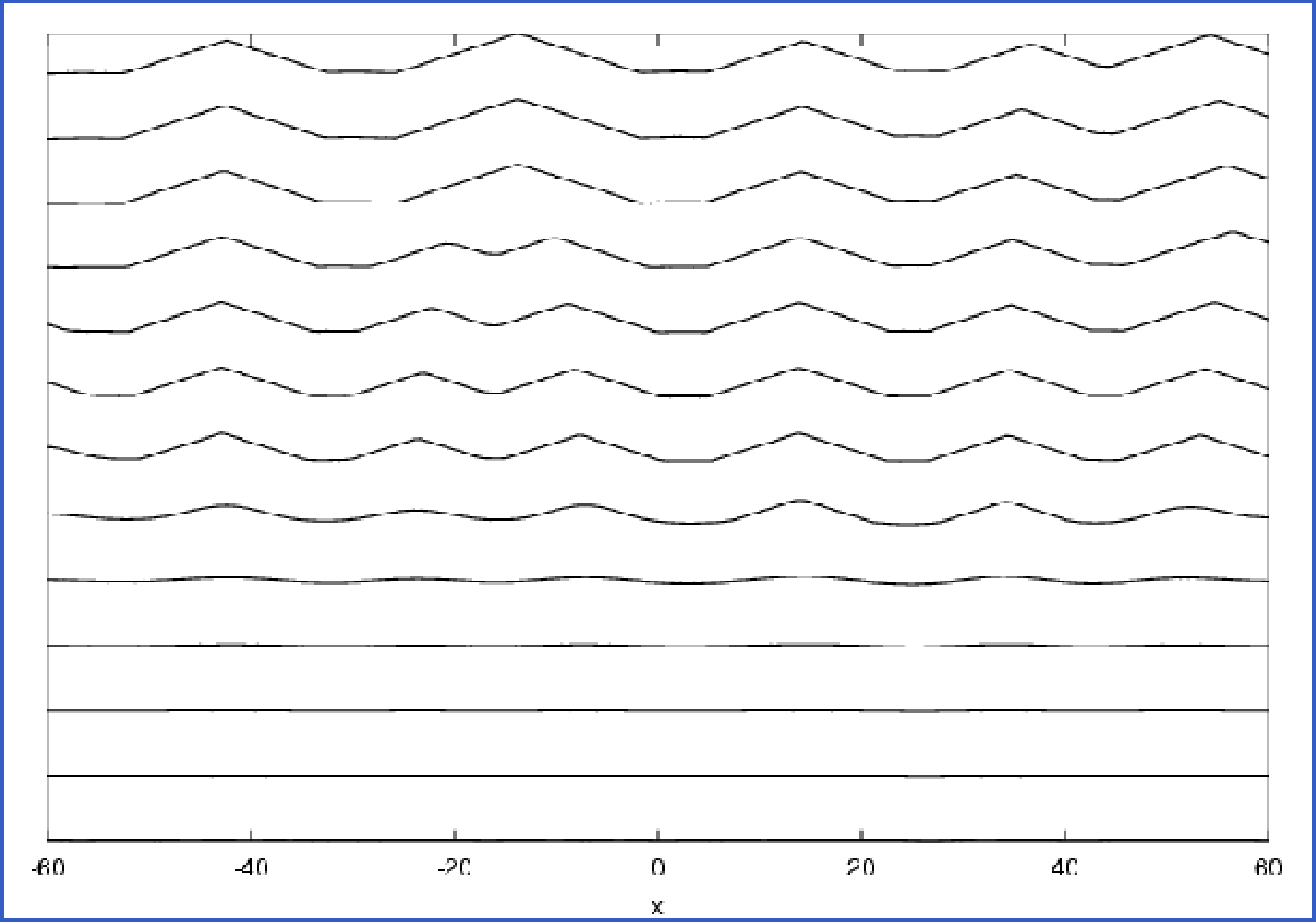
animation

Examples of long time evolution : Oscillating Flow



$l_K = 1/U_s'(0)$ number of ripples and maximum height

Examples of long time evolution : Oscillating Flow



$l_K = 1/U_s'(0)$ and slope limitation (very simple avalanche)

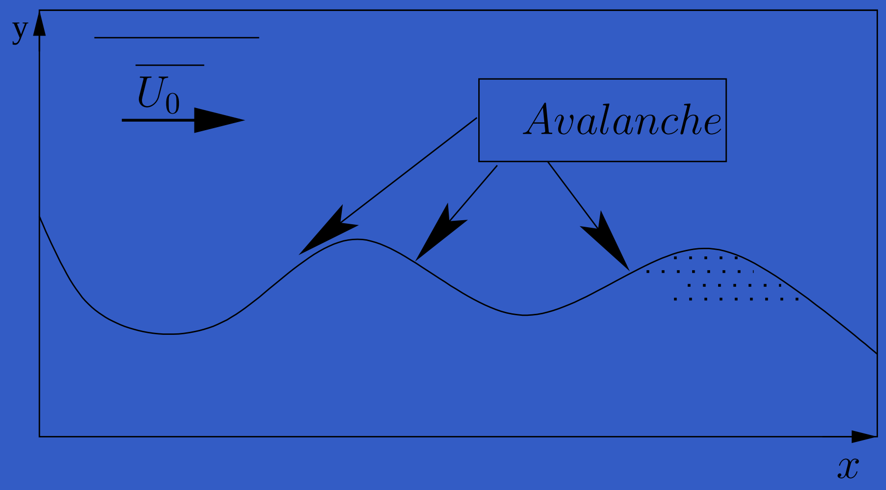
animation

conclusion

- Analytical solution of skin friction in an asymptotical framework
- Stability analysis of different flows with various linear matter flux
- Long time numerical evolution leading to coarsening

perspectives

- An full avalanche model upstream and downstream from each bump



- comparison with experiments (G. Rousseaux and H. Caps)