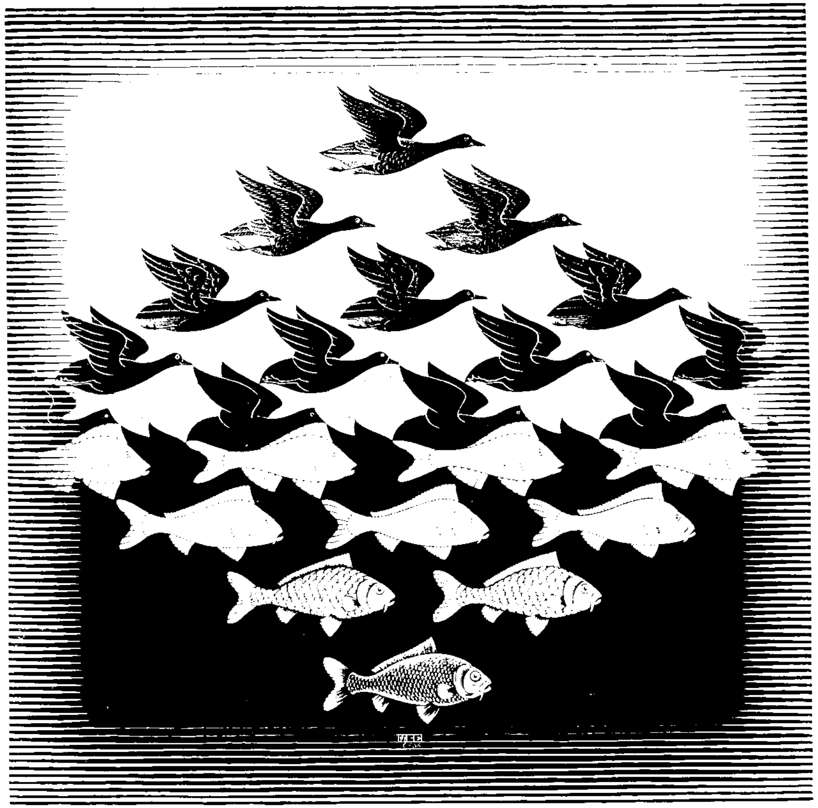

SKY AND WATER I, 1938
by M. C. Escher

Courtesy of Vorpal Galleries, San Francisco, Laguna Beach, New York, and Chicago, and the Escher Foundation, Haags Gemeentemuseum, The Hague

This woodcut by the Dutch artist gives a graphic impression of the "imperceptibly smooth blending" of one flow into another (p. 89) that is the heart of the method of matched asymptotic expansions discussed in Chapter 5.

PERTURBATION METHODS IN FLUID MECHANICS

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Chapter VIII

VISCOUS FLOW AT LOW REYNOLDS NUMBER

8.1. Introduction

We consider now incompressible flow past a body at low Reynolds number, as exemplified by the sphere and circular cylinder (Fig. 8.1).

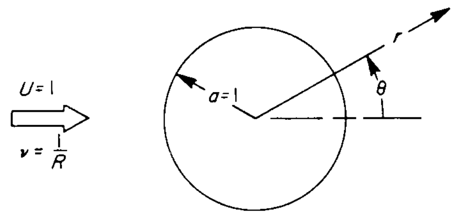

Fig. 8.1. Notation for sphere and circle.

Every high-school student learns that Millikan calculated the drag of an oil drop using the approximation developed by Stokes in 1851:

$$C_D = \frac{D}{\rho U^2 a^2} \sim \frac{6\pi}{R} \quad \text{as} \quad R \equiv \frac{Ua}{\nu} \to 0 \tag{8.1a}$$

A second approximation was found by Oseen in 1910:

$$C_D \sim \frac{6\pi}{R}\,(1 + \tfrac{3}{8}R) \tag{8.1b}$$

However, only in 1957 was it shown how further terms [cf. (1.4)] can be calculated using the method of matched asymptotic expansions.

The classical warning of singular behavior is absent; the highest derivatives are retained in the Navier-Stokes equations in the limit $R \to 0$. However, the problem contains two characteristic lengths: the radius a and the viscous length ν/U. Their ratio is the Reynolds number, so that in the limit $R \to 0$ the viscous length becomes vastly greater than the radius of the body. Hence singular behavior can be anticipated according to the physical criterion advanced in Section 5.3.

Viscous flows at low Reynolds number are easily observed experimentally, in contrast with those at high Reynolds number. Fig. 8.2 shows the sequence of flow patterns for a sphere or circular cylinder as

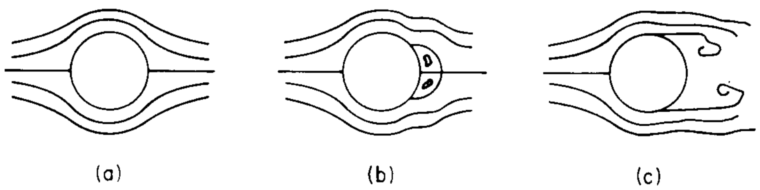

Fig. 8.2. Flow patterns for sphere or circle at low Reynolds number. (a) No eddies. (b) Standing eddies. (c) Unsteady flow.

the Reynolds number increases. At very low speeds the streamline pattern is almost symmetrical fore and aft. A closed recirculating wake or standing eddy makes its appearance at about $R = 10$ for a sphere and $R = 2.5$ for a circle (our Reynolds number being based upon radius rather than diameter). One may imagine that the eddies always exist inside the body, and at these Reynolds numbers penetrate through its surface (cf. Fig. 8.3). The flow becomes unsteady, with oscillations of

