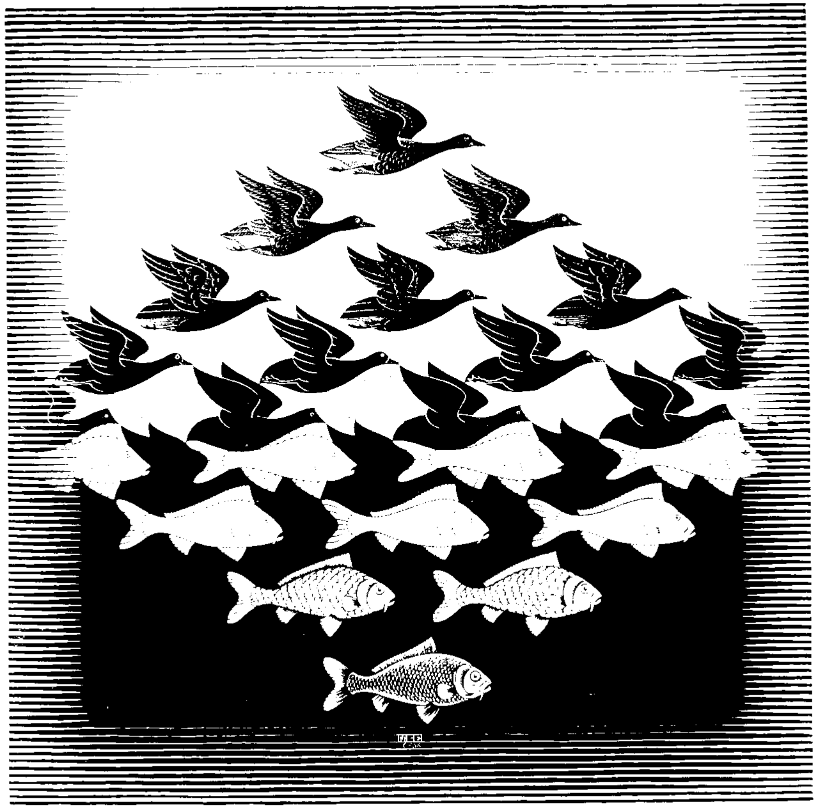

SKY AND WATER I, 1938
by M. C. Escher

Courtesy of Vorpal Galleries, San Francisco, Laguna Beach, New York, and Chicago, and the Escher Foundation, Haags Gemeentemuseum, The Hague

This woodcut by the Dutch artist gives a graphic impression of the "imperceptibly smooth blending" of one flow into another (p. 89) that is the heart of the method of matched asymptotic expansions discussed in Chapter 5.

PERTURBATION METHODS IN FLUID MECHANICS

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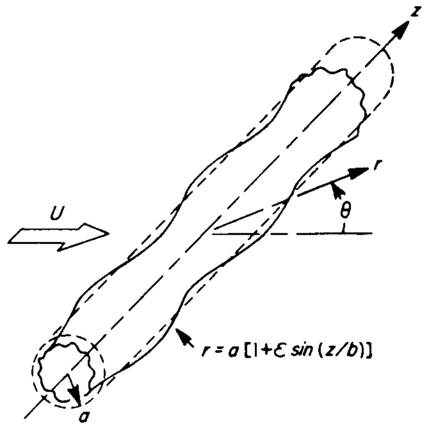

Fig. 2.5. Infinite corrugated cylinder.

2.4. *Circle in parabolic shear.* A circular cylinder of radius a is symmetrically placed in a parallel stream of incompressible inviscid fluid having the parabolic velocity profile $u = U(1 - \frac{1}{2}\varepsilon y^2/a^2)$. Find an exact implicit expression for the vorticity $\omega(\psi)$, and expand to give ω as a series, keeping terms of order ε^2. Carry out a perturbation solution for the flow, showing that a difficulty arises in the term of order ε because no solution can be found for which the velocity disturbances produced by the body disappear far upstream.

See Note 3

Chapter III

THE TECHNIQUES OF PERTURBATION THEORY

3.1. Introduction; Limit Processes

The examples in the preceding chapter have served to introduce various techniques for handling perturbation problems. We now seek to classify and generalize those that are of common utility. We begin with some matters of notation, definitions, and relevant processes of analysis.

We are concerned with finding approximate solutions of the equations of fluid motion that are close to the exact solutions in some useful sense. This involves various kinds of equality, which in decreasing degree of identification will be expressed by the following symbols:

$\equiv$ identical with
$=$ equal to
$\sim$ asymptotically equal to (in some given limit) (3.1)
$\approx$ approximately equal to (in any useful sense)
$\propto$ proportional to

As discussed in Chapter I, we consider approximations that depend upon a *limit process*, the result becoming exact as a perturbation quantity approaches zero or some other critical value. One often encounters a *double* or *multiple limit process*, in which two or more perturbation quantities approach their limits simultaneously. Because the order of carrying out several limits cannot in general be interchanged, one must frequently specify the relative rates of approach. This specification provides a *similarity parameter* for the problem. The following are some familiar examples:

(a) Plane transonic small-disturbance theory for a wing of thickness ratio ε (von Kármán, 1947):

$$\left.\begin{array}{l}\varepsilon \to 0\\ M \to 1\end{array}\right\} \quad \frac{M-1}{\varepsilon^{2/3}} = O(1)$$

(b) Hypersonic small-disturbance theory for a body of thickness ratio ε (Hayes and Probstein, 1959, p. 36):

$$\left.\begin{array}{l} \varepsilon \to 0 \\ M \to \infty \end{array}\right\} \quad \frac{1}{M\varepsilon} = O(1)$$

(c) Newton-Busemann approximation for hypersonic flow past a blunt body (Cole, 1957):

$$\left.\begin{array}{l} M \to \infty \\ \gamma \to 1 \end{array}\right\} \quad \frac{1}{(\gamma - 1)M^2} = O(1)$$

(d) Hypersonic small-disturbance form of Newton-Busemann approximation for a slender body of thickness ratio ε (Cole, 1957):

$$\left.\begin{array}{l} \varepsilon \to 0 \\ M \to \infty \\ \gamma \to 1 \end{array}\right\} \quad \frac{1}{(\gamma - 1)M^2\varepsilon^2} = O(1)$$

In the last example one might have anticipated two similarity parameters, but only one is found to be significant.

A perturbation quantity is never uniquely defined. For example, the thickness parameter for a slender body may be taken as its thickness ratio, maximum slope, mean slope, etc. Of course it may also be changed by a constant multiplier, as in referring Reynolds number to radius rather than diameter for a sphere. One should be alert to the possibility of exploiting this freedom by replacing the obvious choice by an alternative that is superior to it in some respect. The possibilities are too diverse to be subject to rules. They can only be suggested here by listing a number of cases where an ingenious choice of the perturbation quantity, usually suggested by extraneous considerations, leads to simplification or improvement of the results:

(a) $(M^2 - 1)$ instead of $(M - 1)$ in transonic small-disturbance theory, so that the result is valid also in the adjacent regimes of subsonic and supersonic flow (Spreiter, 1953),

(b) $1/\sqrt{M^2 - 1}$ instead of $1/M$ in hypersonic small-disturbance theory, so that the result is valid also in the adjacent regime of supersonic flow (Van Dyke, 1951),

(c) $(\gamma - 1)/(\gamma + 1)$ instead of $(\gamma - 1)$ in the Newton-Busemann approximation for hypersonic flow, because it can be identified with the density ratio across a strong shock wave (Hayes and Probstein, 1959, p. 7),

(d) $(\log R/4 + \gamma - \frac{1}{2})$ instead of $\log R$ in viscous flow past a circle at low Reynolds number (γ being Euler's constant), because the first two terms of the Stokes expansion are thereby combined into one (Kaplun, 1957; see Section 8.7),

(e) $\varepsilon/(1 - \varepsilon)$ instead of ε, ε being the density ratio across a normal shock, in the Newton-Busemann approximation for the standoff distance of the detached shock wave on a blunt body in supersonic flow, because it then becomes infinite for $M \to 1$, as it should (Serbin, 1958),

(f) $(A + BR^{-1/2} + \cdots)^2$ instead of $(A^2 + 2ABR^{-1/2} + \cdots)$ for the drag of a bluff body in laminar flow at high Reynolds numbers, which is suggested by theory and agrees better with known results (Imai, 1957b),

(g) $2\pi/(1 - 2/A - \cdots)$ instead of $2\pi(1 - 2/A + \cdots)$ for the lift-curve slope of an elliptic wing of high aspect ratio A, because it then vanishes for $A \to 0$, as it should (see Chapter IX),

(h) $\xi\sqrt{1 - \xi^2}$, ξ being a parabolic coordinate, instead of x in the Blasius series for the boundary layer on a parabola, because the radius of convergence is thereby extended to infinity (see Chapter X),

(i) $\varepsilon/(2 - \varepsilon)$ instead of ε in free-streamline theory (Garabedian, 1956), where $2 + \varepsilon$ is the number of space dimensions, because the radius of convergence is thereby increased.

3.2. Gauge Functions and Order Symbols

The solution of a problem in fluid mechanics will depend upon the coordinates, say x, y, z, t, and also upon various parameters. One or more of these quantities may, by appropriate redefinition, be regarded as vanishingly small in a perturbation solution. We consider the behavior of the solution as it depends upon one such perturbation quantity, with the other coordinates and parameters fixed. Thus we seek to describe the way in which a function $f(\varepsilon)$ behaves as ε approaches zero. An analogous situation has already arisen in the upstream boundary conditions (2.3b), (2.6b), etc., where it was necessary to describe the behavior of the solution far from the body.

There are a number of possible descriptions, of varying degrees of precision. We discuss six of them, in increasing order of refinement. First, one may simply state whether or not a limit *exists*. For example, $\sin 2\varepsilon$ has a limit as $\varepsilon \to 0$, whereas $\sin 2/\varepsilon$ has not. However, we are concerned only with problems where a limit is believed to exist.

Second, one may describe the limiting value *qualitatively*. There are three possibilities: the function may be

$$\left.\begin{array}{lll}\text{(a) vanishing:} & f(\varepsilon)\to 0 \\ \text{(b) bounded:} & f(\varepsilon)<\infty \\ \text{(c) infinite:} & f(\varepsilon)\to\infty\end{array}\right\} \quad \text{as} \quad \varepsilon\to 0$$

It is a peculiarity of this mode of description that the first case is included in the second; a function that vanishes is also bounded. However, one would naturally use the first description when possible, because it is more precise.

Third, one may describe the limiting value *quantitatively*. There are again three possibilities, only the second having been refined:

$$\left.\begin{array}{lll}\text{(a)} & \lim f(\varepsilon)=0 & \\ \text{(b)} & \lim f(\varepsilon)=c, & \text{a constant} \\ \text{(c)} & \lim f(\varepsilon)=\infty & \end{array}\right\} \quad \text{as} \quad \varepsilon\to 0$$

Fourth, one may describe *qualitatively* the *rate* at which the limit is approached. Only cases (a) and (c) above are thus refined. This can be done by comparing with a set of *gauge functions*. These are functions that are so familiar that their limiting behavior can be regarded as known intuitively. The comparison is made using the order symbols O ("big oh") and o ("little oh"). They provide an indispensible means for keeping account of the degree of approximation in a perturbation solution.

The symbol O is used if comparing $f(\varepsilon)$ with some gauge function $\delta(\varepsilon)$ shows that the ratio $f(\varepsilon)/\delta(\varepsilon)$ remains bounded as $\varepsilon\to 0$. One writes

$$f(\varepsilon)=O[\delta(\varepsilon)] \quad \text{as} \quad \varepsilon\to 0 \quad \text{if} \quad \lim_{\varepsilon\to 0}\frac{f(\varepsilon)}{\delta(\varepsilon)}<\infty \tag{3.2}$$

The symbol o is used instead if the ratio tends to zero. One writes

$$f(\varepsilon)=o[\delta(\varepsilon)] \quad \text{as} \quad \varepsilon\to 0 \quad \text{if} \quad \lim_{\varepsilon\to 0}\frac{f(\varepsilon)}{\delta(\varepsilon)}=0 \tag{3.3}$$

Some examples are

$$\begin{array}{ll} \sin 2\varepsilon=O(\varepsilon), & 1-\cos\varepsilon=O(\varepsilon^2)=o(\varepsilon) \\ \sqrt{1-\varepsilon^2}=O(1), & \sec^{-1}(1+\varepsilon)=O(\varepsilon^{1/2})=o(1) \\ \cot\varepsilon=O\left(\dfrac{1}{\varepsilon}\right), & \exp(-1/\varepsilon)=o(\varepsilon^m) \quad \text{for all } m \end{array} \tag{3.4}$$

Like the perturbation quantity ε itself, the gauge functions are not unique, and a choice other than the obvious one may occasionally be

advantageous. For example, it might under some circumstances prove useful to replace the first case in (3.4) by the equivalents

$$\sin 2\varepsilon=O(2\varepsilon), \qquad O(\tan\varepsilon), \qquad O\left(\frac{\varepsilon}{1+\varepsilon}\right), \qquad \text{etc.}$$

One ordinarily chooses the real powers of ε as gauge functions, because they have the most familiar properties. However, this set is not complete. It fails, for example, to describe $\log 1/\varepsilon$, which becomes infinite as ε tends to zero, but more slowly than any power of ε. The powers of ε must therefore be supplemented, when necessary, by its logarithm, exponential, log log, etc., or their equivalents. Examples are

$$\begin{array}{ll} \operatorname{sech}^{-1}\varepsilon=O\left(\log\dfrac{1}{\varepsilon}\right), & \cosh^{-1} K_0(\varepsilon)=O\left(\log\log\dfrac{1}{\varepsilon}\right) \\ \cosh\dfrac{1}{\varepsilon}=O(e^{1/\varepsilon}), & \exp\left(-\cosh\dfrac{1}{\varepsilon}\right)=O(\exp[-\tfrac{1}{2}e^{1/\varepsilon}]) \end{array} \tag{3.5}$$

Often, as in (1.4), one writes $\log\varepsilon$ where $\log 1/\varepsilon$ would be more appropriate.

