Continuum-mechanical description of a dense, sheared, granular matter

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- Steady flows, simple situations: flow over an inclined planes, rotating drum (invariant by translation) for free surface or Couette flow for confined situations.
- Volume fraction domain of validity: $\phi_m < \phi < \phi_M$

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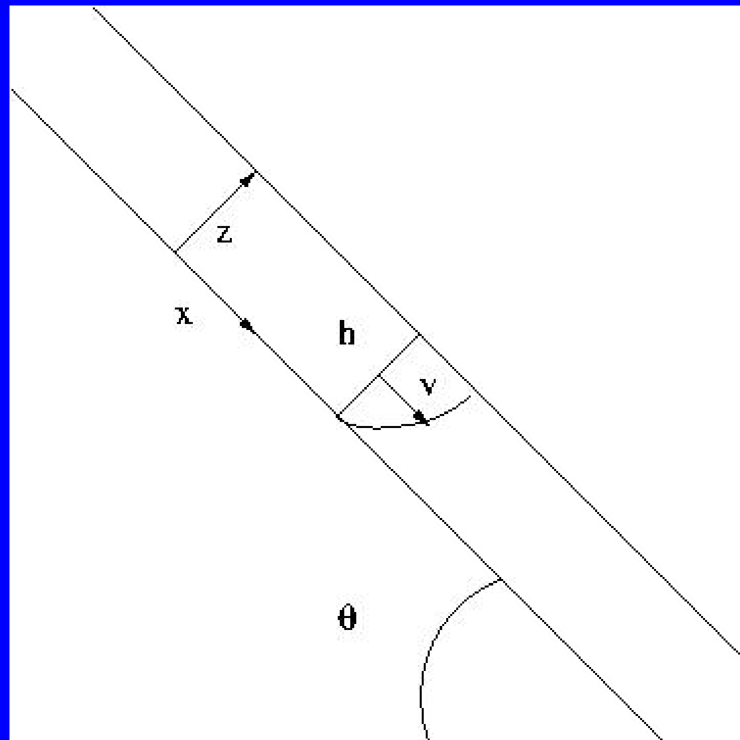
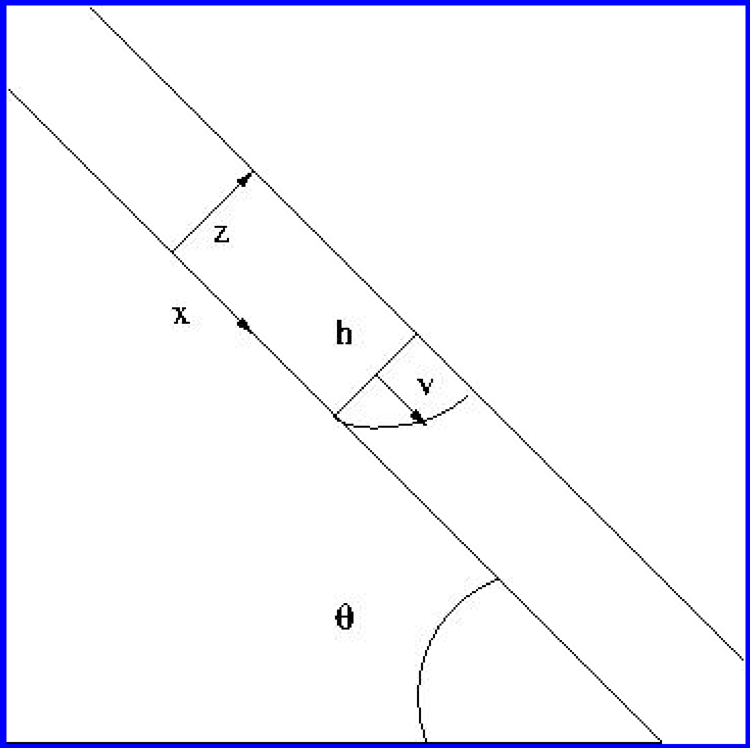
No further assumptions are made here, such as constant density (no dilatance) or Saint-Venant approach.