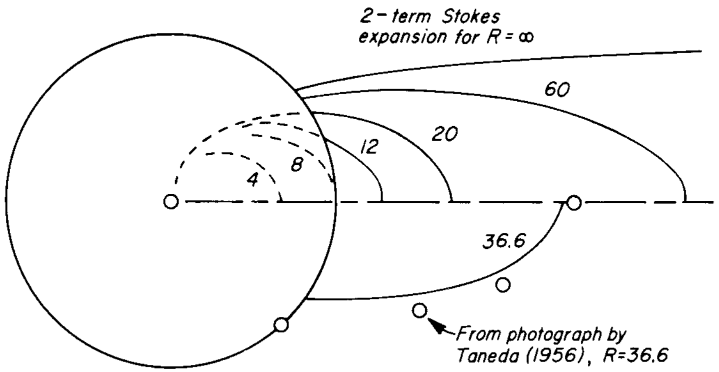

Fig. 8.3. Shape of standing eddy behind sphere.

the downstream part of the wake, at about $R = 65$ for a sphere and $R = 15$ for a circle. The flow becomes irregular, with separation of vortices from the rear of the body, above about $R = 100$ for a sphere and $R = 20$ for a circle.

8.2. Stokes' Solution for Sphere and Circle

Stokes reasoned that at low speeds the inertia forces, represented by the convective terms in the Navier-Stokes equations, are ineffective because they are quadratic in the velocity. Hence at low Reynolds number the pressure forces must be nearly balanced by viscous forces alone. As a first approximation, Stokes neglected the convective terms. In plane flow the result is the biharmonic equation for the stream function:

$$\nabla^4\psi = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)^2 \psi = 0 \tag{8.2a}$$

This follows formally from (7.2a) by letting $R \to 0$. In axisymmetric flow, the corresponding result is

$$\left[\frac{\partial^2}{\partial r^2} + \frac{\sin\theta}{r^2}\frac{\partial}{\partial \theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\right)\right]^2 \psi = 0 \tag{8.2b}$$

Let lengths be made dimensionless by reference to the radius a, and velocities by reference to the free-stream speed U (Fig. 8.2). Then the boundary conditions of zero velocity at the surface are

$$\psi(1, \theta) = \psi_r(1, \theta) = 0 \tag{8.3a}$$

and the condition of uniform flow upstream is

$$\psi(r, \theta) \sim \begin{cases} r \sin\theta, & \text{plane} \\ \frac{1}{2} r^2 \sin^2\theta, & \text{axisymmetric} \end{cases} \quad \text{as} \quad r \to \infty \tag{8.3b}$$

For the circle a symmetry condition must be added to rule out circulation.

Consider first the sphere. The upstream condition (8.3b) suggests separating variables, seeking a solution of the form $\psi = \sin^2\theta f(r)$. This leads to

$$\psi = \sin^2\theta \left\{ r^4, r^2, r, \frac{1}{r} \right\} \tag{8.4}$$

The upstream condition shows that no term in r^4 can be tolerated, and that the coefficient of the term in r^2 is $\frac{1}{2}$. Then the surface conditions (8.3a) fix the coefficients of r and $1/r$, giving Stokes' approximation:

$$\psi \sim \frac{1}{4}\left(2r^2 - 3r + \frac{1}{r}\right) \sin^2\theta \tag{8.5}$$

The first term is the uniform stream, and the third a dipole at the center of the sphere, both representing irrotational flows. The second term, which contains all the vorticity, has been dubbed a "stokeslet" by Hancock (1953), who used a linear distribution of these three elements to simulate a swimming worm. For inviscid flow the stokeslet is absent, and the coefficient of the dipole is $-\frac{1}{2}$ instead of $\frac{1}{4}$. Calculating the skin friction gives two-thirds of the drag (8.1a), the remaining third being pressure drag (Tomotika and Aoi, 1950). The flow pattern is symmetric fore and aft, and is therefore free of eddies as in Fig. 8.2a.

Consider now the circular cylinder. The upstream condition (8.3b) suggests seeking a solution of the form $\psi = \sin\theta\, f(r)$, which leads to

$$\psi = \sin\theta \left\{ r^3, r\log r, r, \frac{1}{r} \right\} \tag{8.6}$$

Imposing the conditions (8.3a) at the surface reduces this to

$$\psi \sim C\left[(1+2k)r\log r - \frac{1}{2}r - \frac{1+k}{2}\frac{1}{r} - \frac{k}{2}r^3\right]\sin\theta \tag{8.7}$$

We can invoke the principle of minimum singularity (Section 4.5), choosing $k = 0$ so that the stream function and velocity grow as slowly as possible with r. This leaves

$$\psi \sim C\left(r\log r - \frac{1}{2}r + \frac{1}{2}\frac{1}{r}\right)\sin\theta \tag{8.8}$$

The second term is the uniform parallel stream, the third a dipole at the origin, and the first a two-dimensional stokeslet containing the vorticity. The solution cannot be completed, however, because no choice of the constant C satisfies the upstream condition (8.3b). The difficulty is that, in contrast with the solution for the sphere, the stokeslet is now more singular at infinity than a uniform stream, and so predicts velocities that are unbounded far from the body.