Neither order symbol necessarily describes the actual rate of approach to the limit, but provides only an upper bound. Thus it is formally correct to replace the first example in (3.4) by

$$\sin 2\varepsilon=O(1), \quad o(1), \quad O(\varepsilon^{1/2}), \quad o(\varepsilon^{1/2}), \quad \text{etc.} \tag{3.6}$$

However, we assume that the sharpest possible estimate is always given. This means, for example, choosing the largest possible power of ε as the gauge function, and using o only when one has insufficient knowledge to use O. Of course the result may still be only an upper bound for lack of sufficient information.

The *mathematical order* expressed by the symbols O and o is formally distinct from *physical order of magnitude*, because no account is kept of constants of proportionality. Therefore $K\varepsilon$ is $O(\varepsilon)$ even if K is ten thousand. In physical problems, however, one has at least a mystical hope, almost invariably realized, that the two concepts are related. Thus if the error in a physical theory is $O(\varepsilon)$ and ε has been sensibly chosen, one can expect that the numerical error will not exceed some moderate multiple of ε: possibly 2ε or even $2\pi\varepsilon$, but almost certainly not 10ε.

The rules for simple operations with order symbols are evident from this physical connection. For example, the order of a product (or ratio) is the product (or ratio) of the orders; the order of a sum or difference is that of the dominant term—i.e., the term of order ε^m having the smallest value of m—etc. Order symbols may be integrated with respect to either ε or another variable. It is not in general permissible to differentiate order relations. Nevertheless, in physical problems one commonly assumes that differentiation with respect to another variable is legitimate, so that the derivative has the same order as its antecedent. For other properties of order symbols see the first chapter of Erdélyi (1956).

3.3. Asymptotic Representations; Asymptotic Series

A fifth scheme is to describe *quantitatively* the rate at which a function approaches its limit. This constitutes a refinement of the fourth scheme—the use of order symbols—just as the third scheme does of the second. We simply restore the constant of proportionality, and write

$$f(\varepsilon) \sim c\delta(\varepsilon) \qquad \text{as} \quad \varepsilon \to 0 \tag{3.7a}$$

if

$$\lim_{\varepsilon \to 0} \frac{f(\varepsilon)}{\delta(\varepsilon)} = c \tag{3.7b}$$

that is, if

$$f(\varepsilon) = c\delta(\varepsilon) + o[\delta(\varepsilon)] \tag{3.7c}$$

This is the *asymptotic form* or *asymptotic representation* of the function, and constitutes the leading term in the asymptotic expansion discussed below. Some examples are

$$\begin{aligned} &\sin 2\varepsilon \sim 2\varepsilon, && \operatorname{sech}^{-1}\varepsilon \sim \log\frac{2}{\varepsilon} \\ &\sqrt{1-\varepsilon^2} \sim 1, && K_0\left(\frac{1}{\varepsilon}\right) \sim \sqrt{\frac{\pi}{2}}\,\varepsilon^{1/2}e^{-1/\varepsilon} \\ &\cot\varepsilon \sim \frac{1}{\varepsilon}, && \int_0^\infty \frac{e^{-t}\,dt}{1+\varepsilon t} \sim 1 \end{aligned} \tag{3.8}$$

Sixth, the preceding description—which is the most precise one possible using only one gauge function—can be refined by adding further terms. Consider the *difference* between the function in question and its asymptotic form as a new function, and determine its asymptotic form. The result can be written

$$f(\varepsilon) \sim c_1\delta_1(\varepsilon) + c_2\delta_2(\varepsilon) \qquad \text{as} \quad \varepsilon \to 0 \tag{3.9a}$$

where the second gauge function $\delta_2(\varepsilon)$ is necessarily of smaller order than the first:

$$\delta_2(\varepsilon) = o[\delta_1(\varepsilon)] \qquad \text{or} \qquad \lim_{\varepsilon\to 0}\frac{\delta_2(\varepsilon)}{\delta_1(\varepsilon)} = 0 \tag{3.9b}$$

and the error is of still smaller order:

$$f(\varepsilon) = c_1\delta_1(\varepsilon) + c_2\delta_2(\varepsilon) + o[\delta_2(\varepsilon)] \tag{3.9c}$$

Further terms can be added by repeating this process. Thus one constructs the *asymptotic expansion* or *asymptotic series* to N terms, written as

$$\begin{aligned} f(\varepsilon) &\sim c_1\delta_1(\varepsilon) + c_2\delta_2(\varepsilon) + \cdots + c_N\delta_N(\varepsilon) \\ &= \sum_{n=1}^{N} c_n\delta_n(\varepsilon) \qquad \text{as} \quad \varepsilon \to 0 \end{aligned} \tag{3.10a}$$

and defined by

$$f(\varepsilon) = \sum_{n=1}^{N} c_n\delta_n(\varepsilon) + o[\delta_N(\varepsilon)] \qquad \text{as} \quad \varepsilon \to 0 \tag{3.10b}$$

If the function $f(\varepsilon)$ were known, together with the gauge functions $\delta_n(\varepsilon)$, the coefficients c_n of the asymptotic expansion could be computed in succession from

$$c_m = \lim_{\varepsilon\to 0}\frac{f(\varepsilon) - \sum_{n=1}^{m-1} c_n\delta_n(\varepsilon)}{\delta_m(\varepsilon)} \tag{3.11}$$

If the gauge functions are all integral positive powers of ε, one speaks of an *asymptotic power series*. As the number N of terms increases without limit, one obtains an *infinite asymptotic series*, which may be either convergent or divergent.

Some examples of asymptotic expansions are

$$\begin{aligned} &\sin 2\varepsilon \sim 2\varepsilon - \frac{4}{3}\varepsilon^3 + \frac{4}{15}\varepsilon^5 + \cdots \\ &\operatorname{sech}^{-1}\varepsilon \sim \log\frac{2}{\varepsilon} - \frac{1}{4}\varepsilon^2 - \frac{3}{32}\varepsilon^4 + \cdots \\ &K_0\left(\frac{1}{\varepsilon}\right) \sim \sqrt{\frac{\pi}{2}}\,\varepsilon^{1/2}e^{-1/\varepsilon}\left(1 - \frac{1}{8}\varepsilon + \frac{9}{128}\varepsilon^2 + \cdots\right) \\ &\int_0^\infty \frac{e^{-t}\,dt}{1+\varepsilon t} \sim 1 - \varepsilon + 2\varepsilon^2 - 6\varepsilon^3 + \cdots = \sum_{n=0}^{\infty}(-1)^n n!\,\varepsilon^n \\ &\log n! \sim (n + \tfrac{1}{2})\log n - n + \log\sqrt{2\pi} + \cdots \end{aligned} \tag{3.12}$$

The first two of these converge if extended indefinitely; the latter three diverge.

It is proper to regard a distant boundary condition as an asymptotic relation. For example, (2.6b) will henceforth be rewritten as

$$\psi \sim U\left[\frac{1}{4}\frac{\varepsilon}{a} r^2(1 - \cos 2\theta) + r \sin\theta\right] \qquad \text{as} \quad r \to \infty \tag{2.6b$'$}$$

It must be understood that this admits the possibility of an error of order $o(r)$. Actually, in the problem in question, the next term in this asymptotic expansion is $O(1)$; the stream function must be left unprescribed far upstream to within a constant, which corresponds physically to the displacement of the stagnation streamline.

3.4. Asymptotic Sequences

The process just described of constructing an expansion term by term is effectively that employed in perturbation solutions, such as those of Chapter II. Thus in each problem a perturbation solution generates a special sequence of gauge functions

$$\delta_1(\varepsilon), \quad \delta_2(\varepsilon), \quad \delta_3(\varepsilon), \quad \cdots \tag{3.13}$$

that are arranged in decreasing order: $\delta_{n+1} = o(\delta_n)$. This is the *asymptotic sequence* associated with the problem. It cannot be prescribed arbitrarily, because it must be sufficiently complete to describe logarithms, for example, if they appear. On the other hand, there are an unlimited number of alternatives to any particular asymptotic sequence:

$$\begin{aligned} \sin 2\varepsilon &\sim 2\varepsilon - \frac{4}{3}\varepsilon^3 + \frac{4}{15}\varepsilon^5 + \cdots \\ &\sim 2\tan\varepsilon - 2\tan^3\varepsilon - 2\tan^5\varepsilon + \cdots \\ &\sim 2\log(1+\varepsilon) + \log(1+\varepsilon^2) - 2\log(1+\varepsilon^3) + \cdots \\ &\sim 6\left(\frac{\varepsilon}{3+2\varepsilon^2}\right) - \frac{756}{5}\left(\frac{\varepsilon}{3+2\varepsilon^2}\right)^5 + \cdots \end{aligned} \tag{3.14}$$

The last two forms illustrate the fact that alternative sequences need not be equivalent: corresponding terms are not of the same order. Both the asymptotic sequence and the asymptotic expansion itself are unique if the perturbation quantity (e.g., ε) and the gauge functions (e.g., ε^m, $\log 1/\varepsilon$, $\log\log 1/\varepsilon$, etc.) are specified.

We have seen that one way of attacking a perturbation problem is to assume the form of a series solution. This requires guessing an appropriate asymptotic sequence. The simplest possibility is that, as in the

examples of Chapter II, it consists of integral powers, ε^n. Fractional powers may also occur, particularly in singular perturbation problems. Examples are:

$1, \varepsilon^{1/2}, \varepsilon, \varepsilon^{3/2}, \ldots$	Unseparated laminar flow over smooth bodies at high Reynolds number R, where $\varepsilon = 1/R$ (Van Dyke, 1962a)
$1, \varepsilon^{3/4}, \ldots$	Separated Oseen flow at high Reynolds number R, $\varepsilon = 1/R$ (Tamada and Miyagi, 1962)

Logarithms may occur at some stage, examples being

$1, \varepsilon, \varepsilon^2 \log\varepsilon, \varepsilon^2, \varepsilon^3\log\varepsilon, \varepsilon^3, \ldots$	Axisymmetric flow at low Reynolds number R, $\varepsilon = R$ (Proudman and Pearson, 1957)
$1, \varepsilon^2\log\varepsilon, \varepsilon^2, \varepsilon^4\log^2\varepsilon, \varepsilon^4\log\varepsilon, \varepsilon^4, \ldots$	Supersonic axisymmetric slender-body theory, ε = thickness parameter (Broderick, 1949)
$\varepsilon\log\varepsilon, \varepsilon, \varepsilon^2\log\varepsilon, \varepsilon^2, \ldots$	Newton-Busemann approximation for plane hypersonic flow past a blunt body, $\varepsilon = (\gamma - 1)/(\gamma + 1)$ (Chester, 1956a)
$(\log\varepsilon)^{-1}, (\log\varepsilon)^{-2}, (\log\varepsilon)^{-3}, \ldots$	Plane viscous flow at low Reynolds number R, $\varepsilon = R$ (Kaplun, 1957; Proudman and Pearson, 1957)
$1, \varepsilon, \varepsilon^2, \varepsilon^3, \varepsilon^4\log\varepsilon, \varepsilon^4, \ldots$	Subsonic thin-airfoil theory for round-nosed profile, ε = thickness parameter (Hantzsche, 1943)
$1, \varepsilon^{1/2}, \varepsilon, \varepsilon^{3/2}\log\varepsilon, \varepsilon^{3/2}, \ldots$	Laminar flow over flat plate at high Reynolds number R, $\varepsilon = 1/R$ (Goldstein, 1956; Imai, 1957a)

In the last two examples, earlier investigators had obtained erroneous solutions because they did not suspect the presence of logarithmic terms. Other examples arising in boundary-layer theory have been discussed by Stewartson (1957), who shows that even log log's occur in the asymptotic solution far downstream on a circular cylinder.

Exponentially small terms are seldom encountered, and are difficult to deal with. The following example shows that the estimate $O(e^{-1/\varepsilon})$ has very little practical value:

$$\lim_{\varepsilon\to 0}\frac{e^{-x/\varepsilon}}{e^{-1/\varepsilon}} = \begin{cases} 0, & x > 1 \\ 1, & x = 1 \\ \infty, & x < 1 \end{cases} \tag{3.15}$$

The question naturally arises how one can be sure of guessing the proper asymptotic sequence. Apparently there are no general rules, but the following principles are of some help:

(a) When in doubt, overguess. A superfluous term will drop out by producing for its coefficient a homogeneous problem, whose solution (if unique!) is zero.

(b) Be ready to suspect the presence of logarithmic terms at the first hint of difficulty.

(c) Iteration will sometimes (but not always!) produce the proper sequence automatically.

One usually has a feeling when the solution is progressing properly; all terms match, and complicated expressions often simplify. With experience, one learns when the absence of such reassurance suggests a re-examination of the assumed form of the series. However, the only fool-proof procedure is to leave the asymptotic sequence unspecified, and to determine it term by term in the course of the solution. This technique will be illustrated in Chapters VII and VIII.