z
x
h
v
θ



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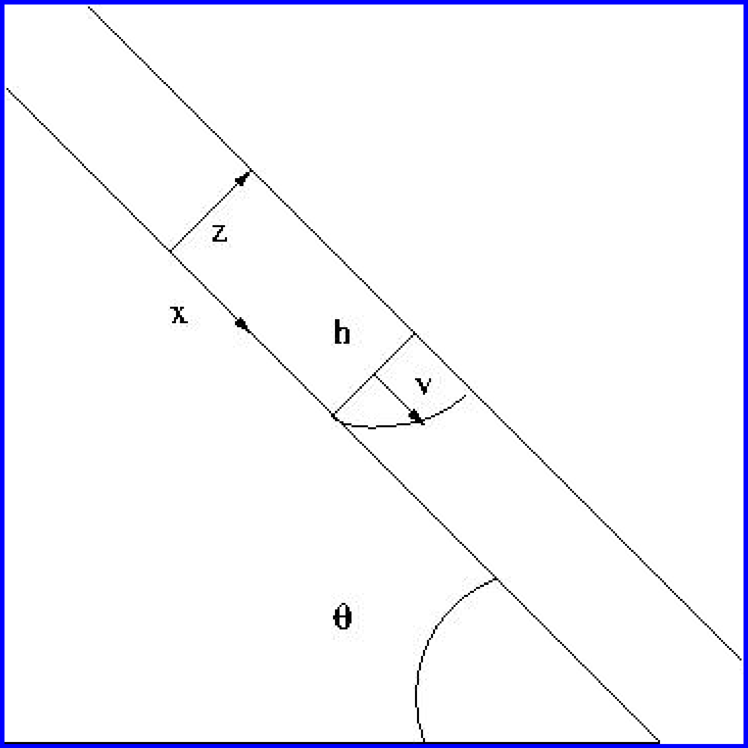

$$\frac{\partial \tau_{xz}}{\partial z} + \phi \rho g \sin(\theta) = 0$$

$$\frac{\partial \tau_{zz}}{\partial z} - \phi \rho g \cos(\theta) = 0$$

Stress description

Normal stress: compaction term + rate-dependent impact stress (Bagnold):

$$\tau_{zz} = -\rho D^2 \mu_N(\phi) \left(\frac{dV}{dz} \right)^2 - \rho g D F(\phi) cos(\theta) \qquad (1)$$

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Shear stress: Coulomb like contribution + shear term:

$$\tau_{xz} = \rho D^2 \mu_T(\phi) \left(\frac{dV}{dz} \right)^2 - \mu(\phi) \tau_{zz} \qquad (2)$$

Boundary conditions

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For confined flows, the density at the boundaries is determined by the knowledge of the applied normal stress.

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- F, μ_T and μ_N diverge as $\phi \longrightarrow \phi_M$
- F and μ_N have to be zero for ϕ_m (zero normal stress tensor at the free surface)
- μ is the only parameter which neither diverges nor annihilates and is believed to have a smooth behaviour. The value of $\mu_T(\phi_m)$ is not prescribed *a priori*.

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$$\frac{D}{g}\left(\frac{dV}{dz}\right)^2 = \frac{F^*}{\mu_T}(sin(\theta) - \mu cos(\theta)) \tag{4}$$

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$$F^*(\phi, \theta) = \frac{F}{1 - \frac{\mu_N}{\mu_T}(tan(\theta) - \mu)}$$

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- Need to know only four coefficient functions F, μ, μ_N and μ_T (from experiments)!

Simple cases

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$F = F_0 \tilde{F}(\tilde{\phi})$ $\mu_T = \mu_0 \tilde{\mu}_T(\tilde{\phi})$

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From experiments (Pouliquen), we have deduced:
$\mu \simeq 0.3$ and $\mu_N/\mu_T \simeq 0.15$.