8.3. The Paradoxes of Stokes and Whitehead

See Note 16

The nonexistence of a solution of Stokes' equation for unbounded plane flow past any body is known as *Stokes' paradox*. Stokes himself (1851) regarded it as an indication that no steady flow exists; a body started from rest would entrain a continually increasing quantity of fluid. However, this explanation is now believed to be incorrect, for reasons discussed in the next section.

Indeed, analogous difficulties arise with three-dimensional bodies, though they are deferred to the second approximation for finite shapes because, as usual, flow disturbances are weaker in three dimensions than two. (For semi-infinite shapes see Exercise 8.2.) Thus Whitehead (1889) failed in his attempt to improve upon Stokes' approximation for the sphere by iteration. The full Navier-Stokes equations give (Goldstein, 1938, p. 115):

$$\frac{1}{R}D^4\psi = \frac{1}{r^2\sin\theta}\left(\psi_\theta\frac{\partial}{\partial r} - \psi_r\frac{\partial}{\partial\theta} + 2\cot\theta\,\psi_r - 2\frac{\psi_\theta}{r}\right)D^2\psi \tag{8.9a}$$

where

$$D^2 = \frac{\partial^2}{\partial r^2} + \frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\right) \tag{8.9b}$$

Substituting the first approximation (8.5) into the convective terms on the right-hand side of (8.9a) that were neglected by Stokes yields the iteration equation

$$\left[\frac{\partial^2}{\partial r^2} - \frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\right)\right]^2\psi = -\frac{9}{4}R\left(\frac{2}{r^2} - \frac{3}{r^3} + \frac{1}{r^5}\right)\sin^2\theta\cos\theta \tag{8.10}$$

A particular integral that satisfies the surface conditions (8.3a) is easily found to be

$$-\frac{3}{32}R\left(2r^2 - 3r + 1 - \frac{1}{r} - \frac{1}{r^2}\right)\sin^2\theta\cos\theta \tag{8.11}$$

However, the velocity does not behave properly at infinity, and no complementary function can be added to correct it. In the next approximation the velocity would become infinite at infinity, as in the first approximation (8.8) for the circular cylinder.

The nonexistence of a second approximation to Stokes' solution for unbounded uniform flow past a three-dimensional body is known as *Whitehead's paradox*. Whitehead himself regarded it as an indication that discontinuities must arise in the flow field associated with the formation of a dead-water wake. However, this explanation too is now known to be incorrect.

See Note 16

8.4. The Oseen Approximation

Just as d'Alembert's paradox was resolved by Prandtl's discovery that flow at high Reynolds number is a singular perturbation problem, so the paradoxes of Stokes and Whitehead were shown by Oseen to arise from the singular nature of flow at low Reynolds number. Whereas the region of nonuniformity is a thin layer near the surface of the body at high Reynolds number, it is the neighborhood of the point at infinity for low Reynolds number. The source of the difficulty can be undertood by examining the relative magnitude of the terms neglected in the Stokes approximation.

Far from the body the nonlinear convective terms are seen from the right-hand side of (8.10) to be of order R/r^2. A typical viscous term—the cross-product in the left-hand side of (8.10)—is, from (8.5),

$$\frac{\partial^2}{\partial r^2}\left[\frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\right)\right]\psi = \left(\frac{3}{r^3} - \frac{6}{r^5}\right)\sin^2\theta = O\left(\frac{1}{r^3}\right) \tag{8.12}$$

Thus the ratio of terms neglected to those retained is

$$\frac{\text{convective}}{\text{viscous}} = O(Rr) \qquad \text{as} \quad r \to \infty \tag{8.13}$$

Although this ratio is small near the body when R is small, it becomes arbitrarily large at sufficiently great distances, no matter how small R may be. Thus the Stokes approximation becomes invalid where Rr is of order unity. This occurs at distances of the order of ν/U, meaning that the viscous length is then the significant reference dimension. The same objection applies *a fortiori* to plane flow, where the incomplete Stokes approximation (8.8) for the circle suggests the estimate

$$\frac{\text{convective}}{\text{viscous}} = O(CRr\log r) \qquad \text{as} \quad r \to \infty \tag{8.14}$$

These nonuniformities are the source of the singular behavior of the Stokes approximation. In three-dimensional flow the difficulty tends to be concealed because the first approximation is sufficiently well behaved; in the region of nonuniformity where $Rr = O(1)$ the velocity has already effectively attained its free-stream value, so that it is possible to impose the upstream boundary condition. This is an exceptional circumstance, which arose previously in the inner solution for a round-nosed airfoil (cf. Section 5.6).

This explanation of the difficulties encountered by Stokes and Whitehead was given by Oseen (1910), who prescribed a cure at the same time. Rather than neglect the convective terms altogether, he approximates them by their linearized forms valid far from the body, where the difficulty arises. For example, in the x-momentum equation in Cartesian coordinates

$$uu_x + vu_y + wu_z + \frac{p_x}{\rho} = \nu(u_{xx} + u_{yy} + u_{zz}) \tag{8.15}$$

Stokes neglects the first three terms altogether, but Oseen approximates

them by Uu_x. In plane flow the dimensionless equation (7.2a) for the stream function then becomes

$$\left(\nabla^2 - R\frac{\partial}{\partial x}\right)\nabla^2\psi = 0 \tag{8.16}$$

This constitutes an *ad hoc* uniformization, of a sort to be discussed further in Section 10.2. The general principle is to identify those terms whose neglect in the straightforward perturbation solution leads to nonuniformity, and to retain them after simplifying them insofar as possible in the region of nonuniformity. If the resulting equations can be solved, the result is a uniformly valid composite approximation, of the sort discussed in Section 5.4.

