HS fluid

P.-Y. Lagrée
CNRS & UPMC Univ Paris 06, UMR 7190,
Institut Jean Le Rond ∂'Alembert, Boîte 162, F-75005 Paris, France
pierre-yves.lagree@upmc.fr ; www.lmm.jussieu.fr/∼lagree

August 7, 2014

0.1 Viscosity

The viscosity is a function of the second principal invariant of the shear strain rate tensor:

$$|D| = \sqrt{\sum_{i,j} D_{ij}D_{ij}}$$

where $D_{ij} = (\partial_i u_j + \partial_j u_i)/2$, it has unit of s^{-1} We use a general Herschel-Bulkley formulation

$$\tau = \tau_y + \mu_N \dot{\gamma}^N$$

which is coded with an effective viscosity of the form:

$$\mu(|D|) = \frac{\tau_y}{2|D|} + \mu_N |D|^{N-1}.$$

Where τ_y is the yield stress: $\tau_y = 56$Pa, $mu_N = 37 Pas^{0.325}$ and the index $N = 0.325$ (note that μ_N has strange units).

0.2 Remarks

The form $\mu(|D|) = \frac{\tau_y}{2|D|} + \mu_N |D|^{N-1}$ gives $\mu(|D|) = \mu_1$ when $N = 1$ and $\tau_y = 0$, the standard viscosity for a Newtonian fluid. The "2" comes from the definition of the relation between rate of deformation and stress: $\underline{\underline{\tau}} = 2\mu \underline{\underline{D}}$ in the tensorial definition.

0.3 Geometry

The problem is a heap of viscoplastic material of height 5cm of length 30cm released on a slope of length 70cm. At this position begins a small step of depth 1cm. The fluid has a density $\rho_0 = 1357 kgm^{-3}$

0.4 Non Dimensional problem

The Navier Stokes equations with Herschel-Bulkley are:

$$\rho_0 \frac{d\overrightarrow{u}}{dt} = -\overrightarrow{\nabla} p - \rho_0 \overrightarrow{g} + \overrightarrow{\nabla} \cdot \underline{\underline{\tau}}$$

with

$$\underline{\underline{\tau}} = (\frac{\tau_y}{2|D|} + \mu_N |D|^{N-1}) \underline{\underline{D}}$$

Using $x = L\bar{x}$, $y = L\bar{y}$, and $(u, v) = U_0(\bar{u}, \bar{v})$ we have $p = \rho_0 U_0^2 \bar{p}$ and $\underline{\underline{D}} = (U_0/L))\underline{\underline{\bar{D}}}$, of course, $t = (L/U_0)\bar{t})$. Let us write the viscous part:

$$\underline{\underline{\tau}} = \rho_0 U_0^2 \left[(\frac{\tau_y}{\rho_0 U_0^2 |\bar{D}|} + 2\frac{\mu_N}{\rho_0 U_0^2} (\frac{U_0}{L})^N |\bar{D}|^{N-1}) \underline{\underline{\bar{D}}} \right]$$

or

$$\underline{\underline{\tau}} = \rho_0 U_0^2 \left[(\frac{\tau_y}{\rho_0 U_0^2 |\bar{D}|} + 2\frac{\mu_N}{\rho_0 U_0 L} (\frac{U_0}{L})^{N-1} |\bar{D}|^{N-1}) \underline{\underline{\bar{D}}} \right]$$

or finally

$$\underline{\underline{\tau}} = \rho_0 U_0^2 \left[(\frac{\bar{\tau}_y}{|\bar{D}|} + \frac{2}{Re_N} |\bar{D}|^{N-1}) \underline{\underline{\bar{D}}} \right]$$

with the yield stress without dimension $\bar{\tau}_y = \dfrac{\tau_y}{\rho_0 U_0^2}$ and a kind of Reynolds number $Re_N = \frac{\rho_0 U_0 L}{\mu_N} (\frac{U_0}{L})^{1-N}$. Hence, with these two parameters the final Navier Stokes is:

$$\frac{d\bar{u}}{d\bar{t}} = -\overrightarrow{\nabla} p - \frac{gL}{U_0^2} \overrightarrow{e}_y + 2\overrightarrow{\nabla} \left[(\frac{\bar{\tau}_y}{2|\bar{D}|} + \frac{1}{Re_N} |\bar{D}|^{N-1}) \underline{\underline{\bar{D}}} \right]$$

of course, if $\tau_y = 0$ and $N = 1$

$$2\overrightarrow{\nabla} \left[(\frac{\bar{\tau}_y}{2|\bar{D}|} + \frac{1}{Re_N} |\bar{D}|^{N-1}) \underline{\underline{\bar{D}}} \right] = 0 + 2\overrightarrow{\nabla} \left[\frac{1}{Re_1} \underline{\underline{\bar{D}}} \right] = \frac{1}{Re_1} \overrightarrow{\nabla}^2 \bar{u}$$

0.5 Non Dimensional problem, choice of U_0, L...

Up to now, it is just a matter of choice.

Let us say we take first $L = 1m$ as longitudinal scale, $x = L\bar{x}$. So the step is for $\bar{x} = 0.7$ We need a time scale T. This is a collapse so that it is natural to put to one in front of the gravity: $\frac{gL}{U_0^2} = 1$. This means that the scales of length and time $L/T^2 = g$, then $T = \sqrt{L/g}$, *i.e.* 0.32s. The velocity scale is $U_0 = L/T$ which is $\sqrt{gL}$, the gravitational term is 1.

Maybe, you have an imposed velocity V_0 in your experiment. So at this point:
• either you use $T = \sqrt{L/g}$, and L, so that the velocity scale L/T is $\sqrt{gL}$ and then the imposed velocity is $V_0/\sqrt{gL}$.
• either you use this velocity as scale, so that V_0 is the fundamental scale and then the time unit is $T = L/V_0$. You then have to change the gravity, which is without dimension is gL/V_0^2.

Anyway, we have now a scale for time T, a scale for length L and a then a velocity L/T

Let us now make τ_y non dimensional, it is simply $\tau_y/((\rho_0(L/T)^2)$, in my case $\tau_y = 56Pa$, $\rho_0 = 1357kg/m^3$, $L = 1m$, $T = \sqrt{L/g}$

$$\tau_y/(\rho_0 gL) = 0.004206$$

Maybe in your case, it is better to have a unit caracteristic velocity and $\tau_y/(\rho_0 V_0^2)$ is the good non dimensional yield.

The parameter $Re_N = \frac{\rho_0 U_0 L}{\mu_N}(\frac{U_0}{L})^{1-N}$ has to be computed now. with my values it is $Re_N = 248$.

0.6 Script

The *Gerris* script is

```
Define L0 1.001
Define alph 20./180*3.1415
Define rhof 0.001
Define nuf 0.0001
Define Mumax 250
Define tauy 0.004206
Define ReN   248
Define N 0.325

(...)
```

```
  SourceViscosity {} {
    double Mu1 = nuf*rhof;
    if (T > 0 && D2 > 0.) {
       Mu1 = (tauy/(2*D2) + 1./ReN*pow(D2,N-1));
       Mu1 = MIN(Mu1,Mumax);
    }
    // Classic trick: use harmonic mean for the dynamic viscosity
    return 1./(T/Mu1 + (1. - T)/nuf/rhof);
   } {
    beta = 1
    tolerance = eps
   }

PhysicalParams { alpha = 1./(T + (rhof)*(1. - T)) }


(...)


   Source {} V  -cos(alph)
   Source {} U   sin(alph)
```