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Granular pressure and the thickness of a layer jamming on a rough incline

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Abstract. – Granular media with a compaction between the random loose and random close packings have very specific features. Among them is the "disorder" pressure which strongly depends on the solid fraction and is pertinent for moving media only. The concept of a granular pressure depending on the solid fraction is not unanimously accepted because dense granular media are often in a frozen state which prevents them from exploring all their possible microstates, a necessary condition for defining a pressure and a compressibility unambiguously. While periodic tapping or cyclic fluidization have already been used for that exploration, we here suggest that a succession of flowing states with velocities slowly decreasing down to zero can also be used for that purpose. And we propose to deduce the pressure in *dense and flowing* granular media from experiments measuring the thickness of the granular layer that remains on a rough incline just after the flow has jammed.

Introduction. – The existence of a pressure in granular media is the simplest way to represent their stiffness. When the concentration of grains is above the random close packing, the granular medium acts as a poro-elastic solid, its pressure is a function of the solid fraction and it involves the elastic constants of the material the grains are made of (see, *e.g.*, [1]). When the granular medium has a smaller compaction, in the range between the random loose and random close packings, the grains can be considered as rigid and the expression of the granular pressure is far less evident. The difficulty comes from the glassy behaviour which makes it quite usual to find these granular media in a *frozen* state concerning their compaction. Hence the feeling that the granular pressure is largely dependent on the way the medium was prepared. However, experiments have been conducted which aim at allowing the granular medium to reach a steady and quasi-equilibrium state concerning its solid fraction. These experiments relied on regular tappings [2,3] or cyclic fluidization [4] favouring the exploration of a maximum of microstates. A systematic exploration of the microstates can also be achieved starting from a flowing granular medium, and *slowly* reducing its velocity down to rest. As a consequence, we suggest that some of the experiments on rough plates (inclined with angle θ)

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which led to define the thickness $h_{stop}(\theta)$ which remains after the flow has jammed [5], can also be used to infer the relation between granular pressure and solid fraction.

The main features of the elastic stress tensor will be reviewed, then those of the "disorder" pressure. How these two contributions combine to build the total granular pressure is debated thereafter. We then deduce the link with the h_{stop} experiments and discuss how the experimental data must be handled to deduce an expression for the granular pressure.

The stress generated by interparticle forces and velocity fluctuations. – Consider a large number of particles with mass m_α, position $\vec{R}^\alpha(t)$ and velocity $\vec{V}^\alpha = \mathrm{d}\vec{R}^\alpha/\mathrm{d}t$, submitted to the forces $\vec{F}^{\alpha\beta}$ exerted by all other particles β. The stress tensor of this granular assembly is known to be [6]

$$\boldsymbol{\sigma} = \Big\langle \sum_\alpha \sum_{\beta\neq\alpha} \frac{\vec{R}^\beta - \vec{R}^\alpha}{2} \otimes \vec{F}^{\alpha\beta}\, \delta(\vec{x} - \vec{R}^\alpha) \Big\rangle - \\ -\Big\langle \sum_\alpha m_\alpha (\vec{V}^\alpha - \vec{V}) \otimes (\vec{V}^\alpha - \vec{V})\, \delta(\vec{x} - \vec{R}^\alpha) \Big\rangle, \tag{1}$$

where the brackets $\langle\ \rangle$ represent a statistical average while $\delta(\vec{x}-\vec{R}^\alpha)$ is a Dirac function located at the center of particle α and $\vec{V}$ is the mean velocity of the grains. For a non-cohesive granular medium the interparticle forces are null except when particle β is in contact with particle α. The above general expression was used in numerical simulations to obtain the stress tensor of a moving granular medium. In a steady shear flow with a shear $\partial V_x/\partial z$, the components σ_{xz} and σ_{zz} of the above stress tensor were found to be rate dependent, with a $(\partial V_x/\partial z)^2$ behaviour (see the second contribution in eq. (4) for example). However, above some minimal concentration of the grains, a rate-independent contribution was also found. According to Aharonov and Sparks [7], Silbert *et al.* [8], O'Hern *et al.* [9] and Head and Doi [10], this rate-independent contribution, called $p^{elastic}$ henceforth, vanishes below a value ϕ_c which is *slightly smaller than the random close packing* ϕ_M and which depends on the microscopic friction between the grains [9] as well as on the orientation of the flow relative to gravity [8]. Above this threshold packing needed to observe the elastic stiffness of the granular assembly, the elastic pressure was found to depend on the volume fraction ϕ and to obey the scaling law