Thus Oseen's equations provide a uniformly valid first approximation for either plane or three-dimensional flow at low Reynolds number. In principle, one could refine the solution by successive approximations, and the result would presumably preserve its uniformity at every stage. In practice, however, although the Oseen equations are linear, their solution is sufficiently complex that no second approximations are known. It is simpler to decompose the composite expansion into its constituent inner and outer expansions, which may subsequently be recombined. This process will be carried out in the following sections.

The Oseen equations possess a second, essentially different, interpretation. At an arbitrary Reynolds number they describe viscous flow at such great distances from a finite body that the velocity has nearly returned to its free-stream value. From this small-disturbance point of view, the Oseen approximation has been used to study the wake far behind a body (Exercises 8.1 and 8.3). In such applications, ψ in (8.16) ordinarily represents a perturbation rather than the full stream function, the distinction affecting only the form of the boundary conditions. This second interpretation of the Oseen approximation of course remains valid at low Reynolds numbers, and will be used in what follows.

The solution of the Oseen equations was given for the sphere by Oseen himself (1910) as See Note 10

$$\psi = \frac{1}{4}\left(2r^2 + \frac{1}{r}\right)\sin^2\theta - \frac{3}{2}\frac{1}{R}(1 + \cos\theta)\left[1 - e^{-\frac{1}{2}Rr(1-\cos\theta)}\right] \tag{8.17}$$

The solution for the circular cylinder was given by Lamb (1911) in terms of Cartesian velocity components. For example, the component normal to the free stream is

$$\psi_x = \frac{1}{\log(4/R) - \gamma + \frac{1}{2}}\left\{\frac{\sin 2\theta}{2r^2} + \frac{2}{R}\frac{\partial}{\partial y}\left[\log Rr + e^{\frac{1}{2}Rr\cos\theta}K_0(\tfrac{1}{2}Rr)\right]\right\} \tag{8.18}$$

Here $\gamma = 0.5772\ldots$ is Euler's constant, and K_0 is the Bessel function.

In both these solutions the surface conditions were satisfied only approximately in a manner appropriate to the underlying assumption that the Reynolds number is small. We shall reconstruct these results later. Solutions for arbitrary Reynolds number were carried out by Goldstein (1929) and Tomotika and Aoi (1950). These more complicated results are of limited value because, contrary to Oseen's own views, the approximation is qualitatively as well as quantitatively invalid at high Reynolds number. For example, the Oseen approximation gives boundary layers whose thickness is of order R^{-1} rather than $R^{-1/2}$ as in Prandtl's correct theory. This discrepancy may be understood physically as arising from the fact that in Oseen's approximation the vorticity generated at the surface by shear is convected *through* rather than along the surface. The detailed flow patterns calculated by Tomotika and Aoi would be of some interest had not Yamada (1954) pointed out that numerical inaccuracy invalidates their qualitative nature even at low Reynolds number. For example, Tomotika and Aoi predict the standing eddies of Fig. 8.2b at arbitrarily low Reynolds number, whereas Yamada shows that they first appear behind the circular cylinder at $R = 1.51$ in the Oseen approximation.

8.5. Second Approximation Far from Sphere

See Note 10

We now improve Stokes' solution for the sphere by applying the method of matched asymptotic expansion. Our analysis follows the spirit of Kaplun and Lagerstrom (1957), but more nearly the notation of Proudman and Pearson (1957).

Let Stokes's approximation (8.5) be the leading term in an asymptotic expansion for small Reynolds number, which we call the *Stokes expansion*. We have seen that this series is invalid far from the body where r is of order R^{-1}. We therefore introduce an appropriate contracted radial coordinate ρ by setting

$$\rho = Rr \tag{8.19}$$

and envision a second asymptotic expansion valid in that distant region. We call it the *Oseen expansion*, because the flow far from the body is a small perturbation of the uniform stream. According to the convention adopted in Section 5.9, the Oseen expansion is the outer, and the Stokes expansion the inner expansion. We choose our notation accordingly except for the radial variable where, because R is not available, we use ρ for the outer and r for the inner variable.

We could, as in the last chapter, write down the two expansions with

their asymptotic sequences left unspecified. However, we prefer to show how matching automatically determines the form of each term in succession when, as in this problem, the matching proceeds according to the standard order (Section 5.9).

Writing the Stokes solution (8.5) in terms of the Oseen variable (8.19) and expanding for small R gives as its 2-term Oseen expansion:

$$\text{2-Oseen (1-Stokes) } \psi = \frac{1}{2}\frac{1}{R^2}\rho^2 \sin^2\theta - \frac{3}{4}\frac{1}{R}\rho \sin^2\theta \tag{8.20}$$

where the first term is the uniform stream. In order to match this, the Oseen expansion must have the form

$$\psi \sim \frac{1}{2}\frac{1}{R^2}\rho^2 \sin^2\theta + \frac{1}{R}\psi_2(\rho, \theta) + \cdots \qquad \text{as} \quad R \to 0 \quad \text{with } \rho \text{ fixed} \tag{8.21}$$