3.5. Convergence and Accuracy of Asymptotic Series

We have seen that an infinite asymptotic series may either converge for some range of ε, or diverge for all ε. In perturbation problems one often neither knows nor cares whether the series converges. This point of view has been persuasively set forth by Jeffreys (1926). It is a fallacy to think that convergence is necessarily of practical value. Mathematical convergence depends upon the behavior of terms of indefinitely high order, whereas in physical problems one can calculate only the first few terms and hope that they rapidly approach the true solution. This requirement may sometimes be better met with a divergent than a convergent series. Thus the expansion

$$J_0(\varepsilon) = 1 - \frac{1}{4}\varepsilon^2 + \frac{1}{64}\varepsilon^4 - \frac{1}{2304}\varepsilon^6 + \cdots \qquad (3.16)$$

for the Bessel function has an infinite radius of convergence, but many terms are needed for accurate results unless ε is small. With $\varepsilon \geq 4$, for example, the first three terms actually increase in magnitude—so that the series appears to be diverging—and at least eight terms are required for three-figure accuracy. On the other hand, the asymptotic expansion

$$J_0\left(\frac{1}{\varepsilon}\right) \sim \sqrt{\frac{2\varepsilon}{\pi}}\left[\left(1 - \frac{9}{128}\varepsilon^2 + \cdots\right)\cos\left(\frac{1}{\varepsilon} - \frac{\pi}{4}\right) + \left(\frac{1}{8}\varepsilon - \frac{75}{1024}\varepsilon^3 + \cdots\right)\sin\left(\frac{1}{\varepsilon} - \frac{\pi}{4}\right)\right] \quad \text{as } \varepsilon \to 0 \qquad (3.17)$$

is divergent for all ε no matter how small, but a few terms give good accuracy for moderately small ε. The first term alone is correct to three significant figures for $(1/\varepsilon) = 4$.

The utility of an asymptotic expansion lies in the fact that the error is, by definition, of the order of the first neglected term, and therefore tends rapidly to zero as ε is reduced. For a fixed value of ε, the error can also be decreased at first by adding terms; but if the series is divergent, a point is eventually reached beyond which additional terms increase the error. This behavior is indicated in Fig. 3.1. These properties are often ideal

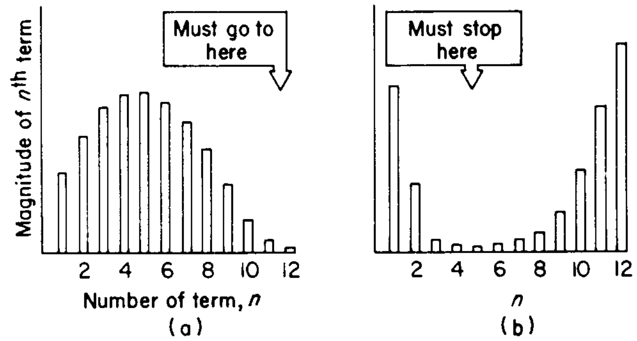

Fig. 3.1. Behavior of terms in series. (a) Slowly convergent series. (b) Divergent asymptotic series.

for practical purposes, particularly in parameter perturbations of the sort exemplified by thin-airfoil theory. Then only small values of ε are of practical interest; and only a few terms are calculated, so that the point of increasing error is not reached.

There are other problems, however, in which one attempts to make ε as large as possible. This is true of such parameter perturbations as expansions for large or small Reynolds numbers. It is almost always true of coordinate perturbations, because one intends to apply the results as far from the origin as possible. Under such circumstances, convergence may be of considerable practical interest. As discussed in Chapter X, one can sometimes improve the rate and radius of convergence, or even render a divergent series convergent.

From a physical point of view, the perturbation quantity ε assumes only positive real values. However, mathematical insight is often gained by envisioning its analytic continuation into the complex plane (Fig. 3.2). This is particularly useful when the solution is a power series in the perturbation quantity. Thus one considers the complex M^2-plane in the Janzen-Rayleigh method, the complex thickness-ratio-plane in thin-

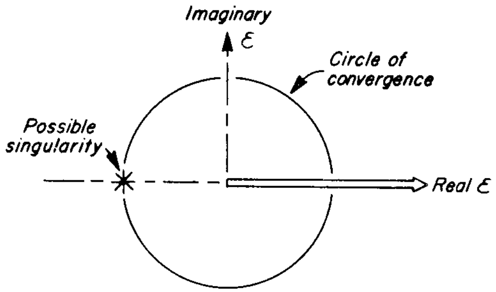

Fig. 3.2. Complex plane of perturbation quantity ε.

airfoil theory, and so on. Then one can take advantage of the powerful unifying viewpoint of complex-variable theory.

Some knowledge of—and feeling for—the principle of analytic continuation is essential. An analytic function has a power series development at every regular point. This converges within a circle that extends to the nearest singularity. A function defined in any region, or even on just a line segment, is ordinarily defined uniquely in a much larger region of the complex plane (possibly on several Riemann sheets), and can be completed by the process of analytic continuation.

See Note 15

Sometimes the first few terms of a perturbation solution suggest that the series converges, but has its radius of convergence limited for no apparent physical reason. According to the principles just outlined, this must result from a singularity in the complex plane of ε elsewhere than on the positive real axis. Several examples to be discussed in Chapter X suggest that in these circumstances the singularity ordinarily lies on the *negative* real axis, and in fact at $\varepsilon = -1$, if the most natural choice of variables has been made (Fig. 3.2). This artificial limitation can be eliminated by shifting the singularity to infinity using a simple conformal mapping, the *Euler transformation*:

$$\bar{\varepsilon} = \frac{\varepsilon}{1 - \varepsilon} \tag{3.18}$$

The radius of convergence is thereby extended to the nearest singularity in the complex plane of $\bar{\varepsilon}$, and the utility of the series often greatly improved.

3.6. Properties of Asymptotic Expansions

In substituting an assumed series solution into a perturbation problem one must carry out such operations as addition, multiplication, and

differentiation. Addition and subtraction are justified in general. Multiplication is valid if the result is an asymptotic expansion. It is not in general permissible to differentiate an asymptotic expansion with respect to either the perturbation quantity or another variable. These and other properties of asymptotic expansions are discussed by Erdélyi (1956). However, it appears that results are not available in sufficient generality to cover such commonly occuring series as those involving logarithmic terms. In practice, therefore, one ordinarily carries out formally such operations as differentiation with respect to another variable without attempting to justify them. When they are not justified, nonuniformities will arise in the solution.

In a physical problem, the coefficients in an asymptotic expansion will depend upon space or time variables other than ε. The series is said to be *uniformly valid* (in space or time) if the error is small uniformly in those variables. Examples of nonuniformity in x are

$$\begin{aligned} \frac{\varepsilon}{\sqrt{x}} &= O(\varepsilon), \qquad \text{but not uniformly near } x = 0 \\ \varepsilon \log x &= O(\varepsilon), \qquad \text{but not uniformly near } x = 0,\ \infty \end{aligned} \tag{3.19}$$

A *singular perturbation problem* is best defined as one in which no single asymptotic expansion is uniformly valid throughout the field of interest. The nonuniformities illustrated in (3.19) arise in practical singular perturbation problems. For example, the first will be encountered at a round leading edge in subsonic thin-airfoil theory, and the second at a sharp leading edge or in plane viscous flow at low Reynolds numbers.

We have observed that each function has a unique asymptotic expansion if the gauge functions or asymptotic sequence is prescribed. On the other hand, the statement is often made that different functions may have the same asymptotic expansion. The extent of this nonuniqueness may be understood by considering the following example:

$$\left.\begin{aligned} &\frac{1}{1+\varepsilon} \\ &\frac{1+e^{-1/\varepsilon}}{1+\varepsilon} \end{aligned}\right\} \sim 1 - \varepsilon + \varepsilon^2 - \cdots = \sum_{n=0} (-1)^n \varepsilon^n \tag{3.20a}$$

With respect to the gauge functions ε^n, these two functions have identical asymptotic expansions to any number of terms. Their difference is so small that it would become evident only if the infinite sequence of powers of ε were exhausted, for example by summing them. It happens that this is easily accomplished in this example by making the Euler transformation (3.18). With respect to powers of the new perturbation quantity $\bar{\varepsilon}$, the two functions have *different* asymptotic expansions (which happen to terminate):

$$\frac{1}{1+\varepsilon} \sim 1 - \bar{\varepsilon}$$

$$\frac{1+e^{-1/\varepsilon}}{1+\varepsilon} \sim 1 - \bar{\varepsilon} + e \cdot e^{-1/\bar{\varepsilon}} - e\bar{\varepsilon}e^{-1/\bar{\varepsilon}} \tag{3.20b}$$

See Note 11

One speaks of $e^{-1/\varepsilon}$ as being ***transcendentally small*** compared with the sequence of powers of ε, because it is $o(\varepsilon^m)$ for any m no matter how large. Similarly, on a different level of magnitude, ε itself is transcendentally small compared with the sequence $(\log \varepsilon)^{-m}$, which arises, for example, in plane viscous flow at low Reynolds numbers. The possibility of dealing with transcendentally small terms will be discussed in Chapter VIII in connection with that problem.

3.7. Successive Approximations

The perturbation problems encountered in fluid mechanics usually involve a system of ordinary or partial differential equations together with appropriate initial and boundary conditions. Integro-differential equations may also arise, as in problems involving thermal radiation. There are two systematic procedures for finding a solution by successive approximations, both of which were illustrated in the previous chapter:

(i) substitution of an assumed series,
(ii) iteration upon a basic approximate solution.

In the first method—which is somewhat more common—the guiding principle is that since the expansion must hold, at least in an asymptotic sense, for arbitrary values of the perturbation quantity ε, terms of like order in ε must separately satisfy each equality. That is, one can equate like powers of ε, terms in $\varepsilon^m \log^n \varepsilon$ having the same values of both m and n, etc.

Each method has its advantages and disadvantages, which can sometimes be exploited by working with a combination of the two. The most important of these differences can be summarized as follows:

(a) Iteration can be started only if an appropriate initial approximation is known. Series expansion is more automatic, because it can generate the basic approximation if one substitutes a series with the asymptotic sequence left unspecified. An example is given in Chapter VII.

(b) Iteration eliminates the need to guess the asymptotic sequence. It is therefore safer than assuming an expansion, unless one leaves the asymptotic sequence unspecified. For example, it will often though not always, in singular perturbation problems produce logarithms in higher-order terms that are missed if a power series is assumed.

(c) Beyond the second term the series expansion is more systematic because it produces only significant results, whereas iteration will, in nonlinear problems, generate some higher-order terms, to which no significance can be attached because others of equal order are missing. For example, in the Janzen-Rayleigh solution for a circle (Section 2.4), the next iteration step would clearly produce not only terms in M^4, which are correct and identical with those given by series expansion, but also some terms in M^6 and M^8 that should be disregarded because they are incomplete.

(d) Iteration will yield in a single step groups of terms of nearly the same order that require several steps in a series expansion. For example, in axisymmetric slender-body theory, each iteration adds one more of the following groups of terms:

$$[1], \quad [\varepsilon^2 \log \varepsilon, \varepsilon^2], \quad [\varepsilon^4 \log^2 \varepsilon, \varepsilon^4 \log \varepsilon, \varepsilon^4], \ldots$$

Whichever method is used, there are certain features common to all perturbation solutions. The basic solution may be linear or nonlinear, but all higher approximations are governed by linear equations with linear boundary conditions. An exception arises in the case of transonic and hypersonic small-disturbance theory, where the double limit process is specifically designed to retain in the perturbation the essential nonlinearity of the problem. In those special cases only the third and subsequent terms in the series satisfy linear problems. Otherwise, because first-order perturbations are linearly independent, they may be superimposed. For example, the three perturbations studied separately in Sections 2.2, 2.3, and 2.4 may be added to give the first-order solution for slightly compressible flow past a slightly distorted circle in a slight shear flow. Higher approximations, however, will be coupled through cross products of the various ε's.

Although the equations governing higher approximations are linear, they ordinarily contain nonconstant coefficients that depend upon the previous approximations. It is often possible to simplify the computation dramatically by taking advantage of known relations for those earlier results. A simple example will appear in Chapter IV in thin-airfoil theory, where both the differential equations and the boundary conditions are used to simplify their counterparts in subsequent approximations.

As in the Janzen-Rayleigh solution of Section 2.4, higher-order problems typically differ from one another only in the appearance of successively more complicated nonhomogeneous terms (depending upon previous approximations) in the differential equations. These are accounted for by finding a particular integral. Usually the best way of seeking a particular integral is to guess it. Only after that attempt has failed should one apply more sophisticated processes of analysis.

One should also be on the alert for the occasional problem in which a *general particular integral* can be found in terms of previous approximations. We illustrate this possibility by two examples from the small-disturbance theory of axisymmetric compressible flow. First, in an approximate linearized theory of supersonic propellers the second-order velocity potential is found to satisfy the nonhomogeneous wave equation

$$\square^2\phi_2 = (1 - M^2)\phi_{2xx} + \phi_{2rr} + \frac{\phi_{2r}}{r} + \frac{\phi_{2\theta\theta}}{r^2} = 2M\phi_{1x\theta} \qquad (3.21a)$$

where the first approximation is a solution of the homogeneous equation $\square^2\phi_1 = 0$. Burns (1951) has noticed that a particular integral is always given in terms of the first-order solution by

$$\phi_{2p} = \frac{M}{1 - M^2}x\phi_{1\theta} \qquad (3.21b)$$

Second, if one seeks to improve the linearized theory of subsonic or supersonic flow past a body of revolution by considering nonlinear terms, the second-order potential satisfies

$$\square^2\phi_2 = M^2\{[2 + (\gamma - 1)M^2]\phi_{1x}\phi_{1xx} + 2\phi_{1r}\phi_{1xr} + \phi_{1r}^2\phi_{1rr}\} \qquad (3.22a)$$

where again $\square^2\phi_1 = 0$. Van Dyke (1952) has found that to within third-order terms a particular integral is given by

$$\phi_{2p} = M^2\left[\phi_{1x}\left(\phi_1 - \frac{\gamma + 1}{2}\frac{M^2}{1 - M^2}r\phi_{1r}\right) - \tfrac{1}{4}r\phi_{1r}^3\right] \qquad (3.22b)$$

Particular integrals of this sort are also ordinarily found by trial rather than by systematic analysis. Analogous solutions of some homogeneous equations were discovered by Lin and Schaaf (1951) for viscous flow.