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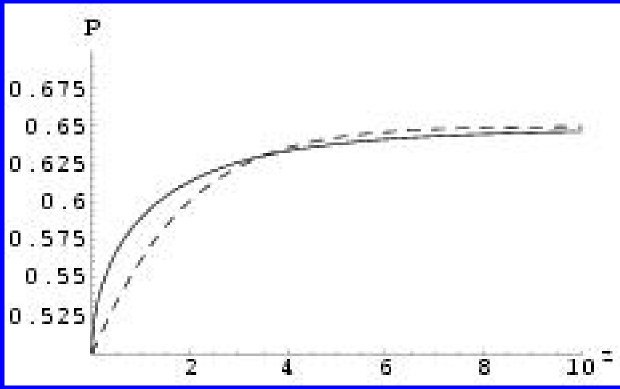
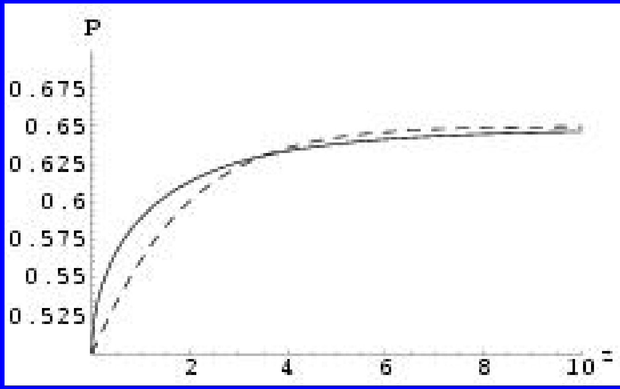
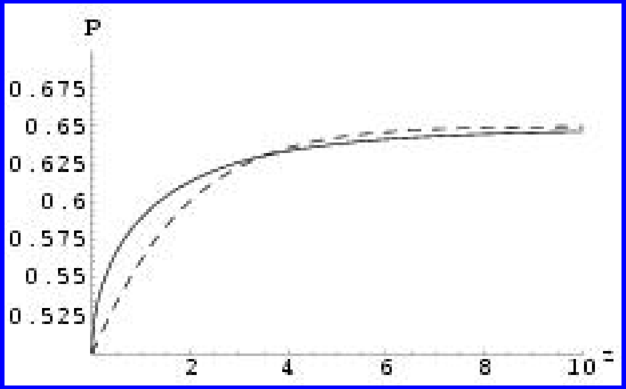
The volume fraction profile is unique when plotted as function of $\tilde{y} = \tilde{h} - \tilde{z}$ ($\alpha = 2$, $\beta = 0.5.\alpha = 2$)


P
0.675
0.65
0.625
0.6
0.575
0.55
0.525
2
4
6
8
10



We have in fact an exact solution for the Log:

$$\frac{1}{\phi_2(\tilde{y})} = \frac{1}{\phi_M} + (\frac{1}{\phi_m} - \frac{1}{\phi_M})e^{\phi_M y}$$



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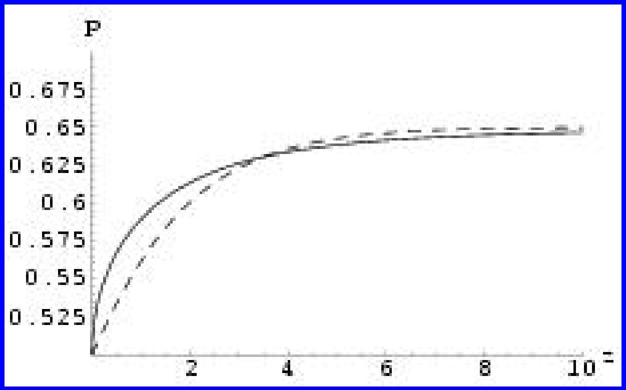

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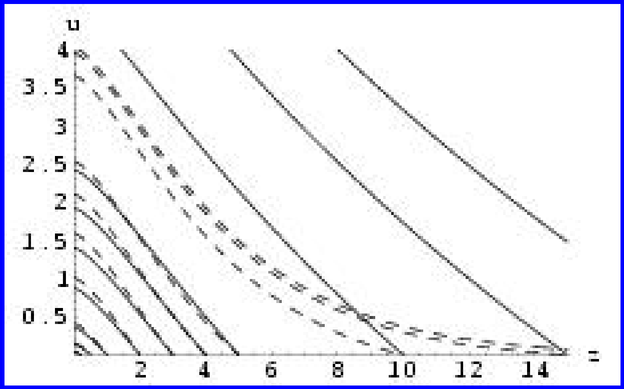
while the algebrical dependance gives:

$$\tilde{\phi} \propto \tilde{y}^{\frac{1}{\alpha}} \quad \tilde{y} \to 0 \quad \tilde{\phi} \sim 1 - \tilde{y}^{-\frac{1}{\beta}} \quad \tilde{y} \to \infty$$

Velocity profiles

For the algebrical model, we have taken $\gamma = 0$ and $\delta = 1$.