Substituting into the full equation (8.9) yields for ψ_2 the classical linearized Oseen equation (8.16) in the form

$$\left(\mathscr{D}^2 - \cos\theta \frac{\partial}{\partial\rho} - \frac{\sin\theta}{\rho}\frac{\partial}{\partial\theta}\right)\mathscr{D}^2\psi_2 = 0 \tag{8.22a}$$

where

$$\mathscr{D}^2 = \frac{\partial^2}{\partial\rho^2} + \frac{\sin\theta}{\rho^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\right) \tag{8.22b}$$

Setting

$$\mathscr{D}^2\psi_2 = e^{\frac{1}{2}Rx}\phi_2 = e^{\frac{1}{2}\rho\cos\theta}\phi_2 \tag{8.23}$$

reduces Eq. (8.22) to

$$(\mathscr{D}^2 - \tfrac{1}{4})\phi_2 = 0 \tag{8.24}$$

Seeking as before a solution of the form $\phi_2 = \sin^2\theta\, f(\rho)$ gives

$$f'' - \left(\frac{2}{\rho^2} + \frac{1}{4}\right)f = 0 \tag{8.25}$$

The solution that vanishes at infinity is

$$f = c_2\left(1 + \frac{2}{\rho}\right)e^{-\frac{1}{2}\rho} \tag{8.26}$$

and any other solution of (8.24) having the proper symmetry is more singular at the origin, and can therefore be rejected as unmatchable by the principle of minimum singularity (Section 5.6).

Thus the original equation for (8.22) for ψ_2 is reduced to

$$\mathscr{D}^2\psi_2 = c_2\left(1 + \frac{2}{\rho}\right)e^{-\frac{1}{2}\rho(1-\cos\theta)} \sin^2\theta \tag{8.27}$$

A particular integral is

$$\psi_2 = -2c_2(1 + \cos\theta)[1 - e^{-\frac{1}{2}\rho(1-\cos\theta)}] \tag{8.28}$$

where the first term is a potential source at the origin, which has been added to cancel the sink in the second term, and so assure zero flux through any surface enclosing the body. Any other complementary function that possesses this property, gives no velocity at infinity, and has the proper symmetry will be more singular at the origin and hence unmatchable.

This is a fundamental solution of the Oseen equation, which describes the disturbance field produced at great distances by any finite three-dimensional nonlifting body. The constant c_2 depends upon certain details of the flow near the body. We find it by applying the asymptotic matching principle (5.24). Writing the Oseen expansion (8.21) in Stokes variables and expanding for small R yields

$$\text{1-Stokes (2-Oseen) } \psi = \tfrac{1}{2}r^2 \sin^2\theta - c_2 r \sin^2\theta \tag{8.29}$$

This matches (8.20) if $c_2 = \frac{3}{4}$.

We have thus found two terms of the Oseen expansion (8.21). When rewritten in Stokes variables this becomes

$$\psi \sim \frac{1}{2}r^2 \sin^2\theta - \frac{3}{2}\frac{1}{R}(1 + \cos\theta)[1 - e^{-\frac{1}{2}Rr(1-\cos\theta)}]$$
$$\text{as} \quad R \to 0 \quad \text{with } Rr \text{ fixed} \tag{8.30}$$

We can construct a uniformly valid composite expansion by combining this with the Stokes approximation (8.5) using the rule (5.32) for additive composition. The result gives a uniform approximation to the *perturbation* field. It is found to be just the solution (8.17) of the Oseen equations given by Oseen himself. This confirms the statement in Section 8.4 that his linearized equations yield a uniform first approximation. Near the body the last term in (8.17) reduces to the stokeslet of the Stokes approximation; it may by analogy be called an "oseenlet."

8.6. Second Approximation near Sphere

We proceed to the second term in the Stokes expansion. The 2-term Stokes expansion of the Oseen expansion (8.21) is found to be

$$\text{2-Stokes (2-Oseen) } \psi = \frac{1}{4}(2r^2 - 3r)\sin^2\theta + \frac{3}{16}Rr^2(1 - \cos\theta)\sin^2\theta \tag{8.31}$$

In order to match this, the Stokes expansion must have the form

$$\psi \sim \frac{1}{4}\left(2r^2 - 3r + \frac{1}{r}\right)\sin^2\theta + R\Psi_2(r, \theta) + \cdots \tag{8.32}$$

The equation for Ψ_2 is evidently Whitehead's (8.10) without the factor R. His particular integral (8.11) remains valid, and the original Stokes approximation (8.5) provides the only complementary function with the proper symmetry that is no more singular at infinity. Thus we set

$$\Psi_2 = C_2\left(2r^2 - 3r + \frac{1}{r}\right)\sin^2\theta - \frac{3}{32}\left(2r^2 - 3r + 1 - \frac{1}{r} - \frac{1}{r^2}\right)\sin^2\theta\cos\theta \tag{8.33}$$

The constant C_2 is found by matching. Carrying out the Oseen expansion of (8.33) yields

$$\text{2-Oseen (2-Stokes) } \psi = \frac{1}{2}\frac{1}{R^2}\rho^2\sin^2\theta$$
$$+ \frac{1}{R}\left(2C_2\rho^2 - \frac{3}{16}\rho^2\cos\theta - \frac{3}{4}\rho\right)\sin^2\theta \tag{8.34}$$

and this matches (8.31) if $C_2 = 3/32$.