3.8. Transfer of Boundary Conditions

Often a boundary condition is imposed at a surface whose position varies slightly with the perturbation quantity ε. The surface may be that of a solid body (as for the slightly distorted circle of Section 2.3), a free streamline, a shock wave, etc. (Fig. 3.3). In order to carry out a systematic expansion scheme, the boundary condition must in each case be expressed in terms of quantitites evaluated at the basic position of the surface, corresponding to $\varepsilon = 0$. Otherwise ε will appear implicitly as well as

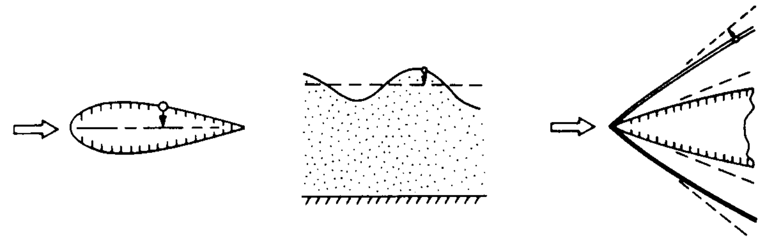

Fig. 3.3. Examples of transfer of boundary conditions.

explicitly in the perturbation expansion, so that the result is unnecessarily complicated, the series is not an asymptotic expansion, and it is not possible to equate like functions of ε.

The transfer of a boundary condition is effected by using a knowledge of the way in which the solution varies in the vicinity of the basic surface. Often the solution is known to be analytic in the coordinates, in which case the transfer is accomplished by expanding in Taylor series about the values at the basic surface. In the first approximation this usually means that the condition is simply shifted from the disturbed to the basic surface. In axisymmetric slender-body theory, on the other hand, the solution is singular on the axis, but the transfer can be carried out using the fact that the velocity potential varies near the axis like $\log r$ or the radial velocity like $1/r$.

After the solution is calculated, values of flow quantities are often required at the body or other surface. These can be found in simplest form by repeating the transfer process, expressing them in terms of values at the basic surface. Both of these transfer processes are illustrated in Chapter IV; see also Exercises 2.1, 2.3, and 3.2.

3.9. Direct Coordinate Expansions

Perturbation problems in which the small quantity is a dimensionless combination involving the coordinates (space or time) rather than the parameters alone have certain special features. A useful discussion of the distinction between parameter and coordinate expansions is given by Chang (1961). The essential point is that no derivatives with respect to a parameter occur, and it is therefore possible to calculate the solution for one value of the parameter without considering other values. Ordinarily one seeks an approximation for either small or large values of one of the coordinates. It is useful to designate these respectively as *direct* and *inverse coordinate expansions.*

A direct coordinate expansion is natural to a problem governed by parabolic or hyperbolic differential equations. One expands the solution for small values of a time-like variable, which can of course be a space coordinate rather than actual time. The perturbation quantity must increase in the positive sense of that variable, following time's arrow. Then there is no backward influence, so that each term in the perturbation series is independent of later ones, and can be calculated in its turn. The result is a perturbation expansion that describes the early stages of the evolution of the solution from a known basic initial state.

The following are typical examples of direct coordinate expansions. Goldstein and Rosenhead (1936) have calculated the growth of the boundary layer on a cylinder set impulsively into motion by expanding in powers of the time, the governing equations being parabolic. Near the stagnation point of a circle, for example, they find the skin friction to be given by

$$\tau \sim \rho\nu^{1/2}U_1^{3/2}x\,\frac{1}{\sqrt{\pi U_1 t}}\,[1 + 1.42442(U_1 t) - 0.21987(U_1 t)^2 + \cdots] \qquad (3.23)$$

where U_1 is the gradient of inviscid surface speed at the nose and x the distance along the surface. As shown in Fig. 3.4, this series obviously

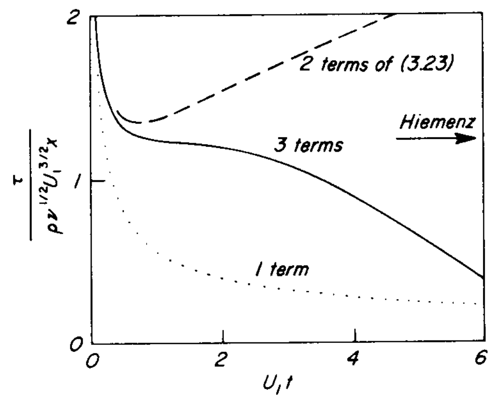

Fig. 3.4. Growth of skin friction near a stagnation point.

diverges for large time, where it should approach Hiemenz' result for steady flow. Such nonuniformity usually arises in direct coordinate expansions (but see Sections 10.6 to 10.8).

A case where a space coordinate assumes the time-like role is Blasius' expansion of the steady boundary layer on a cylinder in powers of the distance x from the stagnation point, the boundary-layer equation (2.25a) being parabolic for steady as well as unsteady motion. The result (1.5) for the parabola is believed to converge only for $x/a < \pi/4$ (Van Dyke, 1964a). Another such case, involving hyperbolic equations, is the axisymmetric Crocco problem: a perturbation of the self-similar solution for supersonic flow past a circular cone yields the initial flow gradients at the tip of an ogive of revolution (Cabannes, 1951).

For elliptic equations, coordinate expansions usually provide only qualitative results. One ordinarily encounters a boundary-value rather than an initial-value problem. Then because of backward influence any local solution depends on remote boundary conditions, and it is not possible to calculate successive terms of an expansion for small values of a coordinate. All that can be achieved is to find the *form* of the expansion, each term being indeterminate by one or more constants. For example, Carrier and Lin (1948) have examined the nature of viscous flow near the leading edge of a flat plate by expanding for small radius. Their series for the stream function is, after correction

See Note 8

$$\psi = 2Ar^{3/2}\left(\cos\frac{\theta}{2} - \cos\frac{3}{2}\theta\right) - Br^{5/2}\left(\cos\frac{\theta}{2} - \cos\frac{5}{2}\theta\right)$$
$$- \frac{A^2}{4\nu}\,r^3\{2\sin^2\theta[\log r\sin\theta + (\theta - \pi)\cos\theta]$$
$$+ \frac{8}{5}(\sin 2\theta - 2\sin\theta) + \frac{4}{3}C\sin^3\theta\} + \cdots \qquad (3.24)$$

where $\theta = 0$ on the plate. The constants $A, B, C, \ldots$ depend upon boundary conditions far outside the range of validity of the expansion, and are therefore undeterminable within the framework of the analysis.

An exception is the unconventional case of an initial-value problem for an elliptic equation. In the inverse problem of supersonic flow past a blunt body (Fig. 3.5), the detached shock wave is prescribed, and one seeks the body that produces it. Although the flow is subsonic near the axis, the stream function can be expanded in powers of the distance downstream of the shock wave, and the coefficients found in succession. Cabannes (1956) has calculated seven terms of the series when $M = 2$.

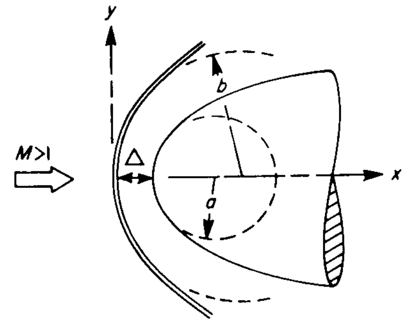

Fig. 3.5. Supersonic flow past a blunt body.

For a paraboloidal shock wave, the stream function is described near the axis by

$$\frac{\psi}{y^2} = \frac{1}{2} - \frac{5}{3}\frac{x}{a} - \frac{85}{18}\left(\frac{x}{a}\right)^2 - \frac{215}{27}\left(\frac{x}{a}\right)^3 - \frac{11,945}{324}\left(\frac{x}{a}\right)^4$$

$$- \frac{7051}{243}\left(\frac{x}{a}\right)^5 - \frac{1,817,909}{2916}\left(\frac{x}{a}\right)^6 - \cdots \tag{3.25}$$

Aside from such rare exceptions, direct coordinate expansions can be effectively applied to elliptic equations only by treating them as if they were parabolic and truncating the series at a finite number of terms.

The blunt-body problem is so intractable that several investigators have been willing to introduce another independent variable in order to render the equations parabolic or hyperbolic. Thus Cabannes (1953) considers an impulsive start, and expands the unsteady flow field in powers of the time. He suggests that the accuracy will increase with Mach number; at $M = \infty$ and for $\gamma = 7/5$ he finds as the ratio of shock-wave standoff distance Δ to nose radius a for any smooth body:

$$\frac{\Delta}{a} = \frac{1}{5}\left(\frac{Ut}{a}\right) - (n-1)\frac{7}{75}\left(\frac{Ut}{a}\right)^2 + \cdots \tag{3.26}$$

Here n is the number of space dimensions: $n = 2$ for plane flow and $n = 3$ for axisymmetric flow. We can expect the result to be the more accurate for plane motion, because it is exact for the one-dimensional piston problem $n = 1$. The series evidently diverges for large time, but Cabannes attempts to estimate its limit as the maximum given by the two known terms. This yields $\Delta/a = 0.107$ for plane flow, compared with accurate numerical calculations of 0.377 for the circular cylinder. We reconsider this discrepancy in Section 10.7.

3.10. Inverse Coordinate Expansions

In contrast to direct expansions, which usually possess a finite radius of convergence, inverse expansions for large values of a coordinate ordinarily appear to be divergent asymptotic series. They also suffer from indeterminacy irrespective of the type of the governing equations. For elliptic equations the situation is the same as that discussed above for a direct expansion. However, the undetermined constants can sometimes be related to simple integral properties of the solution. Thus the first few constants in the expansion for subsonic flow far from a finite body can be identified with its lift, drag, moment, etc. (Imai, 1953; Chang, 1961). For parabolic and hyperbolic equations, indeterminacy arises because the expansion runs contrary to the direction of the time-like variable. Eigensolutions therefore appear at an early stage, whose constant multipliers depend upon certain details of the previous history. Occasionally a few of these can be supplied without detailed knowledge of past events by invoking some global conservation principle (cf. Section 4.5). More often a sequence of constants remains undetermined.

The form of an inverse coordinate expansion varies widely with the type of the equations, the number of space dimensions, and the extent of the body. In some problems the leading term of the expansion is obvious; for example, it is evidently the undisturbed stream for the steady flow far from a finite body, the conical motion for flow far downstream on a blunted cone, and the corresponding steady motion for the flow long after an impulsive start. Then one perturbs that basic solution to find how it is approached. The approach is sometimes algebraic, in inverse powers of the large coordinate, as for noncirculatory potential flow far from a finite body (Imai, 1953). It very often involves logarithmic as well as algebraic terms, as for circulatory potential motion or viscous flow far from a body (Chang, 1961). In time-dependent problems it is often exponential, as for unsteady viscous or free-streamline flows (Kelly, 1962; Curle, 1956).

We quote one example. The Blasius series for the boundary layer (Section 2.6) is a direct coordinate expansion for small distances from the stagnation point. On a parabolic cylinder we can supplement that approximation by an inverse expansion for large distances. It is evident that the leading term is the solution for the flat plate, because far downstream the nose radius is negligible compared with the dimensions of interest. Perturbing that basic solution yields as the complement to (1.5) for the coefficient of skin friction

$$c_f \sim 0.6641\sqrt{\frac{\nu}{Ux}}\left[1 + 0.3006\,\frac{\log(2x/a)}{x/a} + \frac{C_1 - 1}{2x/a} + \cdots\right] \tag{3.27}$$

Here x is the abscissa, and C_1 is an undetermined constant multiplying the first of an infinite sequence of eigensolutions for the flat-plate solution (Section 7.6).

Sometimes the leading term is by no means obvious. Free-streamline motion provides an example of the complications that may appear. In plane flow the width of the deadwater region increases far downstream like $x^{1/2}$, but in axisymmetric flow it has the unlikely growth of $x^{1/2}(\log x)^{-1/4}$ (Levinson, 1946).

3.11. Change of Type and of Characteristics

A curious feature of perturbation methods is that they may spuriously change the type of the governing partial differential equations. A striking example is Prandtl's boundary-layer approximation. Although the Navier-Stokes equations are elliptic, they are replaced by parabolic equations inside the boundary layer, and by elliptic or hyperbolic ones outside, according as the flow is subsonic or supersonic. Again, in the theory of surface waves the elliptic Laplace equation is replaced by a nonlinear hyperbolic equation in the shallow-water approximation (Stoker, 1957). Conversely, the hyperbolic equations of inviscid supersonic motion become elliptic in the slender-body approximation (Ward, 1955).