$$p^{elastic} \sim E(\phi - \phi_c)^\alpha\ , \tag{2}$$

where E is the bulk modulus of the material the particles are made of, while $\alpha = 1$ for Hookean contact forces and $\alpha = 3/2$ for Hertzian ones. Due to the very large value of E, the compressibility of the granular medium is almost infinite for volume fractions less than the threshold ϕ_c and vanishingly small above it. It is thus not surprising that numerical simulations based on expression (1) of the stress tensor predict an almost uniform volume fraction $\phi \approx \phi_c$ all over the granular material. However, many experiments exist on surface flows of dense granular materials in which the solid fraction profile was observed to be non-uniform (see, for example, [11, 12] and [13]) and extended in the range between the random loose packing ϕ_m at the surface and some higher volume fraction (presumably ϕ_c) far from it. Is it possible that a second contribution exists to the rate-independent pressure, giving the flowing granular material a finite compressibility for solid fractions in the range between ϕ_m and ϕ_c?

The configuration pressure and the disorder pressure. – Consider a large volume containing many *rigid* spheres with a high enough volume fraction $\tilde{\phi}$ for a contact network to invade the whole volume. Let $\Omega(\tilde{\phi})\mathrm{d}\tilde{\phi}$ be the number of different spatial configurations of these spheres in the range between $\tilde{\phi}$ and $\tilde{\phi} + \mathrm{d}\tilde{\phi}$. To belong to $\Omega(\tilde{\phi})$, a configuration must display a large enough number of contacts, but no forces at the contact points, hence giving *no contribution to the mechanical stress* (1). The density of micro-states $\Omega(\tilde{\phi})$ is thus a *purely geometric concept*. If instead of rigid spheres we were considering soft ones, we would say we are counting the number of configurations with zero energy ("incipient" contacts), yet able to resist an infinitesimal external pressure load. This density of states presumably vanishes below a minimum compaction $\phi_{min} \simeq 0.40$ (the gel threshold) and above the maximum compaction $\phi_{max} \simeq 0.74$ (the most compact crystalline configuration). At some intermediate volume fraction the density of microstates displays a large maximum value. This intermediate compaction with the maximum number of microstates is likely to be the loosest random packing ϕ_m. According to Onoda and Liniger [14] its value for spherical grains is $\phi_m = 0.555$. Introduce now a number P which represents a *non-dimensional measure of the configuration pressure* and define the partition function

$$Z(P) = \int_{\phi_{min}}^{\phi_{max}} e^{P\tilde{\phi}}\, \Omega(\tilde{\phi})\, \mathrm{d}\tilde{\phi}.$$

The mean volume fraction is then related to the configuration pressure in the form $\phi = \frac{\mathrm{d}}{\mathrm{d}P}$ Log Z. This relation can be used to obtain both $P(\phi)$ and the variance of the density fluctuations $\langle(\tilde{\phi} - \phi)^2\rangle = \mathrm{d}\phi/\mathrm{d}P$. Another useful quantity is the *configuration entropy* $S =$ Log $Z - P\phi$. This configuration entropy is a function of the mean volume fraction and the non-dimensional configuration pressure is nothing but $P = -\frac{\mathrm{d}S}{\mathrm{d}\phi}$. Many works starting from Kanatani [15] and revived by Edwards and Oakeshott [16] strived to find an explicit form for the configuration entropy. The general trends of their results are the following: for a vanishing pressure, the mean volume fraction is the one with the maximum number of microstates, *i.e.* ϕ_m, while for an infinite pressure, the volume fraction is ϕ_{max} and the compressibility $\mathrm{d}\phi/\mathrm{d}P$ vanishes. The simplest expression meeting these conditions is the one deduced from the entropy of the lattice-gas model (see, *e.g.*, [17])