Thus we have found two terms of the Stokes expansion for the stream function in the vicinity of the sphere:

$$\psi \sim \frac{1}{4}(r-1)^2\sin^2\theta\left[\left(1 + \frac{3}{8}R\right)\left(2 + \frac{1}{r}\right) - \frac{3}{8}R\left(2 + \frac{1}{r} + \frac{1}{r^2}\right)\cos\theta\right] \tag{8.35}$$

This vanishes not only on the sphere and along the axis of symmetry, but also along the curve

$$\cos\theta = \left(\frac{8}{3}\frac{1}{R} + 1\right)\frac{2r^2 + r}{2r^2 + r + 1} \tag{8.36}$$

This is the approximate description of the boundary of the standing eddy. It is plotted in Fig. 8.3. The eddy appears only at Reynolds numbers so large that one would not have expected the Stokes expansion to have any validity. Nevertheless, the lower half of Fig. 8.3 shows striking agreement with the experimental observations of Taneda (1956) at $R = 36.6$. The downstream end of the eddy lies at

$$r_e = \tfrac{1}{4}(\sqrt{1 + 3R} - 1) \tag{8.37}$$

Therefore the eddy first appears in the flow field at $R = 8$. Despite its magnitude, this agrees well with the value of 12 measured by Taneda and the value of 8.5 calculated numerically by Jenson (1959) using the full Navier-Stokes equations. Indeed, Fig. 8.4 shows that good agreement

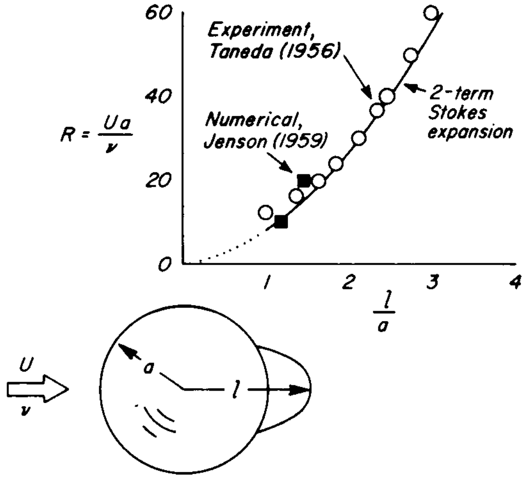

Fig. 8.4. Length of eddy behind sphere.

persists out to $R = 60$, which is about the limit for observation of steady flow. These remarkable results call for corroboration through examination of the effect of further terms in the Stokes expansion.

Higher approximations can be found by continuing the preceding analysis. Proudman and Pearson (1957) have carried it far enough to show that the next Stokes approximation contains a term in $R^2 \log R$ as well as R^2, and that logarithms are thereby introduced also into the Oseen expansion beginning with $R^3 \log R$. They have calculated only the term in $R^2 \log R$ in the Stokes expansion. This provides the drag formula (1.4) of Chapter I.

See Note 10

According to these results, the Reynolds number at which the eddy first appears is a solution of the transcendental equation

$$1 - \frac{1}{8}R + \frac{9}{40}R^2 \log R + O(R^2) = 0 \tag{8.38}$$

Unfortunately, this has no real root if three terms are retained. The logarithmic term limits the Stokes expansion to values of R small compared with 1. Until some method is found of enlarging the range of applicability of such a series (cf. Section 10.7), we cannot say whether the striking predictions of the second Stokes approximation are more than coincidence.

Because of symmetry, the second term in square brackets in (8.35) contributes nothing to the drag, which according to the first term is then $(1 - \frac{3}{8}R)$ times the Stokes value. But the second term is the particular integral of Whitehead for the nonlinear terms. For that reason, the Oseen approximation, which neglects the nonlinear terms near the body, nevertheless gives the drag correct to second order at least for symmetric shapes (Chester, 1962).

8.7. Higher Approximations for Circle

See Note 10

Stokes' paradox for plane flow is more striking than Whitehead's for three dimensions. Its resolution by the method of matched asymptotic expansions is correspondingly more dramatic, despite the practical shortcoming that the solution cannot be carried to nearly as great accuracy. We treat the typical example of the circular cylinder, using a synthesis of the work of Kaplun (1957) and of Proudman and Pearson (1957).

The analysis largely parallels that for the sphere, but interesting differences appear. In particular, the matching is marginal, and the asymptotic sequence correspondingly slow. It was for this problem that Kaplun and Lagerstrom (1957) devised their sophisticated apparatus of intermediate limits and expansions, and the intermediate matching principle (Section 5.8). However, we shall see that the asymptotic matching principle (5.24) is entirely adequate, although the simple limit matching principle (5.22) does not hold.

We reconsider the solution (8.8) of the biharmonic equation as the first term in a Stokes expansion:

$$\psi \sim \Delta_1(R)\left(r \log r - \frac{1}{2}r + \frac{1}{2}\frac{1}{r}\right) \sin\theta \quad \text{as} \quad R \to 0 \quad \text{with } r \text{ fixed} \tag{8.39}$$

The multiplier Δ_1 must be allowed to depend upon Reynolds number because our asymptotic sequence is unspecified. Although this approximation cannot satisfy the condition (8.3b) of a uniform stream at infinity, it can be *matched* to the uniform stream, regarded as the first term of an Oseen expansion (Lagerstrom and Cole, 1955, Section 6.3). Again the Oseen variable is taken as $\rho = Rr$ so that lengths are referred to the viscous dimension ν/U rather than the radius a. Then the Oseen expansion begins with the free stream in the form

$$\psi \sim \frac{1}{R}\rho \sin\theta + \cdots \quad \text{as} \quad R \to 0 \quad \text{with} \quad \rho > 0 \quad \text{fixed} \tag{8.40}$$

Writing the Stokes approximation (8.39) in Oseen variables and expanding gives

$$\text{1-Oseen (1-Stokes)}\ \psi = \frac{\Delta_1(R)}{R} \log\frac{1}{R}\, \rho \sin\theta \tag{8.41}$$

This matches (8.40) if $\Delta_1(R) = (\log 1/R)^{-1}$, or more generally if $\Delta_1(R) = (\log 1/R + k)^{-1}$, where k is any constant; and we shall later exploit this freedom.