These changes of type imply significant changes in the regions of influence and dependence, and in the boundary conditions required. Thus Prandtl's boundary-layer equations, because they are parabolic, can be integrated step-by-step downstream. The backward influence of their elliptic antecedents has been suppressed, but will reassert itself in higher approximations (cf. Chapter VII). Similarly, the Kutta-Joukowski condition must be abandoned at a subsonic trailing edge in thin-wing theory, because its upstream influence has been lost.

Some assumption of smoothness underlies any such change of type. The true type of the equations must be recognized wherever that assumption is violated. Otherwise the perturbation solution will break down at least locally. Thus boundary-layer theory is invalid near a corner, as are the shallow-water and slender-body approximations. These thus become singular perturbation problems as the result of discontinuities in the boundary conditions.

Often the tendency toward change of type is incomplete; the perturbation equations merely become "less hyperbolic" or "more elliptic." For hyperbolic equations this means that the characteristic surfaces are changed. An example is supersonic small-disturbance theory, where at each stage the true characteristics are approximated in the perturbation equations by the free-stream Mach cones. Again this defect is inconsequential if the body is sufficiently smooth, but otherwise leads to nonuniformities (cf. Chapter VI and Exercise 3.4).

EXERCISES

3.1. *Modified hypersonic similarity rule.* According to hypersonic small-disturbance theory the pressure coefficient on a slender wedge or cone of semivertex angle ε has the form

$$C_{p_s} \approx \varepsilon^2 f(M\varepsilon)$$

Devise an alternative form that can be expected to be superior for thick bodies in view of Newtonian impact theory, according to which the pressure coefficient at any point on a body in hypersonic flight is twice the square of the sine of the angle of the surface with the stream. Exhibit the degree of improvement realized by making numerical comparison with the full solutions at $M = \infty$. Investigate whether the result can be extended further to thick bodies at lower speeds if one is guided by the supersonic similarity rule

$$C_{p_s} \approx \frac{1}{M^2 - 1} F(\sqrt{M^2 - 1}\,\varepsilon)$$

3.2. *Transfer of tangency condition.* A small sphere pulsates in still air, its radius varying with time as $\varepsilon f(t)$, and thereby produces weak outgoing waves whose velocity potential satisfies the acoustic equation $\phi_{tt} = c^2 \nabla^2 \phi$. Show that if the function f is sufficiently smooth the tangency condition can approximately be transferred to the origin as

$$\lim_{r \to 0} r^2 \phi_r = \frac{1}{3} \frac{d}{dt} [\varepsilon f(t)]^3$$

Calculate ϕ using this condition. How smooth must f be? What happens to the solution if that restriction is violated?

3.3. *Estimate of ultimate value from coordinate expansion.* Test Cabannes' idea for estimating the value of (3.26) at infinite time by applying it to (3.23) and (1.5), where the answers are known. Should one choose the value at the inflection point when no extremum exists? Try to devise a better or more rational scheme of this sort.

3.4. *Effect of change of characteristics.* The following problem is a mathematical model of steady supersonic flow over the upper surface of a thin airfoil:

$$\phi_{yy} - \phi_{xx} = \varepsilon \phi_{xx}, \qquad \phi(0, y) = \phi_x(0, y) = 0 \quad \text{for} \quad y > 0$$
$$\phi_y(x, 0) = \varepsilon f'(x), \qquad \text{where} \quad f(x) = 0 \quad \text{for} \quad x < 0$$

Then the matching principle (4.36) is to be applied with $m = n = 2$, for which we find

2-term outer expansion: $$q \sim U\left\{1 + \frac{2}{\pi}\varepsilon\left[2 + (1-s)\log\frac{s}{2-s}\right]\right\} \tag{4.51a}$$

rewritten in inner variables: $$= U\left\{1 + \frac{2}{\pi}\varepsilon\left[2 + (1 - e^{-1/\varepsilon}S)\log\frac{e^{-1/\varepsilon}S}{2 - e^{-1/\varepsilon}S}\right]\right\} \tag{4.51b}$$

expanded for small ε: $$= U\left[\left(1 - \frac{2}{\pi}\right) + \frac{2}{\pi}\varepsilon(2 + \log S - \log 2) + \cdots\right] \tag{4.51c}$$

2-term inner expansion: $$= U\left[\left(1 - \frac{2}{\pi}\right) + \frac{2}{\pi}\varepsilon(2 + \log S - \log 2)\right] \tag{4.51d}$$

rewritten in outer variables: $$= U\left[1 + \frac{2}{\pi}\varepsilon\left(2 + \log\frac{s}{2}\right)\right] \tag{4.51e}$$

2-term inner expansion: $$q \sim U_i S^{\tan^{-1}2\varepsilon/(\pi - \tan^{-1}2\varepsilon)} \tag{4.52a}$$

rewritten in outer variables: $$= U_i(e^{1/\varepsilon}s)^{\tan^{-1}2\varepsilon/(\pi - \tan^{-1}2\varepsilon)} \tag{4.52b}$$

expanded for small ε: $$= e^{2/\pi}U_i\left[1 - \frac{2}{\pi}\varepsilon\left(\log s - \frac{2}{\pi}\right) + \cdots\right] \tag{4.52c}$$

2-term outer expansion: $$= e^{2/\pi}U_i\left[1 - \frac{2}{\pi}\varepsilon\left(\log s - \frac{2}{\pi}\right)\right] \tag{4.52d}$$

rewritten in inner variables: $$= e^{2/\pi}U_i\left[\left(1 - \frac{2}{\pi}\right) + \frac{2}{\pi}\varepsilon\left(\log S + \frac{2}{\pi}\right)\right] \tag{4.52e}$$

The rule (4.50) has been used in obtaining (4.52c). The additional last step (4.51e) or (4.52e) is required because the comparison of the two results must be made in terms of either outer or inner variables alone. Then equating (4.51e) and (4.52d) yields (but see exercise 4.5)

See Note 4

$$U_i = e^{-2/\pi}U\left[1 + \frac{2}{\pi}\varepsilon\left(2 - \log 2 - \frac{2}{\pi}\right) + \cdots\right] \tag{4.53}$$

This completes the inner expansion (4.49) to second order. The reader can, using (4.24), find the next term in (4.53) by matching with $m = n = 3$.

4.12. A Shifting Correction for Round Edges

We return to the singularity at a round edge, and describe an alternative way of correcting it. This serves to introduce a second general method of handling singular perturbation problems, which is discussed in detail in Chapter VI.

Consider the simplest round-nosed shape, the parabola. We must examine the complete velocity field, because the surface speed suffers distortion through transfer of the boundary conditions. For the parabola $y = \pm\varepsilon\sqrt{2x}$ of nose radius ε^2 the first-order complex velocity can be extracted from that (4.12) for the ellipse as

$$\phi_x - i\phi_y = 1 - \frac{i\varepsilon}{\sqrt{2z}} + O(\varepsilon^2), \qquad z = x + iy \tag{4.54}$$

We compare with the corresponding exact result which, by conformal mapping or separation of variables in parabolic coordinates, is found to be

$$\phi_x - i\phi_y = 1 - \frac{i\varepsilon}{\sqrt{2(z - \frac{1}{2}\varepsilon^2)}} \tag{4.55}$$

We see that the first approximation (4.54) becomes exact if the origin of coordinates is simply shifted by $\frac{1}{2}\varepsilon^2$. This removes the square-root singularity from the vertex and places it inside the parabola at the focus, which is the singular point of the conformal mapping. Thus thin-airfoil theory is seen to give the exact source distribution for a parabola, but displaced by half its nose radius.

This remarkable property must hold approximately for any round-nosed airfoil. For that reason Munk (1922) advocated modifying the abscissae to shift the leading edge of any profile by half its radius. It is clear that a simple translation is not correct for a finite airfoil, because that would leave the trailing edge misplaced. For the ellipse of Fig. 4.2, a uniform contraction of the x-coordinate evidently provides the required shift at both ends, as do an unlimited number of more complicated strainings. To retain the benefits of complex-variable theory, we strain instead the single variable $z = x + iy$.

Lighthill (1949a) has proposed the following principle for finding strained coordinates that restore uniform validity in a wide class of singular perturbation problems:

> Higher approximations shall be
> no more singular than the first. (4.56)

We illustrate this principle by application to the thin-airfoil expansion and check that it satisfies the second-order similarity rule for airfoils (Hayes, 1955):

If at

$$M = 0, \qquad C_p = \qquad \varepsilon C_{p1}(x) + \varepsilon^2 C_{p2}(x) + \cdots$$

then for

$$M \geqslant 0, \qquad C_p = \frac{\varepsilon}{\sqrt{1 - M^2}} C_{p1}(x) + \varepsilon^2 \frac{(\gamma + 1)M^4 + 4(1 - M^2)}{4(1 - M^2)^2} C_{p2}(x) + \cdots$$

Chapter V

THE METHOD OF MATCHED ASYMPTOTIC EXPANSIONS

5.1. Historical Introduction

We introduced in the preceding chapter a method for treating singular perturbation problems that is a generalization of the boundary-layer theory of Prandtl (1905). This has in the past been called the method of "inner and outer expansions" or of "double asymptotic expansions." We prefer to follow Bretherton (1962) in speaking of the *method of matched asymptotic expansions*.

The ideas underlying the method have grown through the years. It was being used in the 1950's by Friedrichs (1953, p. 126; 1954, p. B-184) and his students. It was systematically developed and applied to viscous flows at the California Institute of Technology. Kaplun (1954) introduced the formal inner and outer limit processes for boundary-layer theory, and the corresponding inner and outer expansions. Later, in studying flows at low Reynolds number, Kaplun and Lagerstrom (1957) made a penetrating analysis of the matching process (see also Lagerstrom, 1957). Kaplun (1957) used those ideas to gain deeper insight into the resolution of the Stokes paradox for plane flow at low Reynolds number. Lagerstrom and Cole (1955) evaluated the method in comparison with new exact solutions of the Navier-Stokes equations for a sliding and expanding circular cylinder. Coles (1957) applied it to some special solutions for the compressible boundary layer.

Proudman and Pearson (1957) applied this expansion method to treat flow past a sphere and circular cylinder at low Reynolds number. Goldstein (1956, 1960) and Imai (1957a) gave the first correct extension of Blasius' boundary-layer solution for the semi-infinite flat plate. Ting (1959) solved the riddle of the course of the viscous shear layer between two streams of different speeds. See Note 16

Following this developmental period, the method of matched asymptotic expansions was applied to a variety of problems in fluid mechanics. Most of the earlier applications were to viscous flows. For example, Germain and Guiraud (1960, 1962) and Chow and Ting (1961) calculated the effect of curvature upon the structure of a shock wave. Murray (1961) and Ting (1960) found the effect of external vorticity upon the boundary layer near and far from the leading edge of a flat plate. Chang (1961) clarified the behavior of viscous flow far from a finite body. Flows at low Reynolds numbers have been analyzed for ellipsoids by Breach (1961), for a spinning sphere by Rubinow and Keller (1961), for a circle in shear flow by Bretherton (1962), and so on.

The method is equally successful for inviscid flows. The preceding chapter gives examples in incompressible flow. Cole and Messiter (1957) have studied transonic flow past slender bodies. Although the method appears to be less popular in the Soviet Union, Bulakh (1961) has used it to correct linearized supersonic conical flow and its higher approximations in the vicinity of the bow shock wave. Similarly, Fraenkel and Watson (1964) have attacked the "pseudotransonic" flow past a triangular wing that occurs when the bow wave lies close to the leading edge. Yakura (1962) has analyzed the entropy layer produced by slightly blunting the tip of a body in hypersonic flow.

Since 1960, applications of the method have proliferated in many fields of fluid mechanics, as well as in other branches of applied mathematics. Some recent examples are discussed in later chapters of this book.

5.2. Nonuniformity of Straightforward Expansion

Before we discuss the details of the method of matched asymptotic expansions, it is useful to inquire how singular perturbation problems arise. What is the source of the nonuniformity? Can we predict whether a given physical problem will lead to regular or singular perturbations?

The classical warning of singular behavior is familiar from Prandtl's boundary-layer theory. A small parameter multiplies one of the highest derivatives in the differential equations. Then in a straightforward perturbation scheme that derivative is lost in the first approximation so that the order of the equations is reduced. One or more boundary conditions must be abandoned, and the approximation breaks down near where they were to be imposed. This happens except in the unlikely circumstance that the original boundary conditions are consistent with the reduced equations.

It is often helpful to examine linear ordinary differential equations as mathematical models that illustrate the essential features of more complicated problems. A simple model that illustrates loss of the highest

derivative in boundary-layer theory is given by Friedrichs (1942) as

$$\varepsilon \frac{d^2f}{dx^2} + \frac{df}{dx} = a, \qquad f(0) = 0, \qquad f(1) = 1 \tag{5.1}$$

The exact solution is

$$f(x;\ \varepsilon) = (1 - a)\frac{1 - e^{-x/\varepsilon}}{1 - e^{-1/\varepsilon}} + ax \tag{5.2}$$

However, setting $\varepsilon = 0$ reduces the differential equation to first order, so that both boundary conditions cannot be satisfied unless it happens that $a = 1$. The exact solution shows that the condition at $x = 0$ must be dropped. Then the approximate solution for small ε is

$$f(x; \varepsilon) \sim (1 - a) + ax \tag{5.3}$$

As indicated in Fig. 5.1, this is a good approximation except within the "boundary layer" where $x = O(\varepsilon)$. Introduction of a magnified inner coordinate X appropriate to that region by setting

$$f(x; \varepsilon) = F(X; \varepsilon), \qquad X = \frac{x}{\varepsilon} \tag{5.4}$$

transforms the original problem (5.1) to

$$\frac{d^2F}{dX^2} + \frac{dF}{dX} = a\varepsilon, \qquad F(0) = 0, \qquad F\left(\frac{1}{\varepsilon}\right) = 1 \tag{5.5}$$

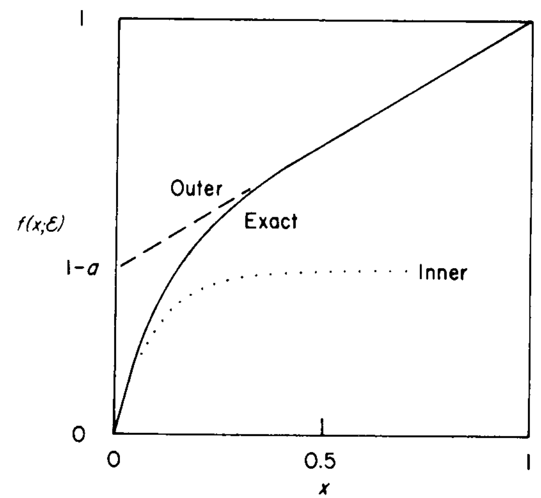

Fig. 5.1. Solution of model problem.

