1 Unsteady boundary layer

1.1 Unsteady boundary layer flow over a semi infinite flat plate impulsively started

Reintroducing the time in the boundary layer equation seems a simple task, the convective time scale reintroduces $\partial/\partial t$. We show a first example which is simple (Stewartson 51 et 73, Smith 70 & 72 et Hall 69). At time $t = 0$ a semi infinite flat plate is impulsively put in motion. We are in the framework of the plate, so that the Ideal Fluid response is instantaneously $u_e = 1$ (the plate slips in the ideal fluid). One has only to introduce the time derivative in the boundary layer equations :

$$\begin{cases} \dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} = 0, \\ \dfrac{\partial u}{\partial t} + u\dfrac{\partial u}{\partial x} + v\dfrac{\partial u}{\partial y} = \dfrac{\partial^2 u}{\partial y^2}, \\ u(x,0,t) = v(x,0,t) = 0, \\ u(x, y > 0, t = 0) = 1 \\ v(x, y > 0, t = 0) = 0 \\ \text{and } u(x, \infty, t > 0) = 1. \end{cases} \tag{1}$$

At a fixed position x we observe for short times the Rayleigh flow (or Stokes first problem):

$$\partial_t u = \partial_y^2 u; \quad u(y > 0, t = 0) = 1, u(0, t) = 0, u(y \to \infty, t) = 1$$

And we guess that for a long time, at a given x, the flow will finally be steady, $\partial u/\partial t = 0$, we will recover the Blasius flow. The good variable is $\tau = t/x$. Depending if it is small or large, we go from Rayleigh to Blasius. Transition occurs for $\tau = 1$, this time correspond to the time necessary so that information which travels at velocity 1 arrives at the considered point.

The solution is numerically computed on figure 1, we use simple finite difference technique.

For $1.5 < \tau < 4$, the difference between the two régimes is noticeable. We see it on the figure 1 (first obtained by Hall 69 with a specific method using similarity variables and valid for $\tau \geq 1$), we plot on this figure $\frac{\partial u(x,y=0,t)}{\partial y}\sqrt{x}$ so that

$$\tau >> 1 \quad \tau_w = .332/\sqrt{x}, \qquad \delta_1 = 1.732\sqrt{x}; \text{ and for } \quad \tau \leq 1 \quad \tau_w = 1/\sqrt{\pi t}, \qquad \delta_1 = 2\sqrt{\frac{1}{\pi t}}.$$

On the next figure we plot $2\sqrt{\frac{1}{\pi}} - \delta_1\sqrt{\frac{1}{t}}$, which is 0 for Rayleigh solution ($\tau \leq 1$) and which is function of τ in the Blasius case ($2\sqrt{\frac{1}{\pi}} - 1.732\sqrt{\frac{1}{\tau}}$, expression valid for $\tau >> 1$.

The analytic study of the problem of the transition between the two régimes is difficult. Stewartson had to do two papers (51 & 73) to solve it. The difficulty comes because there is an "essential singularity" in the developments around $\tau = 1$, it means that all the terms of the Taylor expansion are zero (just like e^{-x^2}, this function has no Taylor expansion in $x = 0$).

1.2 Unsteady boundary layer flow over a semi infinite flat plate impulsively started, integral point of view

The unsteady system may be written in integral form ($\partial_x u = -\partial_y v$),

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \frac{\partial u}{\partial t} + \frac{\partial u^2}{\partial x} + \frac{\partial vu}{\partial y} =$$
$$\frac{\partial u}{\partial t} + \frac{\partial (u^2 - u)}{\partial x} + \frac{\partial u}{\partial x} + \frac{\partial vu}{\partial y} =$$
$$\frac{\partial u}{\partial t} + \frac{\partial (u^2 - u)}{\partial x} + \frac{\partial (v(u-1))}{\partial y} = -\frac{\partial^2 u}{\partial y^2},$$