The overlap is so slight in this case that in order to match we have had to accept a relative error of order Δ_1, which is enormous compared with the error of order R in the Stokes approximation for the sphere. We are thereby committed to a slow expansion in powers of Δ_1, of which an infinite number of terms correspond to only the first term for the sphere.

Expanding the Stokes approximation (8.39) further in Oseen variables yields

$$\text{2-Oseen (1-Stokes)}\ \psi = \frac{1}{R}[1 - \Delta_1(R)\,(\log\rho - k - \tfrac{1}{2})]\rho \sin\theta \tag{8.42}$$

This requires, in order to match, that the Oseen expansion (8.40) continue as

$$\psi \sim \frac{1}{R}[\rho \sin\theta + \Delta_1(R)\psi_2(\rho, \theta) + \cdots] \tag{8.43}$$

Substituting this into the full equation (7.2a) shows, of course, that ψ_2 satisfies the linearized Oseen equation (8.16). The appropriate solution can be found by proceeding as for the sphere (Proudman and Pearson, 1957). However, the stream function is disadvantageous here because it can be written only as an infinite series, whereas the velocity components are closed expressions.

We evidently seek the plane counterpart of (8.28), the oseenlet representing the disturbances produced by an infinitesimal drag at the origin. This fundamental solution, due to Oseen (Rosenhead, 1963, p. 183), gives as Cartesian velocity components

$$u_2 \equiv \frac{\partial\psi_2}{\partial(\rho\sin\theta)} = 2c_2\left\{\frac{\partial}{\partial(\rho\cos\theta)}[\log\rho + e^{\frac{1}{2}\rho\cos\theta}K_0(\tfrac{1}{2}\rho)] - e^{\frac{1}{2}\rho\cos\theta}K_0(\tfrac{1}{2}\rho)\right\} \tag{8.44a}$$

$$v_2 \equiv -\frac{\partial\psi_2}{\partial(\rho\cos\theta)} = 2c_2\frac{\partial}{\partial(\rho\sin\theta)}[\log\rho + e^{\frac{1}{2}\rho\cos\theta}K_0(\tfrac{1}{2}\rho)] \tag{8.44b}$$

The term in $\log\rho$ is again a potential source at the origin that cancels the sink in the term involving the Bessel function K_0. For small ρ these are approximately

$$\frac{\partial\psi_2}{\partial(\rho\sin\theta)} \sim -c_2\left(\log\frac{4}{\rho} - \gamma + \cos^2\theta\right) + O(\rho\log\rho) \tag{8.45a}$$

$$\frac{\partial\psi_2}{\partial(\rho\cos\theta)} \sim c_2 \sin\theta\cos\theta + O(\rho\log\rho) \tag{8.45b}$$

where Euler's constant $\gamma = 0.5772\ldots$, and integrating gives

$$\psi_2 \sim -c_2\left(\log\frac{4}{\rho} + 1 - \gamma\right)\rho\sin\theta + O(\rho^2\log\rho) \tag{8.46}$$

Using this, we find that the Oseen expansion (8.43) behaves near the body like

$$\text{1-Stokes (2-Oseen)}\ \psi = \frac{1}{R}\left[\rho\sin\theta - c_2\Delta_1(R)\left(\log\frac{\rho}{4} - \gamma - 1\right)\rho\sin\theta\right] \tag{8.47}$$

Then matching with (8.42) according to the asymptotic matching principle gives $c_2 = 1$.

The second term in the Stokes expansion—and indeed the term of any finite order—is evidently again a solution of the biharmonic equation, because the nonlinear terms of order R are transcendentally small on the scale of powers of $\Delta_1(R)$. Matching, or applying the principle of minimum singularity, shows that each is simply a multiple of the first approximation (8.8). It is convenient, following Kaplun (1957), to make the second term vanish by choosing the constant k so that (8.42) and (8.47) match perfectly: $k = \log 4 - \gamma - \frac{1}{2}$. Then the Stokes expansion assumes the form

$$\psi \sim \left(\Delta_1 + \sum_{n=3}^{\infty} a_n\Delta_1^{\,n}\right)\left(r\log r - \frac{1}{2}r + \frac{1}{2}\frac{1}{r}\right)\sin\theta \tag{8.48a}$$

where

$$\Delta_1 = \left(\log\frac{4}{R} - \gamma + \frac{1}{2}\right)^{-1} = \left(\log\frac{3.703}{R}\right)^{-1} \tag{8.48b}$$

Forming a uniformly valid two-term composite expansion by additive composition of (8.43) and (8.48) reproduces Lamb's solution (8.18) of the linearized Oseen equation.

Kaplun (1957) has carried the process through one more cycle to find the coefficient of the third term in the Stokes expansion (8.48) as $a_3 \approx -0.87$. Hence he finds for the drag coefficient

See Note 11

$$C_D = \frac{D}{\rho U^2 a} \sim \frac{4\pi}{R}[\Delta_1(R) - 0.87\Delta_1^3(R) + O(\Delta_1^4)] \tag{8.49}$$

See Note 15 The first term is the classical result of Lamb (1911, 1932). Comparison with the measurements of Tritton (1959) in Fig. 8.5 shows the limited utility of this result. Also shown for comparison is Tomotika and Aoi's (1950) full numerical solution of the linearized Oseen equation (8.16).