If we now set $\varepsilon = 0$, the solution of the differential equation that satisfies the inner boundary condition is any multiple of $(1 - e^{-X})$. Imposing the outer boundary condition would give the multiplicative factor as unity, but the exact solution shows that this is incorrect. The outer boundary condition must be abandoned for the inner solution just as the inner condition was dropped from the outer solution. Instead, the inner solution must be matched to the outer solution using the matching principle (4.36). Thus one finds the uniform first approximation for small ε:

$$f(x; \varepsilon) \sim \begin{cases} (1 - a) + ax & \text{as } \varepsilon \to 0 \text{ with } x > 0 \text{ fixed} & (5.6a) \\ (1 - a)(1 - e^{-X}) & \text{as } \varepsilon \to 0 \text{ with } X = \dfrac{x}{\varepsilon} \text{ fixed} & (5.6b) \end{cases}$$

The warning provided by loss of a highest derivative is more often than not absent from a singular perturbation problem. In the thin-airfoil theory of Chapter IV, the nonuniformity arises not from the differential equation but from the boundary conditions. Likewise, for viscous flow at low Reynolds numbers the highest derivatives are all preserved in the approximate equations of Stokes; the nonuniformity is associated rather with the infinite extent of the fluid. Evidently it would be useful to have a more reliable indication of nonuniformity.

5.3. A Physical Criterion for Uniformity

In physical problems a more general warning of singular behavior can be based upon dimensional reasoning. We have seen that inherent nonuniformity will be suppressed by exceptional boundary conditions, so that one can give only necessary and not sufficient conditions for nonuniformity. We therefore state the following rule instead as a positive test for uniformity:

> A perturbation solution is uniformly valid in the space and time coordinates unless the perturbation quantity ε is the ratio of two lengths or two times. (5.7)

This criterion may be understood by considering first a parameter perturbation. The geometry of the problem will be characterized by a typical major dimension that we may call the *primary reference length*. Examples are the radius of the circular cylinder in Chapter II, and the chord length of the thin airfoil in Chapter IV. This length is the natural basis for forming dimensionless coordinates—a characteristic speed also being required in unsteady flows—and these constitute the straightforward outer variables. Nonuniform behavior is possible only if the parameters in the problem provide another *secondary reference length* whose ratio to the first tends to zero or infinity as ε vanishes. This second length, if properly chosen, is the basis for the inner variables.

A familiar problem involving two disparate lengths is potential flow past a thin round-nosed airfoil. The geometry is characterized by the chord length except close to the leading edge, where it is dominated by the nose radius. Since the ratio of these two lengths vanishes with the square of the thickness ratio, our criterion (5.7) suggests that the thin-airfoil solution could be singular. In Chapter IV this possibility was seen to be realized, outer variables being based on the chord and inner variables on the nose radius.

Our criterion shows that a coordinate perturbation is never safe from nonuniformity in the remaining coordinates. The latter may be made dimensionless using either the primary reference length or the perturbation coordinate; and because the ratio of these two lengths tends to zero or infinity, they provide the scales for an inner and an outer expansion.

The following are examples of parameter perturbations that involve only one characteristic length scale, and are therefore necessarily regular according to our criterion. Slightly compressible flow past a circle (Section 2.4) contains the radius as the only characteristic length, the perturbation parameter being formed from a ratio of speeds or energies rather than lengths. The slightly distorted circle in potential flow (Section 2.3) involves two lengths, but they are of the same order of magnitude, their ratio approaching unity rather than zero or infinity in the limit $\varepsilon \to 0$.

The following parameter perturbations involve two disparate lengths, and do as a consequence exhibit singular behavior (Fig. 5.2). A lifting wing is characterized by its chord and its span, and their ratio vanishes

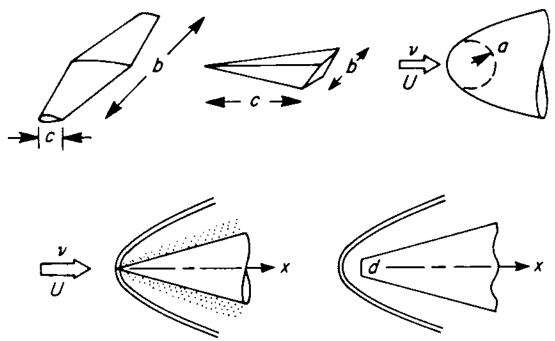

Fig. 5.2. Singular perturbation problems involving two disparate lengths.

in Prandtl's lifting-line theory (cf. Section 9.2) and becomes infinite in the slender-wing approximation. A body in viscous flow is characterized by not only a geometric dimension, but also the viscous length ν/U; and their ratio is the Reynolds number, which vanishes in Stokes flow (Chapter VIII) and becomes infinite in boundary-layer theory (Chapter VII). This example illustrates that the secondary reference length is not always a geometric dimension.

The following are some coordinate perturbations that are singular (Fig. 5.2). Viscous supersonic flow over a cone or a wedge (including the flat plate) can be solved approximately for distances from the vertex large compared with the viscous length ν/U. Flow of a gas undergoing vibrational or chemical reactions can be treated similarly using a characteristic relaxation length, as can the effect of slight blunting. The impulsive start of a body through viscous fluid can be expanded in powers of time, referred to a characteristic length and speed. Viscous flow far from a body can be expanded in inverse powers of the radius, referred to a typical dimension.

Although the following examples involve two disparate lengths, they are nevertheless regular perturbation problems. Potential flow past a wavy wall, a cusp-nosed airfoil, or any thin shape free of stagnation points is a regular perturbation in the thickness parameter. Uniform shear flow past a circle (Section 2.2) shows the superficial symptoms of non-uniformity at infinity, in that the ratio of the perturbation to the basic solution in (2.11) grows like εr; but the result is exact and therefore uniformly valid. However, any other shear distribution would lead to a singular perturbation problem (Exercises 2.4 and 5.7). In an inviscid fluid the expansion for flow far from a body is regular, as is that for an impulsive start. Even the archetypical nonuniformity of Prandtl's boundary-layer theory disappears if in place of a fixed solid surface one prescribes a distensible skin moving at just the speed of the potential flow.

A problem may involve more than two disparate lengths, associated with a multiple limit process; then multiple nonuniformities are possible. In the case of three layers one may speak of the *outer*, *middle*, and *inner expansions*. An example is viscous flow at high Reynolds number and high or low Prandtl number, where the thermal boundary layer is much thicker or thinner than the viscous layer. An inviscid example is analyzed in Section 9.9.

A coordinate perturbation may sometimes be replaced by a parameter perturbation. For example, the expansion for distances far from the tip of a blunted wedge (Fig. 5.2) may be regarded instead as the solution for a wedge of finite length whose nose thickness tends to zero (Section 9.9). Thus in conical geometry an angle plays the role of a length. Consequently, nonuniformity is possible in conical flows if the perturbation parameter is the ratio of two angles. Examples are a flat elliptic cone, where the straightforward perturbation solution evidently fails at the leading edges just as in thin-airfoil theory, and a circular cone at infinitesimal angle of attack, which exhibits near its surface the vortical layer of Ferri (1950).

5.4. The Role of Composite and Inner Expansions

We have seen that a singular perturbation flow problem typically involves two disparate lengths. As a result, the straightforward perturbation solution with coordinates referred to the primary reference length fails in regions where the secondary reference length is the relevant dimension. The secondary reference length is not always the obvious one. It is clearly the chord for a wing of high aspect ratio, the thickness for a flat-nosed airfoil, and the viscous length ν/U for flow at low Reynolds number. However, at high Reynolds number it is the square root of the product of the viscous and geometric lengths. For a round-nosed airfoil it is not the thickness but the nose radius, which is proportional to the square of the thickness divided by the chord. For a thin airfoil in supersonic flow (Section 6.4) it is the mean radius of curvature of the profile, which is proportional to the square of the chord divided by the thickness. The sharp-nosed airfoil in subsonic flow is a delicate borderline case between uniformity and nonuniformity in which the region of non-uniformity, being exponentially small, is not directly related to any physical dimension. A similar observation applies to the vortical layer on an inclined cone in supersonic flow.

The straightforward perturbation solution yields an asymptotic expansion of the form

$$f(x, y, z; \varepsilon) \sim \sum \delta_n(\varepsilon) f_n(x, y, z) \qquad \text{as} \quad \varepsilon \to 0 \quad \text{with} \quad x, y, z \text{ fixed} \tag{5.8}$$

Here the $\delta_n(\varepsilon)$ are an appropriate asymptotic sequence, and x, y, z are the coordinates made dimensionless using the primary reference length. This expansion is valid wherever the functions f_n are regular. They will become singular at any point within the flow field where phenomena are dominated by the secondary rather than the primary reference length. This point lies at infinity if the secondary length is the larger. A modified expansion, in order to be uniformly valid, must depend also upon the coordinates made dimensionless by the secondary reference length. Because the ratio of primary and secondary lengths is a function of ε, this amounts to depending also upon ε. Thus a uniformly valid expansion must have the more complicated form

$$f(x, y, z; \varepsilon) \approx \sum \delta_n(\varepsilon) g_n(x, y, z; \varepsilon) \qquad \text{uniformly as} \quad \varepsilon \to 0 \tag{5.9}$$

Because the perturbation parameter ε now appears implicitly in the function g_n as well as explicitly in the asymptotic sequence δ_n, this is not an asymptotic expansion in the usual sense. We call it a *composite expansion*. Such expansions have been discussed in connection with singular perturbation problems by Erdélyi (1961), who calls them "generalized asymptotic expansions." There are two objections to working with composite expansions. First, they are difficult to manipulate; evidently such familiar operations as equating like powers of ε must be reconsidered and, as will be seen later, a composite series is not uniquely determined. Second, they unnecessarily combine the complications of both the straightforward expansion and the region of nonuniformity (see, however, Sections 10.3 and 10.4).

It is simpler to isolate the difficulties associated with the nonuniformity by constructing a supplementary *inner expansion* valid in its vicinity. This is accomplished by introducing new *inner coordinates* that are of order unity in the region of nonuniformity. Then the inner expansion has the form

$$f(x, y, z; \varepsilon) \sim \sum \Delta_n(\varepsilon) F_n(X, Y, Z) \qquad \text{as} \quad \varepsilon \to 0 \quad \text{with} \quad X, Y, Z \text{ fixed} \tag{5.10}$$

We always denote inner variables by capital letters. Here the asymptotic sequence $\Delta_n(\varepsilon)$ must be allowed to differ from the asymptotic sequence $\delta_n(\varepsilon)$ for the outer expansion; often they are identical, but Section 6.3 gives an example in which they are different. If the region of nonuniformity is the neighborhood of a point in the finite plane, the inner coordinates X, Y, Z are ordinarily the coordinates made dimensionless using the secondary reference length. If the nonuniformity occurs along a line, as in boundary-layer theory, only the normal coordinate is changed. If it occurs at infinity, the coordinates must sometimes be stretched by different functions of ε in different directions. Like the outer expansion, the inner expansion is a conventional asymptotic series, so that the usual operations are valid.

It is often a useful preliminary to introduce dependent as well as independent variables that are of order unity in the inner and outer regions, so that the leading terms in the asymptotic sequences δ_n and Δ_n are unity. The degree of stretching is in general different for the independent and dependent variables. Following Kaplun (1954) and Lagerstrom and Cole (1955), we may formalize the procedure by defining:

Outer variables: Dimensionless independent and dependent variables based upon the primary reference quantities in the problem.

Outer limit: The limit as the perturbation parameter ε tends to zero with the outer variables fixed.

Outer expansion: The asymptotic expansion for $\varepsilon \to 0$ with outer variables fixed. Obtained in principle from the exact result by successive application of the outer limit process in conjunction with an appropriate outer asymptotic sequence.

Inner variables: Dimensionless independent and dependent variables stretched by appropriate functions of ε so as to be of order unity in the region of nonuniformity of the outer expansion.

Inner limit: The limit as $\varepsilon \to 0$ with inner variables fixed.

Inner expansion: The asymptotic expansion for $\varepsilon \to 0$ with inner variables fixed. Obtained in principle from the exact result by successive application of the inner limit process in conjunction with an appropriate inner asymptotic sequence.

Composite expansion: Any series that reduces to the outer expansion when expanded asymptotically for $\varepsilon \to 0$ in outer variables, and to the inner expansion in inner variables.