$$P \sim \mathrm{Log}\frac{\phi_{max} - \phi_m}{\phi_{max} - \phi}.$$

Note that the above pressure stems from the total number of different configurations, including both random and crystalline ones. It is also possible to select random configurations only. In this case one introduces the random close packing ϕ_M above which all configurations display some cristalline order. For spheres, it is generally admitted that $\phi_M = 0.635$. A plausible expression for the disorder pressure is $p^{disorder} \sim \mathrm{Log}(\phi_M - \phi_m)/(\phi_M - \phi)$ an expression restricted to the range $\phi_m < \phi < \phi_M$. The gradient of the disorder pressure acts as a diffusion force which pushes the grains towards regions of smaller volume fractions, those with a larger number of microstates belonging to $\Omega(\tilde{\phi})$. For the grains to have a chance to explore all microstates, the best solution is a steady flow. This is why the concept of a disorder granular pressure is pertinent for dynamic situations only. The disorder pressure confers the granular medium a compressibility which decreases when the volume fraction increases. With the above lattice-gas expression for the pressure, the compressibility is proportional to $\phi_M - \phi$, in agreement with the experimental results of Nowak *et al.* [2] but *not* with those of Schroeter *et al.* [4], which display a minimum of the compressibility for a compaction between ϕ_m and ϕ_M. Besides its ϕ-dependence, a second issue to be considered is the scaling of the disorder pressure. Since neither the grain elastic properties nor the Brownian motion is involved in $p^{disorder}$, one must discard any elastic modulus or the thermal energy as candidates. We are thus led to write the disorder pressure as

$$p^{disorder} \ = \ P_0 \,\mathrm{Log}\frac{\phi_M - \phi_m}{\phi_M - \phi}, \tag{3}$$

where P_0 is some yet undetermined characteristic pressure. This constitutive relation for the disorder pressure will hold in all circumstances. The characteristic pressure is expected to compare with the self-weight pressure that exists under several granular layers and which leads to volume fractions below ϕ_M. In contrast, the pressure load considered in soil mechanics compares with the elastic bulk modulus and leads to volume fractions higher than the random close packing. In that domain, the notion of disorder pressure is irrelevant.

The rate-independent granular pressure. – In a steady shear $\partial V_x/\partial z$ the normal stress σ_{zz} has been modelled as [18–20]

$$\sigma_{zz} = p(\phi) + \rho_p D^2 \mu_N(\phi)(\partial V_x/\partial z)^2, \tag{4}$$

where $\mu_N(\phi)$ is a function of the solid fraction which depicts the relative importance of dilatancy effects, ρ_p is the mass density of the grains and D is the grain size. We are here interested in the rate-independent contribution to the normal stress. We would like the granular pressure $p(\phi)$ to represent as far as possible all what was said above about the disorder pressure and the elastic pressure. It is clear that $p(\phi)$ would coincide with $p^{disorder}$ if ϕ_c were equal to ϕ_M. Unfortunately, this equality appears to hold only for horizontal flows [8] and frictionless grains [9]. In all other cases, ϕ_c *is slightly smaller than* ϕ_M and because of the tremendous increase of $p^{elastic}$ above ϕ_c, the granular pressure will strongly increase (and for us will diverge) at ϕ_c instead of ϕ_M (see fig. 1). For this reason we will test two different