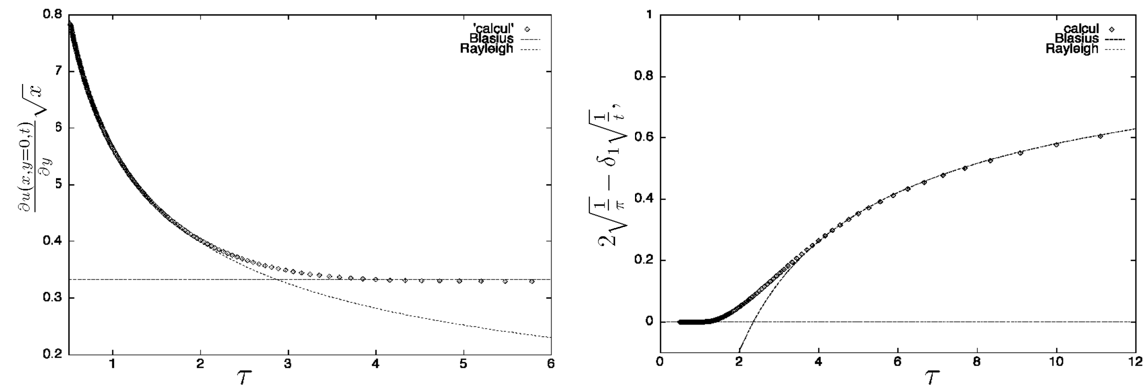

Figure 1: Unsteady numerical solution in finite differences of the unsteady boundary layer equation. We observe the transition from Rayleigh infinite flat plate impulsive solution to the Blasius steady solution. Left, shear times $\sqrt{x}$ at the wall, from Rayleigh, at small τ, to the constant Blasius value. Right, plot of $2\sqrt{\frac{1}{\pi}} - \delta_1\sqrt{\frac{1}{t}}$, (points) compared to the Balsius value $2\sqrt{\frac{1}{\pi}} - 1.732\sqrt{\frac{1}{\tau}}$, line, as a function of τ

were we have defined the displacement thickness, the momentum thickness and the shape factor

$$\delta_1 = \int_0^\infty (1-u)dy, \quad \delta_2 = \int_0^\infty u(1-u)dy \text{ and } H = \frac{\delta_1}{\delta_2},$$

and defining a function f_2 linked to the skin friction as: $\frac{\partial u}{\partial y} = f_2 \frac{H}{\delta_1}$. Then by integration, and by boundary condition in 0 and ∞

$$\frac{\partial}{\partial t}\delta_1 + \frac{\partial}{\partial x}\frac{\delta_1}{H} = \frac{f_2 H}{\delta_1}$$

We see a convection equation $\partial_t \delta_1 + H^{-1}\partial_x \delta_1$, of velocity $1/H$. This velocity is the velocity of propagation of the information of the existence of the leading edge of the semi infinite flat plate.

For small time, at a given position x from the nose, we are in the Rayleigh-Stokes problem: there is up to now no information that the plate is not infinite ∂_x is zero, we have only

$$\frac{\partial}{\partial t}\delta_1 = \frac{f_2 H}{\delta_1}$$

which gives the square root behavior of δ_1 in time

$$\delta_1 = \sqrt{2 f_2 H}\sqrt{t}$$

using the closure, this gives $f_2 = 0.22$, $H = 2.59$ and$\delta_1 = 1.06\sqrt{t}$ (Stokes value 1.12)

For long time, at a given position x from the nose, we are in the Blasius problem: there is no more the unsteady ∂_t term, we have only

$$\frac{\partial}{\partial x}\frac{\delta_1}{H} = \frac{f_2 H}{\delta_1}$$

which gives the square root behavior of δ_1 in space

$$\delta_1 = \sqrt{2 f_2} H \sqrt{x}$$

using the closure, this gives $f_2 = 0.22$, $H = 2.59$ and $\delta_1 = 1.7\sqrt{x}$ (Blasius value 1.732)

Of course, we see that if $\tau = t/x$, then we go for small τ from $\delta_1 = \sqrt{2 f_2 H}\sqrt{t}$ to $\delta_1 = \sqrt{2 f_2} H\sqrt{x}$ at large τ. The propagation of the information of the existence of the leading edge of the plate is at velocity $1/H$. As $H \simeq 2.6$, we obtain the same estimate than previously on τ when solving the full problem.