The technique of matching two complementary asymptotic expansions reduces a singular perturbation problem to its simplest possible elements. If the first inner problem is nevertheless found to be "impossible," then one may suppose that the problem itself is intractable. For example, it is clear that extending the thin-airfoil theory of Chapter IV to subsonic compressible flow leads, in the case of a round-nosed airfoil, to the inner problem of subsonic flow past a parabolic cylinder, for which no complete solution is known. Again, viscous flow past a cusp-nosed airfoil at high Reynolds number leads to the inner problem of viscous flow past a semi-infinite flat plate, for which only partial solutions exist. An advantage of the technique is that even in these "impossible" situations one can make use of numerical solutions or even of experimental measurements for the inner solution. Thus in his lifting-line theory Prandtl advocates the use of experimental airfoil-section data for what will be seen in Section 9.2 to be the inner solution.

5.5. Choice of Inner Variables

A crucial step in the method of matched asymptotic expansions is the choice of inner variables. One faces the questions:

(a) Which independent variables should be stretched?
(b) How should they be stretched?
(c) How should the dependent variables be stretched?

Answering the first question depends upon recognizing the singular nature of the problem, including the location of the nonuniformity and its "shape"—that is, whether it is the neighborhood of a point, line, or surface.

The degree of stretching required is usually evident when it is possible to calculate several terms of the outer expansion. For example, the formal thin-airfoil solution for an ellipse was seen in Section 4.4 to be invalid within a distance of order ε^2 from the leading edge, and the inner coordinates were magnified accordingly. Physical insight may suggest or confirm the proper stretching as the scale of the secondary reference length in the problem.

Otherwise, the stretching can be sought by trial. The guiding principles are that the inner problem shall have the least possible degeneracy, that it must include in the first approximation any essential elements omitted in the first outer solution, and that the inner and outer solutions shall match. As an example, consider the model problem (5.1). Trying an arbitrary stretching of the independent variable only, we set

$$f(x;\varepsilon) = F(X;\varepsilon), \qquad X = \frac{x}{\sigma(\varepsilon)} \tag{5.11}$$

The problem becomes

$$\frac{d^2F}{dX^2} - \frac{\sigma(\varepsilon)}{\varepsilon}\frac{dF}{dX} = \frac{\sigma^2(\varepsilon)}{\varepsilon}a, \qquad F(0) = 0, \qquad F\Big(\frac{1}{\sigma}\Big) = 1 \tag{5.12}$$

Because the highest derivative was lost in the outer limit, d^2F/dX^2 must now be preserved in the inner limit. This means that the factor $\sigma(\varepsilon)/\varepsilon$ multiplying dF/dX must not become infinite as $\varepsilon \to 0$. If it vanishes, the solution satisfying the inner boundary condition (which must also be preserved) is simply a multiple of X; but this cannot be matched with the outer solution (5.3). The remaining possibility is that $\sigma(\varepsilon)/\varepsilon$ approaches a constant; this also yields the least degenerate differential equation. Taking the constant as unity without loss of generality gives the previous results (5.4) and (5.5). It was unnecessary to stretch the dependent variable in this example because the first inner problem is homogeneous; but in general one must admit separate stretching of each dependent as well as each independent variable.

The inner variables are almost always, as in the preceding examples, formed by linear stretching. An exception arises in the problem of the vortical layer on an inclined cone (Munson, 1964). This is illustrated by the following model problem:

$$x^m\frac{df}{dx} - \varepsilon f = 0, \qquad f(1) = 1 \tag{5.13}$$

Iterating or substituting a series in powers of ε yields the straightforward (outer) perturbation solution:

$$f(x;\varepsilon) \sim 1 + \frac{\varepsilon}{1-m}(x^{1-m} - 1) + O(\varepsilon)^2, \qquad m \neq 1 \tag{5.14a}$$

$$\sim 1 + \varepsilon \log x + O(\varepsilon^2), \qquad m = 1 \tag{5.14b}$$

This is singular at $x = 0$ for $m \geqslant 1$ and at $x = \infty$ for $m \leqslant 1$. It seems likely that for $m \neq 1$ an appropriate inner coordinate is

$$X = x\varepsilon^{-1/(m-1)} \tag{5.15}$$

and this is confirmed by examining either the resulting inner equation

$$X^m\frac{df}{dX} - f = 0 \tag{5.16}$$

or the exact solution

$$f(x;\varepsilon) = \exp\Big(-\frac{\varepsilon}{1-m}\Big)\exp\Big(\frac{\varepsilon x^{1-m}}{1-m}\Big), \qquad m \neq 1 \tag{5.17a}$$

$$= x^\varepsilon, \qquad m = 1 \tag{5.17b}$$

In the special case $m = 1$, (5.14b) indicates that the region of nonuniformity near the origin is exponentially small: $x = O(e^{-1/\varepsilon})$. One might suppose that, just as for the sharp-nosed airfoil of Section 4.7, an appropriate inner variable would therefore be given by the corresponding linear stretching:

$$X = xe^{1/\varepsilon} \tag{5.18}$$

However, this leaves the transformed differential equation

$$X\frac{df}{dX} - \varepsilon f = 0 \tag{5.19}$$

unchanged, so that simple stretching is ineffective. Instead, the exact solution (5.17b) shows that the proper inner coordinate is given by the nonlinear distortion

$$X = x^\varepsilon \tag{5.20}$$

which transforms the differential equation to

$$X \frac{df}{dX} - f = 0 \tag{5.21}$$

Thus it appears that a fractional-power transformation is required when nonuniformity in an exponentially small region arises not from the boundary conditions but from a homogeneous operator of the form $x\,\partial/\partial x$ in the differential equation.

5.6. The Role of Matching

The method of matched asymptotic expansions involves loss of boundary conditions. An outer expansion cannot be expected to satisfy conditions that are imposed in the inner region; conversely, the inner expansion will not in general satisfy distant conditions. Thus it is an exceptional circumstance that the inner solution for the elliptic airfoil (Section 4.10) happens to satisfy the upstream boundary condition; the inner solution for a sharp edge does not (Section 4.11). Hence insufficient boundary conditions are generally available for either the outer or inner expansion. The missing conditions are supplied by matching the two expansions.

For partial differential equations a useful preliminary to matching is application of the principle of minimum singularity (Section 4.5). Experience has shown that of the admissible solutions only the one that is least singular in its region of nonuniformity can be matched to the complementary expansion. For example, the inner solution for a round-nosed thin airfoil was seen in Section 4.9 to be a symmetric flow past the osculating parabola. Figure 5.3 shows the first two of an unlimited

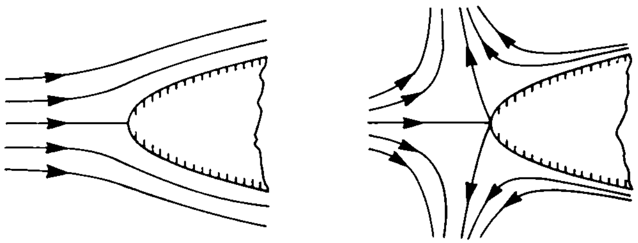

Fig. 5.3. Alternative symmetric flows past parabola.

number of possibilities. All but the first give unbounded speeds at infinity, and consequently cannot be matched with the thin-airfoil expansion (Exercise 4.2).

Although the principle of minimum singularity often reduces the number of possibilities, it cannot always single out a unique flow pattern. For example, it rules out source eigensolutions in the linearized solution for a round-nosed airfoil, but not in the second approximation. This is as it should be, because a source eigensolution must actually occur in the second-order outer solution for a smooth profile that differs from an ellipse only in the vicinity of the leading and trailing edges (Fig. 5.4).

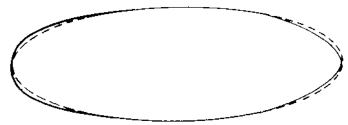

Fig. 5.4. Airfoil having same linearized solution as ellipse.

Matching is the crucial feature of the method. The possibility of matching rests on the existence of an *overlap domain* where both the inner and outer expansions are valid. By virtue of the overlap, one can obtain exact relations between finite partial sums. This remarkable achievement is possible only for a parameter perturbation that is nonuniform in the coordinates, or for a coordinate perturbation that is nonuniform in the other coordinates. One cannot match two different parameter perturbations, such as expansions for large and small values of Reynolds number or of Mach number. Neither can one match two different coordinate expansions, such as for small and large time or distance. Such series may overlap in the sense that they have a common region of convergence, but the process of analytical continuation yields only approximate relations from any finite number of terms (cf. Section 10.9).

Matching may also be contrasted with what we shall call numerical *patching*. This consists in joining two series by forcing their values and perhaps several derivatives to agree at an arbitrary intermediate boundary. Although the result may be of practical utility—or even numerically indistinguishable from that of matching—patching is esthetically displeasing, and ordinarily no simpler. Also, matching is more systematic than patching in higher approximations. Our view is that patching should be avoided whenever it can be replaced by matching, which provides an imperceptibly smooth blending of the two solutions.

5.7. Matching Principles

The existence of an overlap domain implies that the inner expansion of the outer expansion should, to appropriate orders, agree with the outer expansion of the inner expansion (Lagerstrom, 1957). This general matching principle can be given various specific formulations. The literature shows that the choice of matching principle is somewhat a matter of the investigator's taste.

See Note 5

In matching his boundary-layer approximation to the outer inviscid flow, Prandtl tacitly applied what we may call the *limit matching principle:*

$$\begin{aligned}&\text{The inner limit of (the outer limit)}\\&= \text{the outer limit of (the inner limit).}\end{aligned} \tag{5.22}$$

Whether this primitive rule is correct, or adequate, depends not only on the problem, but also on the choice of independent variables being matched. It is evidently valid for the tangential velocity in the boundary layer, which must for large values of its argument approach the inviscid surface speed. However, it is invalid for the normal velocity or stream function, where the first repeated limit in (5.22) is zero but the second is infinite. The same difficulty arises in plane flow at low Reynolds number (Chapter VIII).

We can improve this simple rule by describing more precisely the limiting behavior of the quantity being matched (cf. Sections 3.2 and 3.3). Instead of mere limits we use asymptotic representations. This gives the matching principle

$$\begin{aligned}&\text{Inner representation of (outer representation)}\\&= \text{outer representation of (inner representation).}\end{aligned} \tag{5.23}$$

Here the outer (or inner) representation means the first *nonzero* term in the asymptotic expansion in outer (or inner) variables. This rule provides matching in cases where the limit principle (5.22) gives only a trivial result. For example, we shall see in Section 8.7 that it suffices for plane flow at low Reynolds number.

The principle is extended to higher approximations by retaining further terms in the asymptotic expansions. We must permit the number of terms to be different in the inner and outer expansions, because the normal matching order (Section 5.9) requires a difference of one in the even-numbered steps. Thus we obtain the *asymptotic matching principle* introduced in Chapter IV: (See Note 4)

$$\begin{aligned}&\text{The } m\text{-term inner expansion of (the } n\text{-term outer expansion)}\\&= \text{the } n\text{-term outer expansion of (the } m\text{-term inner expansion).}\end{aligned} \tag{5.24}$$

Here m and n are any two integers; in practice m is usually chosen as either n or $n + 1$.

This matching principle appears to suffice for any problem to which the method of matched asymptotic expansions can successfully be applied. It will be used throughout this book. In the following section, however, we describe an alternative principle that provides deeper insight into the nature of the overlap domain.

5.8. Intermediate Matching

In the outer limit process of thin-airfoil theory (Chapter IV) a point remains a fixed distance from the leading edge as the thickness ratio ε tends to zero, whereas in the inner limit process for the elliptic airfoil the distance decreases like ε^2. It is by no means obvious that the two limit processes can be interchanged, because there is a gap between the inner and outer regions. That a gap exists is clarified by considering a point whose distance from the nose decreases only like ε (Fig. 5.5). This point ultimately emerges from any vicinity of the leading edge, and is at the same time excluded from the region of validity of the outer solution.

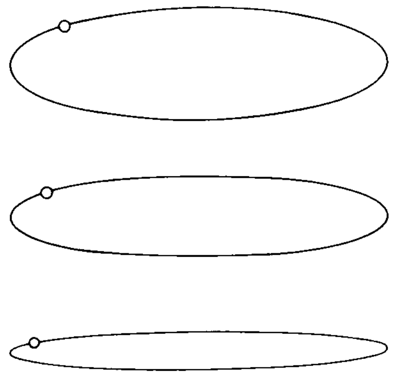

Fig. 5.5. Intermediate limit process for elliptic airfoil.

To bridge this gap, Kaplun (1957) has introduced the concept of a continuum of *intermediate limits*, lying between the inner and outer limits. Although he considers a very general class of limits, it will suffice for purposes of illustration to consider only those associated with powers of the small parameter. If s is the outer variable associated with a non- (See Note 5) uniformity at $s = 0$, we introduce an *intermediate variable*

$$\tilde{s} = \frac{s}{\varepsilon^{\alpha}}, \qquad 0 < \alpha < \alpha_i \tag{5.25}$$

Here $\alpha = 0$ gives the outer and $\alpha = \alpha_i$ the inner limit; for example, $\alpha_i = 2$ for the elliptic airfoil. The limit as $\varepsilon \to 0$ with $\tilde{s}$ fixed is called the intermediate limit; and its repeated application in conjunction with an appropriate asymptotic sequence yields the *intermediate expansion.*

Carrying out the intermediate limit in the differential equations and boundary conditions yields the *intermediate problem.* Although we have introduced very many limit processes, they lead to only a few different problems. All intermediate limits yield essentially a single intermediate problem, which is often the same as the inner problem. For example, setting $\tilde{s} = s/\varepsilon^{\alpha}$ and $\tilde{y} = y/\varepsilon^{\alpha}$ in Eq. (4.33a) for the elliptic airfoil and letting $\varepsilon \to 0$ gives

$$\tilde{y} \sim \pm \varepsilon^{1-\alpha/2}\sqrt{2\tilde{s}} \tag{5.26}$$

Thus the intermediate problem is that of symmetric flow past a parabola of nose radius $\varepsilon^{2-\alpha}$.