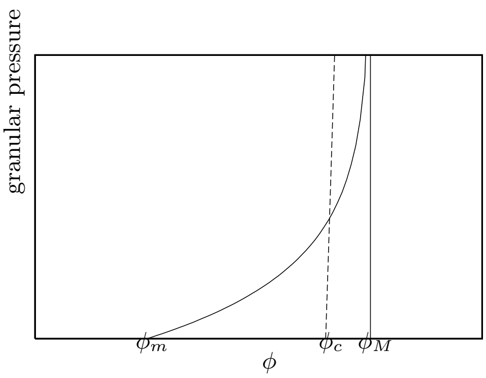

Fig. 1

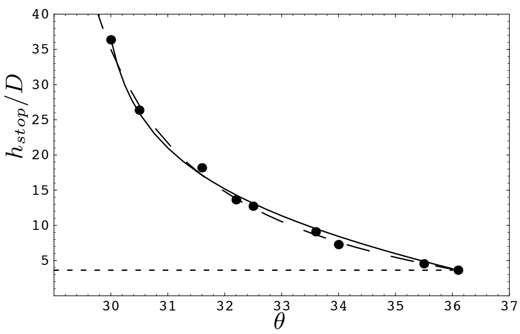

Fig. 2

Fig. 1 – Schematic representation of the elastic pressure (dashed line) and disorder pressure (plain line) as a function of the compaction in the range between the random loose packing ϕ_m and the random close packing ϕ_M. The disorder pressure vanishes below ϕ_m and diverges at ϕ_M while the elastic pressure vanishes below ϕ_c and strongly increases above.

Fig. 2 – Dependence of h_{stop}/D on the inclination θ. Comparison between experimental results (points) for sand over carpets (from [22]) and fitting curves deduced from the granular pressure (6) (plain) and (5) (dashed). Experimental results with $h_{stop}/D \leq 3.6$ were discarded and the value $\theta_{max} \approx 36.1$ was adopted.

expressions for the granular pressure:

$$p(\phi) \;=\; P_0 \,\mathrm{Log}\frac{\phi_c - \phi_m}{\phi_c - \phi} \tag{5}$$

and

$$p(\phi) \;=\; P_0 \,\frac{(\phi - \phi_m)^n}{(\phi_c - \phi)^m}, \tag{6}$$

with positive but yet unknown exponents m and n for the second expression. Note that these expressions for the rate-independent granular pressure hold in the very small solid fraction range $\phi_m < \phi < \phi_c$, and for flowing media only. Expression (5), with ϕ_M instead of ϕ_c, was already adopted in previous works [18–20]. We now acknowledge this replacement of ϕ_c by ϕ_M was not justified and we want to test the pertinence of the granular pressure (5) or (6) in an unsteady flow that leads to jamming.

The maximum thickness of a granular layer that jams on a rough incline. – Consider a layer of granular material flowing down a rough inclined plate. Upon gently reducing the inclination with a constant thickness h, the layer ultimately stops at some angle θ. Since the flow velocity was slowly reduced to zero, the granular layer had time to explore a lot (if not all) of the microstates involved in $p^{disorder}$. It is thus likely that the peculiar jammed state which the layer arrives at is described by the rate-independent pressure p defined above. The mechanical equilibrium of the *freshly* jammed layer is thus expressed by

$$0 = -\frac{\partial p}{\partial z} + \phi \rho_p g \cos\theta \quad \text{and} \quad \tan\theta \;=\; \min\left[\mu(z)\right], \tag{7}$$

where the z-axis is orthogonal to the free surface of the layer and points downwards while μ is the macroscopic friction coefficient (which is possibly non-uniform over the layer thickness), and g is the acceleration of gravity. Substituting expression (5) or (6), one deduces that the solid fraction profile $\phi(z)$ increases from ϕ_m at the free surface up to values close to ϕ_c at a depth of order L with

$$L = \frac{P_0}{\phi_c \rho_p g \cos\theta}\,. \tag{8}$$

Concerning the particular expression (5), the whole profile is exponential-like and given by

$$\phi(z) = \frac{\phi_c}{1 + (\frac{\phi_c}{\phi_m} - 1)e^{-\frac{z}{L}}}\,. \tag{9}$$

A similar exponential profile was already observed in experiments [11–13] and was also obtained in simulations of a frustrated lattice-gas model [21]. On the contrary, recent numerical simulations [8] which do not see any rate-independent pressure below ϕ_c predicted a flat profile at this volume fraction. We come back to this discrepancy in the concluding section.