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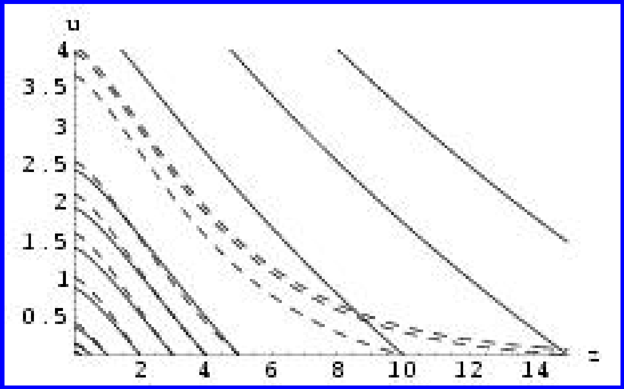

The two models show different regimes: one reaches an asymptotic velocity profile while the other one show no saturation of the velocity.

Such behaviours can all be obtained with the algebrical model, with a proper choice of the exponents.

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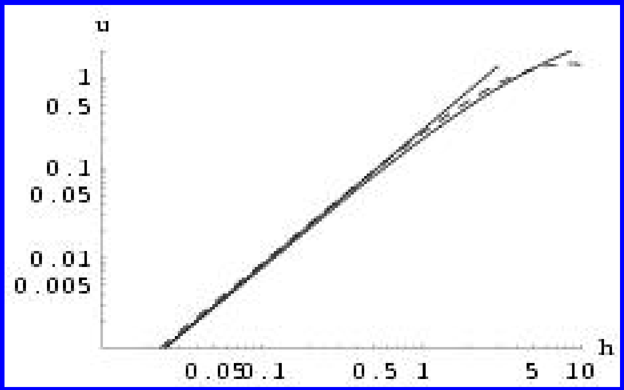

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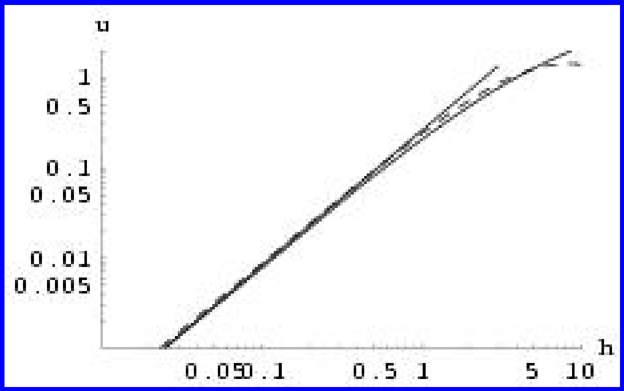

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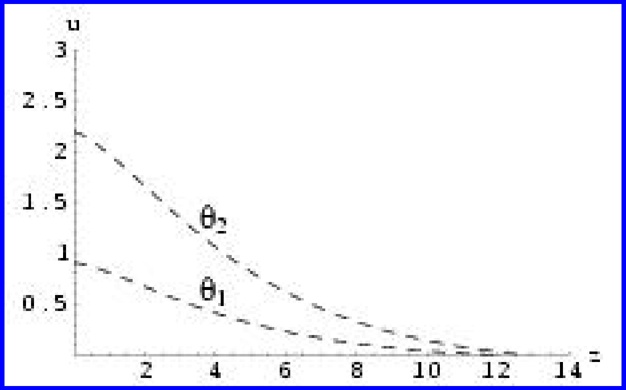
For the logarithmic model:



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- The approach adapts easily to confined shear flows: the boundary conditions account for given external normal and shear stresses. The localization of a sheared layer is obtained.

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