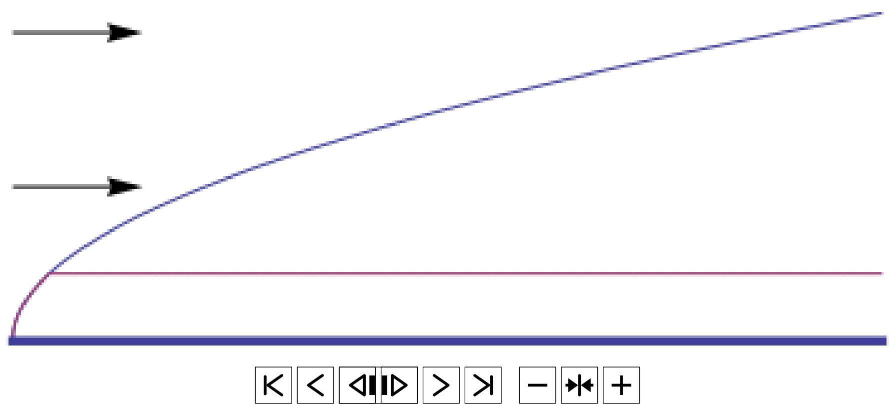

Figure (moovie): Boundary layer formation on an impulsively started semi infinite flat plate, the given external velocity is 1, solution obtained from equation $\frac{\partial}{\partial t}\delta_1 + \frac{\partial}{\partial x}\frac{\delta_1}{H} = \frac{f_2 H}{\delta_1}$ at small times the displacement thickness increases with $\sqrt{t}$ at large time it increases in $\sqrt{x}$ t from 0.1 to 2.5. [click to launch the movie, QuickTime Adobe/ Reader required].

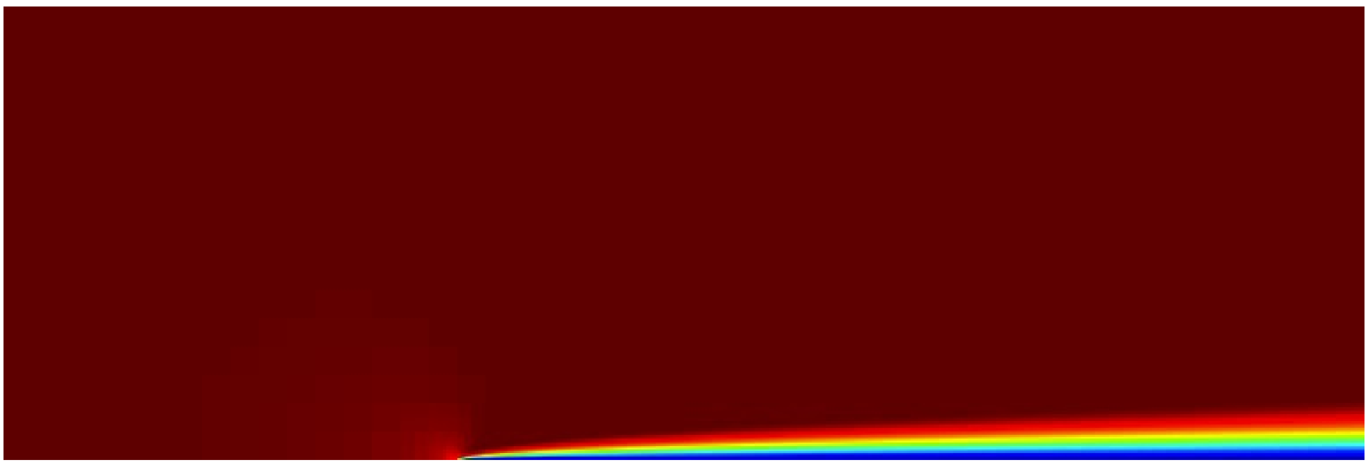

Figure 2: (moovie): Boundary layer formation on an impulsively started semi infinite flat plate, the given external velocity is 1, solution obtained from *Gerris* [click to launch the movie, QuickTime Adobe/ Reader required]..