Knowing the compaction profile, let us now focus on the layer thickness h. Da Cruz [22] has deduced from numerical simulations of dense flows a very important (and a bit counter-intuitive) result: the macroscopic friction coefficient *decreases* almost linearly with the solid fraction and can be written as

$$\mu \;=\; \mu_{max} - (\mu_{max} - \mu_{min})\,\frac{\phi - \phi_m}{\phi_c - \phi_m}. \tag{10}$$

Since the compaction increases with the distance from the free surface, *the minimum value of* μ *happens very close to the rough plate*, that is to say for $z \simeq h$. Just after jamming, we thus have: $\tan\theta = \mu_{max} - (\mu_{max} - \mu_{min}) \frac{\phi(h_{stop})-\phi_m}{\phi_c-\phi_m}$. For the special expression (5) and related profile (9), one obtains a rather simple expression,

$$\frac{h_{stop}}{L} = \mathrm{Log}\left[1 + \frac{\phi_c}{\phi_m}\frac{\mu_{max} - \tan\theta}{\tan\theta - \mu_{min}}\right] . \tag{11}$$

When the granular pressure is given by (6) there is no simple analytical result for h_{stop} but two asymptotic results:

$$\frac{h_{stop}}{L} \approx \left(\frac{\mu_{max} - \mu_{min}}{\tan\theta - \mu_{min}}\right)^m \qquad \text{and} \qquad \frac{h_{stop}}{L} \approx \frac{\phi_c}{\phi_m}\left(\frac{\mu_{max} - \tan\theta}{\mu_{max} - \mu_{min}}\right)^n , \tag{12}$$

which hold when $\tan\theta$ is slightly larger than μ_{min} and slightly smaller than μ_{max}, respectively.

Experimental results. – Systematic measurements of the layer thickness were initiated by Pouliquen [5] who obtained the thickness $h_{start}(\theta)$ for an initially static layer and $h_{stop}(\theta)$ for an initially flowing layer. Since the disorder pressure is relevant to moving media only, we are interested in h_{stop} exclusively. The experimental results were fitted with [5, 22]

$$\frac{h_{stop}}{BD} = \mathrm{Log}\left[\frac{\mu_{max} - \mu_{min}}{\tan\theta - \mu_{min}}\right] \qquad \text{or} \qquad \frac{h_{stop}}{BD} = \frac{\mu_{max} - \tan\theta}{\tan\theta - \mu_{min}} , \tag{13}$$

where D is the grain size while B is a number to be deduced from experiments. To deduce the *bulk* granular pressure from these experiments on rough inclines, one must discard experiments performed with relatively smooth plates, and more generally those for which the curves $h_{stop}(\theta)$ are strongly modified upon changing the plate roughness. And concerning those with a high enough roughness, we must exclude some boundary layer of thickness δ and consider $h-\delta$ as the relevant thickness for bulk behaviour. Accordingly, we were led to discard all the experiments performed with glass beads because the friction generated by the beads glued on the incline is only slightly larger than the friction in the bulk. But we considered as significative the experiments with sand flowing on carpets of various roughnesses. And for these experiments with sand we discarded a boundary layer with thickness estimated to $\delta \simeq 4D$. Because of the scarcity of data, we could not fully discriminate between expression (5) and expression (6) with $m = 1$ and $n = 1$ (see fig. 2). However, it was possible to give an order of magnitude to the reference pressure P_0. If it can be taken for granted that h_{stop} scales with the grain size D, then the reference pressure must scale with $\rho_p g D$ and in fact, the fit with experimental data of sand over carpets was obtained with

$$P_0 \approx 5\,\rho_p g D , \tag{14}$$

together with $\mu_{max} \approx 0.7$ and $\mu_{min} \approx 0.5$.