The *intermediate solution* is the solution of the intermediate problem. Its difference from the full solution must vanish uniformly in the intermediate limit. Thus for the elliptic airfoil the intermediate solution for surface speed is, from (4.28),

$$q \sim U_i\sqrt{\frac{\tilde{s}}{\tilde{s} - \frac{1}{2}\varepsilon^{2-\alpha}}} \tag{5.27}$$

The denominator cannot be expanded, because the result would not be uniform near the stagnation point. This example illustrates that the intermediate solution is not necessarily the intermediate limit of the full solution—which is here simply U_i—but may have a more complex structure.

Although the gap between inner and outer limits has been bridged by the intermediate solution, it is not yet apparent that there exists an overlap domain. This is assured by Kaplun's *extension theorem*, which asserts that the range of validity of the inner or outer limit extends at least slightly into the intermediate range. We forego the proof of this theorem, whose truth will be evident in specific examples. Thus we can match the intermediate expansion with the outer expansion at one end of the range and with the inner expansion at the other end. Often the intermediate expansion is identical with the inner expansion—as in our example of the elliptic airfoil—or is contained in it as a special case. Then we can simply match the inner and outer expansions in the outer overlap domain.

Matching requires that in the overlap domain the difference between the outer (or inner) and intermediate solutions vanish in the intermediate limit. Thus for the elliptic airfoil we match the intermediate solution (5.27) to the outer uniform stream by considering

$$\lim_{\varepsilon\to 0,\ \tilde{s}\text{ fixed}} \left[U - U_i\sqrt{\frac{\tilde{s}}{\tilde{s} - \frac{1}{2}\varepsilon^{2-\alpha}}}\right] = \lim_{\varepsilon\to 0}\,[U - U_i + O(\varepsilon^{2-\alpha})] \tag{5.28}$$

This vanishes if $U_i = U$, and $\alpha < 2$. Hence the outer overlap domain is $0 \leqslant \alpha \leqslant 2$.

We may call the extension of this rule to higher approximations the *intermediate matching principle*:

> In some overlap domain the intermediate expansion of the difference between the outer (or inner) expansion and the intermediate expansion must vanish to the appropriate order. (5.29)

For example, consider two terms each of the intermediate and outer expansions for the speed on an elliptic airfoil, where in the latter we

admit the source eigensolution of (4.32a). The difference between the two expansions is, in intermediate variables

$$D = U\left[1 + \varepsilon + \frac{\varepsilon C_1}{2\varepsilon^{\alpha}\tilde{s} - \varepsilon^{2\alpha}\tilde{s}^2}\right] - U_i\sqrt{\frac{\tilde{s}}{\tilde{s} + \frac{1}{2}\varepsilon^{2-\alpha}}} \tag{5.30}$$

and expanding gives

$$D \sim U(1 + \varepsilon) - U_i + C_1U\left(\frac{\varepsilon^{1-\alpha}}{2s} + \frac{\varepsilon}{4}\right) + O(\varepsilon^{1+\alpha}, \varepsilon^{2-\alpha}) \tag{5.31}$$

This vanishes to order ε—that is, to second order in powers of ε—if $U_i = U(1 + \varepsilon)$, $C_1 = 0$, and $0 < \alpha < 1$. The first two of these results were found by asymptotic matching in Chapter IV. The third means that the outer overlap domain has shrunk to half its previous width.

5.9. Matching Order

All our previous discussion suggests complete symmetry between the inner and outer limits, so that the two terms could be interchanged throughout. However, we have heretofore used "outer" always to denote the straightforward or basic approximation, and we insist on adhering to this convention. More precisely, we assign the terms so that the outer solution is, to first order, independent of the inner. The test is to consider a first-order change in each, and see whether the other is affected. For example, in thin-airfoil theory the free stream is disturbed only slightly by doubling the nose radius, whereas the flow near the nose is drastically altered by doubling the free-stream speed.

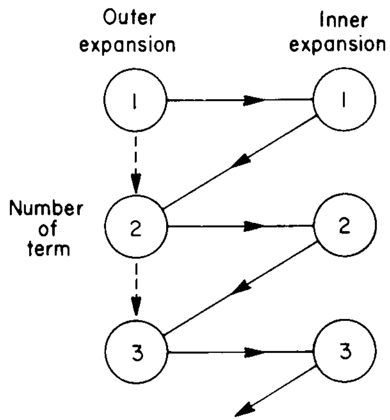

Fig. 5.6. Matching order for inner and outer expansions.

In general, matching must proceed step by step as indicated by the solid arrows in Fig. 5.6. The basic solution dominates the inner solution, which in turn exerts a secondary influence on the outer expansion, and so on. This order is inviolable in the direct problem of boundary-layer theory, for example.

One can sometimes short-circuit the standard matching order. An obvious case is an initial-value problem, where all the boundary conditions are imposed in the outer region. Then one can calculate an unlimited number of terms of the outer expansion, as indicated by dotted arrows in Fig. 5.6, and subsequently match with the inner expansion to complete the solution. This situation can arise in fluid mechanics from inverse formulation of a problem, an example being given in Section 9.9.

The same bypassing of the inner expansion occurs in a more subtle case, when the nonuniformity is so weak that it does not affect the outer flow. An example is the biconvex airfoil of Sections 4.7 and 4.11. For a round-nosed airfoil, on the other hand, the example of Fig. 5.4 shows that only two terms can be calculated before one must resort to matching with the inner expansion.

When the standard order is followed, matching will indicate each new term in the asymptotic sequence, which therefore need not be guessed in advance. For example, rewriting any number of terms of the thin-airfoil expansion (4.14b) for the ellipse in inner variables and expanding for small ε shows at each stage that the next term in the inner expansion is of the order of the next higher power of ε. An example where the asymptotic sequences are different for the inner and outer expansions is discussed in Section 6.3.

5.10. Construction of Composite Expansions

Representing the solution of a singular perturbation problem by an inner and an outer expansion may raise awkward practical questions of where to shift from one to the other. A crude device would be to change where the two curves cross, but the result would have spurious corners. Moreover, for the elliptic airfoil, for example, the first-order inner and outer solutions for surface speed do not meet (Fig. 5.7).

Fortunately, since the two expansions have a common region of validity, it is easy to construct from them a single uniformly valid expansion. The result is necessarily more complex than either of its constituents, and is in fact a composite expansion in the sense of Section 5.4. The construction can in principle be carried out in a variety of ways. The results may be different, because a composite expansion is not unique; but they will all be equivalent to the order of accuracy retained.

Two essentially different methods have been used in practice. The first may be called *additive composition*. The sum of the inner and outer expansions is corrected by subtracting the part they have in common, so that it is not counted twice. The common part can sometimes be

found by inspection. Otherwise, it may be calculated as the inner expansion of the outer expansion, or vice versa. Thus, in an obvious

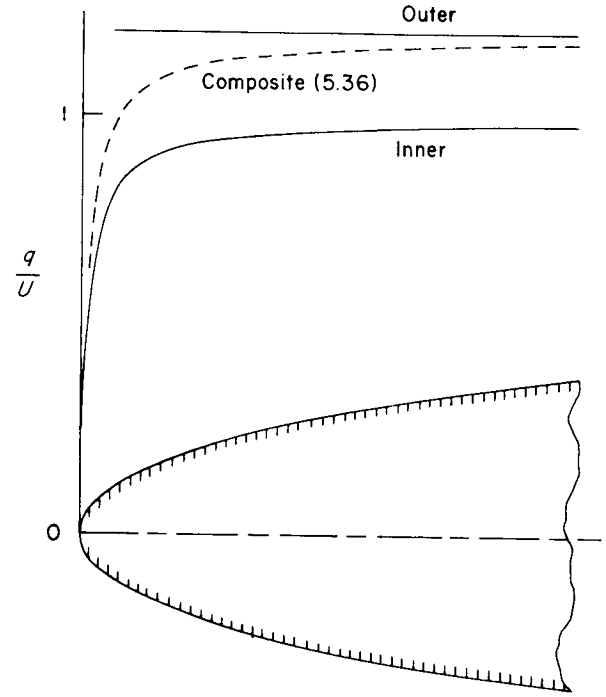

Fig. 5.7. First-order inner and outer solutions for speed on thin ellipse.

notation, where $f_i^{(m)}$ means the m-term inner expansion, and so on, the rule for additive composition is

$$f_c^{(m,n)} = \begin{cases} f_i^{(m)} + f_o^{(n)} - [f_o^{(n)}]_i^{(m)} \\ f_o^{(n)} + f_i^{(m)} - [f_i^{(m)}]_o^{(n)} \end{cases} \tag{5.32}$$

One can verify this rule by taking the m-term inner and n-term outer expansions of both sides. Using the asymptotic matching principle (5.24) shows that the inner and outer expansions are reproduced in their respective regions.

Working with differences, though conceptually somewhat different, yields the same rule. The outer expansion is made uniformly valid by adding to it the solution of the inner problem for the *difference* between the exact solution and its outer expansion, or the inner expansion is corrected analogously. This may be written symbolically

$$f_c^{(m,n)} = \begin{cases} f_o^{(n)} + [f - f_o^{(n)}]_i^{(m)} \\ f_i^{(m)} + [f - f_i^{(m)}]_o^{(n)} \end{cases} \tag{5.33}$$

The asymptotic matching principle shows that these are equivalent to the additive rule (5.32).

The second method may be called *multiplicative composition*. The outer expansion is multiplied by a correction factor consisting of the ratio of the inner expansion to its outer expansion, or the inner expansion is treated similarly. This gives

$$f_c^{(m,n)} = f_o^{(n)} \frac{f_i^{(m)}}{[f_i^{(m)}]_o^{(n)}} = f_i^{(m)} \frac{f_o^{(n)}}{[f_o^{(n)}]_i^{(m)}}$$

$$= \frac{f_o^{(n)} f_i^{(m)}}{[f_o^{(n)}]_i^{(m)} = [f_i^{(m)}]_o^{(n)}} \tag{5.34}$$

The first form is recognized as providing the multiplicative correction factor that was applied to round-nosed airfoils in Section 4.8. The last form exhibits the inherent symmetry between the inner and outer limit processes.

The additive and multiplicative rules of composition are related by the fact that the ratio of two quantities near unity can be expanded into a sum using the binomial theorem. The additive rule is usually simpler to apply; the multiplicative one sometimes gives simpler results. Either can be used even when the inner problem cannot be solved analytically, the solution being known only from numerical computation or experiment.

See Note 6

We illustrate these two methods for the surface speed on a thin elliptic airfoil. From two terms of the outer expansion (4.13) and one term of the inner expansion (4.46) we obtain as the uniform first-order perturbation solution, by additive composition (5.32)

$$\frac{q}{U} \sim \sqrt{\frac{2s}{2s + \varepsilon^2}} + \varepsilon \tag{5.35}$$

and by multiplicative composition (5.34)

$$\frac{q}{U} \sim (1 + \varepsilon) \sqrt{\frac{2s}{2s + \varepsilon^2}} \tag{5.36}$$

Here $s = 1 + x$ or $1 - x$ according as we correct the nonuniformity at the leading or trailing edge; a truly uniform solution is obtained by treating each edge in turn (Exercise 5.3). Other kinds of multiple nonuniformity (cf. Section 9.13) can likewise be handled by repeated application of the rules for composition.

A composite expansion has at least the accuracy of each of its constituents. Thus (5.35) and (5.36) are in error by no more than $O(\varepsilon^2)$ away from the edge and $O(\varepsilon)$ near the edge. In fact the additive result (5.35) is evidently in error by precisely ε at the stagnation point. The multiplicative result (5.36) has the advantage of being exact there. Extending the composite result by using two terms of the inner as well as the outer expansion leads again to (5.36) for either addition or multiplication. This means that by coincidence the error in (5.36) is actually no greater than $O(\varepsilon^2)$ everywhere. Figure 5.7 shows the improvement resulting from use of the composite expansion.

EXERCISES

5.1. *Uniform approximation for Friedrichs' model.* Form in two different ways a composite expansion from the solution (5.6). Discuss the difference, and compare with the exact solution. Consider higher approximations.

5.2. *Uniform approximation for biconvex airfoil.* Construct an approximation for the surface speed on a thin biconvex airfoil in incompressible flow that is uniformly valid to order ε except at the trailing edge.

5.3. *Composite rules for two nonuniformities.* Devise rules, analogous to (5.32) and (5.34), for constructing composite expansions in the case of two separated nonuniformities—as for a thin airfoil with stagnation leading and trailing edges. Apply your results to the surface speed on an elliptic airfoil, and compare with the exact solution.

5.4. *Outer, middle, and inner expansions.* Show that a perturbation solution of the problem

$$x^3 \frac{dy}{dx} = \varepsilon[(1 + \varepsilon)x + 2\varepsilon^2]y^2, \qquad y(1) = 1 - \varepsilon$$

for $0 \leqslant x \leqslant 1$ requires three matched expansions. Calculate in succession two terms of the straightforward (outer) expansion, two terms of the middle expansion, and one term of the inner expansion. Choose new magnified variables as suggested by the preceding expansion, and match at each step.