From a formal mathematical point of view, we should exhaust all the

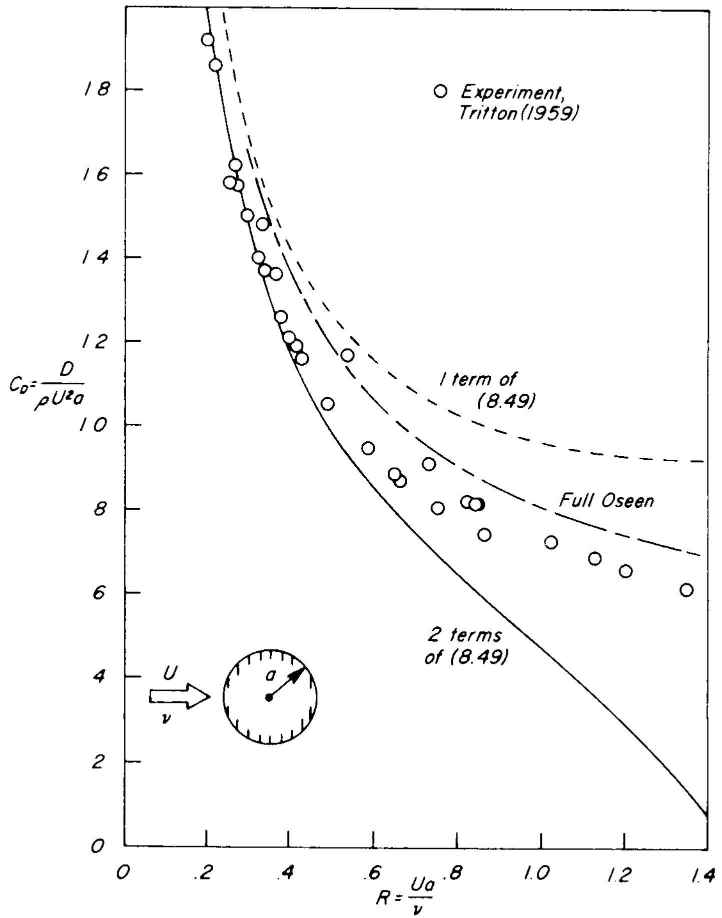

Fig. 8.5. Drag of circular cylinder at low Reynolds number.

powers of Δ_1 in (8.48) before considering nonlinear corrections to the Stokes equation, which are relatively of the transcendentally small order R. In practice, however, such terms are significant. The first such term (Exercise 8.5) is of order R, which is greater than Δ_1^4 for $R > 0.00008$. Proudman and Pearson (1957) discuss briefly how these terms could be calculated. They would be needed to show any asymmetry in the flow pattern, such as the emergence of standing eddies. See Note 11

EXERCISES

8.1. *Oseen solution for flat plate and plane wake.* In parabolic coordinates (7.51) the Navier-Stokes equations give for the stream function in plane flow:

$$\left[\nu\left(\frac{\partial^2}{\partial\xi^2} + \frac{\partial^2}{\partial\eta^2}\right) + \psi_\xi \frac{\partial}{\partial\eta} - \psi_\eta \frac{\partial}{\partial\xi}\right] \frac{\psi_{\xi\xi} + \psi_{\eta\eta}}{\xi^2 + \eta^2} = 0$$

Derive the corresponding linearized equation (8.16) of Oseen. Find by separation of variables the Oseen solution for a semi-infinite flat plate in a uniform stream, and for the flow far from a finite nonlifting body. Show that in the first case the boundary-layer approximation in parabolic coordinates is the full Oseen solution. Compare the skin friction with the known value for the Navier-Stokes equations far downstream. In the second case express the constant multipliers for both the term representing the wake and that for potential flow in terms of the drag of the body. Relate the solution to (8.44). [The first case was originally treated in a more complicated way by Lewis and Carrier (1949); for the second, see Imai (1951) and Chang (1961).]

8.2. *Viscous flow past slender paraboloid.* In paraboloidal coordinates (cf. Exercise 8.1), the Navier-Stokes equations give for axisymmetric flow See Note 3

$$\left[\nu(\xi^2 + \eta^2)D^2 + \frac{1}{\xi\eta}\left(\psi_\xi \frac{\partial}{\partial\eta} - \psi_\eta \frac{\partial}{\partial\xi}\right) + \frac{2}{\xi^2\eta^2}(\eta\psi_\eta - \xi\psi_\xi)\right] D^2\psi = 0$$

where

$$D^2 \equiv \frac{\xi\eta}{\xi^2 + \eta^2}\left[\frac{\partial}{\partial\xi}\left(\frac{1}{\xi\eta}\frac{\partial}{\partial\xi}\right) + \frac{\partial}{\partial\eta}\left(\frac{1}{\xi\eta}\frac{\partial}{\partial\eta}\right)\right]$$

Find the Stokes solution for the paraboloid of revolution. Show that it can be matched to the uniform stream in the same marginal way as for the circular cylinder. Calculate the second term in the Oseen expansion. Describe how the process would continue. What light does it shed on the accuracy of the known Oseen approximation for the elliptic paraboloid (Wilkinson, 1955)? How does