Conclusions. – A rate-independent granular pressure depending on solid fraction was proposed long ago for poro-elastic media [1]. It corresponds to $p^{elastic}$ and vanishes below the threshold solid fraction ϕ_c. The statistical properties of granular materials in the range between the random loose packing ϕ_m and the random close packing ϕ_M [15, 16] can be represented by a second rate-independent pressure $p^{disorder}$. Since $\phi_m < \phi_c \leq \phi_M$, we proposed that granular media with solid fraction in the range $\phi_m < \phi \leq \phi_c$ can be endowed with a granular pressure $p(\phi)$ similar to $p^{disorder}$ but diverging at ϕ_c instead of ϕ_M. It must be stressed that for $p^{disorder}$ to be observed, the granular material must be able to explore all its microstates. This is certainly the case of *steady flows* and in fact, with expression (5) for the granular pressure, we have found already very good agreement with the experimental velocity and solid fraction profiles observed in steady free surface or confined flows [19, 20]. In the present letter we assumed that the exploration of microstates can also be achieved in *unsteady flows* provided their velocity is smoothly decreasing down to zero and we proposed to infer the granular pressure from measurements of $h_{stop}(\theta)$. With expression (5) we predicted result (11) which fits quite well with the experimental results. Moreover the scaling of h_{stop} with the grain size D has confirmed our previous guess [19, 20]: the granular pressure (and $p^{disorder}$) scales with $\rho_p g D$. We have thus many reasons to believe in the existence of a granular pressure of entropic origin in the range $\phi_m < \phi \leq \phi_c$. Thereby we do not understand why up to date simulations are unable to observe $p^{disorder}$ and the related finite compressibility for solid fractions between ϕ_m and ϕ_c. We have, however, a suggestion. As observed in some experiments (*e.g.*, [23]), the mean velocity and the granular temperature reach their steady state *much more rapidly* than the solid fraction. It is thus possible that the number of time-steps performed in the simulations is not large enough to let the solid fraction reach its true steady profile.

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REFERENCES

[1] Baer M. R. and Nunziato J. W., *Int. J. Multiphase Flow*, **12** (1986) 861.
[2] Nowak E. R., Knight J. B., Ben-Naim E., Jaeger H. M. and Nagel S. R., *Phys. Rev. E*, **57** (1998) 1971.
[3] Philippe P. and Bideau D., *Europhys. Lett.*, **60** (2002) 677.
[4] Schroeter M., Goldman D. I. and Swinney H. L., *Phys. Rev. E*, **71** (2005) 030301.
[5] Pouliquen O., *Phys. Fluids*, **11** (1999) 542-548.
[6] Irving J. H. and Kirkwood J. G., *J. Chem. Phys.*, **18** (1950) 817.
[7] Aharonov E. and Sparks D., *Phys. Rev. E*, **60** (1999) 6890.
[8] Silbert L. E., Ertas D., Grest G. S., Halsey T. C., Levine D. and Plimpton S. J., *Phys. Rev. E*, **64** (2001) 051302.
[9] O'Hern C. S., Silbert L. E., Liu A. J. and Nagel S. R., *Phys. Rev. E*, **68** (2003) 011306.
[10] Head D. and Doi M., preprint (2005).
[11] Clement E. and Rajchenbach J., *Europhys. Lett.*, **16** (1991) 133.
[12] Rajchenbach J., *Adv. Phys.*, **49** (2000) 229.
[13] Ancey C., *Phys. Rev. E*, **65** (2001) 011304.
[14] Onoda G. Y. and Liniger E. G., *Phys. Rev. Lett.*, **64** (1990) 2727.
[15] Kanatani K., *Lett. Appl. Eng. Sci.*, **18** (1980) 989.
[16] Edwards S. F. and Oakeshott R. B., *Physica A*, **157** (1989) 1080.
[17] Witten T. A., *Structured Fluids* (Oxford University Press) 2004, sect. 2.
[18] Savage S. B., *J. Fluid. Mech.*, **377** (1998) 1.
[19] Josserand C., Lagrée P.-Y. and Lhuillier D., *Eur. Phys. J. E*, **14** (2004) 127.
[20] Josserand C., Lagrée P.-Y. and Lhuillier D., *Trends in Applications of Mathematics in Mechanics*, edited by Wang Y. and Hutter K. (Shaker Verlag, Aachen) 2005, pp. 157-164.
[21] Nicodemi M., Coniglio A. and Herrmann H. J., *J. Phys. A*, **30** (1997) L379.
[22] Da Cruz F., *Ecoulements de grains secs: frottement et blocage*, PhD Thesis, ENPC 2004. Available from `http://pastel.paristech.org/archive/00000946/`.
[23] Charru F., Mouilleron H. and Eiff O., *J. Fluid Mech.*, **519** (2004) 55.