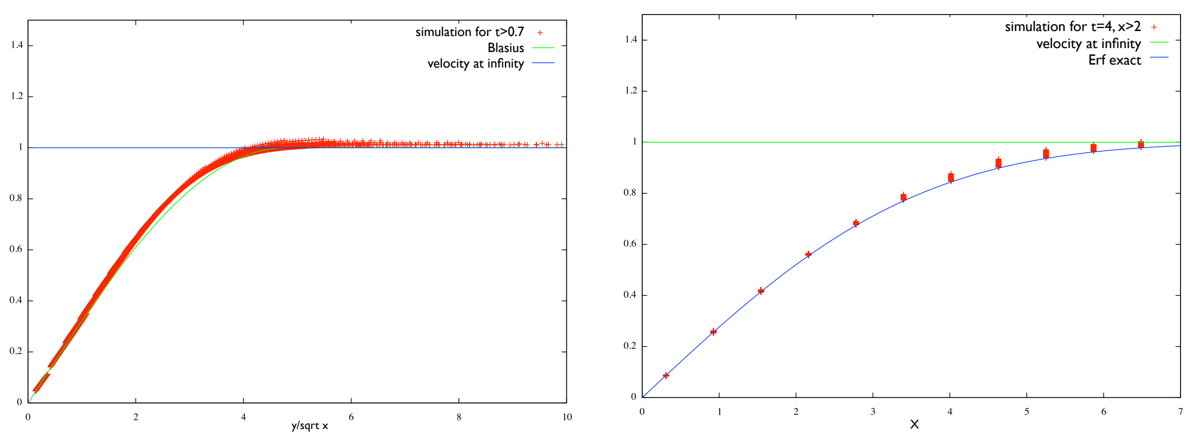

Figure 3: Navier Stokes computation by *Gerris* at $Re = 1000$, left we have the selfsimilar Blasius profile (superposition of several profiles plotted with $\bar{y}(Re/\bar{x})^{1/2}$). Right the erf solution.

2 Annex 3: Navier Stokes

Navier Stokes computation with *Gerris*

```
#######################################################################
# Blasius par PYL, sauver dans "blasius.gfs"
# lancer avec
# gerris2D -DRe=1000. blasius.gfs | gfsview2D v.gfv
# 29/09/10
# valeur du Reynolds
#Define Re 100000.
# definition de 3 boites avec 2 connections
3 2 GfsSimulation GfsBox GfsGEdge{
# met le coin gauche decalle - > paque 2 est en 0,0
    x = -0.5 y = 0.5  } {
  SourceViscosity {} 1./Re
  PhysicalParams { L = 2 }
#  Time { end = 1000  dtmax = 0.001}
#   precision 2**(-4.) = 1/16=0.06   5-> 32  0.03  6 ->  0.015625 2**(-8.) = 0.00390625   pr 2**(-8
   Refine 6
# temps initial 0
  Init {} { U = 1
            V = 0 }
  Init {istep = 1}{
    dyU = dy("U"); }
# AdaptGradient { istep = 1 } { cmax = .1 maxlevel = 6 } U
  AdaptVorticity { istep = 1 } { maxlevel = 8 minlevel = 4 cmax = 1e-2 }
# sortie tous les 20 pas de calculs du temps en cours
  OutputTime { istep = 20 } stderr
# valeurs qui vont sortir pour entrer dans gfsview
# tous les 30 pas de calcul
  OutputSimulation { istep = 30 } stdout
  OutputLocation { step = 0.1 } vals.data  cut.dat
  OutputSimulation { step = 0.25 } SIM/sim-%g.txt { format = text }
  EventScript { step = 0.25 } { cp SIM/sim-$GfsTime.txt sim.data}
  OutputPPM { step= 0.05 } { ppm2mpeg > blastok.mpg } { min = 0 max = 1 v = Velocity }
# p[0:10][0:1.5]"< awk '{if($1>.7){print $0}}' sim.data" u ($2/sqrt($1/1000)):6,sin(pi*x/2/4.79)*1.
# p[0:10][0:1.5]"< awk '{if($1>.9){print $0}}' sim.data" u ($2/sqrt($1/1000)):6,sin(pi*x/2/4.79)*1.
# p[0:5][0:1.5]"< awk '{if($1>0){print $0}}' SIM/sim-3.txt" u ($2*sqrt(1000)):6,1,erf(x/2/sqrt(3))
#p[][:] "< awk '{if($2<0.01){print $0}}' sim.data" u ($1):($9),.33/sqrt(x/1000)
# arret lorsque la variation de U devient "petite"
  EventStop { istep = 10 } U 1.e-4 DU}
#conditions aux limites
# first box free stream
GfsBox {
left = Boundary {
            BcDirichlet U 1
            BcDirichlet V 0 }
    bottom = Boundary {
            BcNeumann U 0
            BcDirichlet V 0 }
     top = Boundary {
```

```
            BcNeumann U 0
            BcNeumann V 0 }
        }
 GfsBox {
# en bas vitesse nulle
# second box the flat plate
        bottom = Boundary {
            BcDirichlet U 0
            BcDirichlet V 0 }
          top = Boundary {
            BcNeumann U 0
            BcNeumann V 0}
         }
 GfsBox {
# thrid box
  bottom = Boundary {
          # BcNeumann U 0
          #the trailing  edge
          BcDirichlet U 0
          # or the plate
           BcDirichlet V 0 }
      top = Boundary {
            BcNeumann U 0
            BcNeumann V 0}

    right = Boundary {
            BcDirichlet P 0
            BcNeumann U 0 }
 }
1 2 right
2 3 right
#######################################################################
```