7 Method of multiple scales

This is a general method applicable to a wide range of problems. The problems are characterised by having two physical processes, each with their own scales, and with the two processes acting simultaneously. This should be contrasted with the method of matched asymptotic expansions which also has two processes with different scales, but with the processes acting separately in different regions.

7.1 van der Pol oscillator

We return to the oscillator which was introduced in §5.6 now to consider the case where the nonlinear friction is a small perturbation to the linear simple harmonic oscillator. It is convenient to study the initial value problem

$$\ddot{x} + \epsilon\dot{x}(x^2 - 1) + x = 0 \qquad \text{in } t \geq 0 \text{ with } \epsilon \searrow 0$$
$$\text{subject to} \quad x = 1 \text{ and } \dot{x} = 0 \quad \text{at } t = 0$$

Treating the problem as a regular one yields the approximation

$$x(t,\epsilon) \quad \sim \quad \cos t + \epsilon\left[\tfrac{3}{8}(t\cos t - \sin t) - \tfrac{1}{32}(\sin 3t - 3\sin t)\right]$$

This expansion is asymptotic for fixed t as $\epsilon \searrow 0$, but breaks down when $t \geq \operatorname{ord}(\epsilon^{-1})$. (It is however possible to prove that the expansion converges!)

The trouble with the naive approximation is that the ϵ-damping changes the amplitude of the oscillation on a time scale ϵ^{-1} by the slow accumulation of small effects. Thus the oscillator has two processes acting on their own time scales. There is the basic oscillation on the time scale of 1 from the inertia causing the restoring force to overshoot the equilibrium position. There is also the slow drift in the amplitude (and possibly the phase) on the time scale ϵ^{-1} due to the small friction. We

recognise these two time scales by introducing two time variables.

$$\tau = t \quad \text{– the } \textit{fast time} \text{ of the oscillation}$$
$$T = \epsilon t \quad \text{– the } \textit{slow time} \text{ of the amplitude drift}$$

The slowly changing features will then be combined into factors which are functions of T, while the rapidly changing features will be combined into factors which are functions of τ. Thus we look for a solution of the form

$$x(t;\epsilon) \;=\; x(\tau, T; \epsilon)$$

As real time t increases the fast time τ increases at the same rate, while the slow time T increases slowly. Thus

$$\frac{d}{dt} \;=\; \left(\frac{\partial}{\partial\tau}\right)_T + \epsilon\left(\frac{\partial}{\partial T}\right)_\tau$$

and so

$$\ddot{x} \;=\; x_{\tau\tau} + 2\epsilon x_{\tau T} + \epsilon^2 x_{TT}$$

We now seek an asymptotic approximation for x allowing in the leading order for the possibility of changes over the long time scale. Thus we pose

$$x(t;\epsilon) \quad \sim \quad x_0(\tau, T) \;+\; \epsilon x_1(\tau, T)$$

with the requirement that the expansion be asymptotic for $T = \operatorname{ord}(1)$. Substituting into the governing equation and comparing coefficients of ϵ^n, we find a sequence of problems.

At ϵ^0:

$$x_{0\tau\tau} + x_0 = 0 \qquad \text{in } t \geq 0$$
$$\text{with} \quad x_0 = 1 \quad \text{and} \quad x_{0\tau} = 0 \quad \text{at } t = 0$$

Integrating with respect to τ, treating T as an independent variable held constant, we obtain a general solution

$$x_0 \;=\; R(T)\cos(\tau + \theta(T))$$

in which the amplitude R and the phase θ are constant as far as the rapid τ variations are concerned, but are allowed to vary over the long T time. The initial conditions give

$$R(0) \;=\; 1 \qquad \text{and} \qquad \theta(0) \;=\; 0$$

Except for this information, R and θ are unknown in the leading order analysis. Knowing that the amplitude is controlled by the action of the small friction over a long time, it is quite clear that we must proceed to the next order.

At ϵ^1:

$$\begin{aligned} x_{1\tau\tau} + x_1 &= -x_{0\tau}(x_0^2 - 1) - 2x_{0\tau T} \qquad \text{in } t \geq 0 \\ &= 2R\theta_T \cos(\tau + \theta) + \left(2R_T + \tfrac{1}{4}R^3 - R\right)\sin(\tau + \theta) \\ &\qquad + \tfrac{1}{4}R^3 \sin 3(\tau + \theta) \end{aligned}$$

from the (partially) known x_0. The initial conditions are

$$x_1 = 0 \quad \text{and} \quad x_{1\tau} = -x_{0T} = -R_T \qquad \text{at } t = 0$$

Again we integrate with respect to τ treating T as a constant. The $\sin 3(\tau + \theta)$ forcing term will induce a $\sin 3(\tau + \theta)$ bounded response in x_1, but the resonating forcing terms $\sin(\tau + \theta)$ and $\cos(\tau + \theta)$ would induce a response in x_1 growing like τ which would break the asymptoticness when $t \geq \text{ord}(\epsilon^{-1})$. Thus to maintain the asymptoticness of the expansion we must exploit the freedom in the undetermined $R(T)$ and $\theta(T)$ in order to insist that the potentially resonating terms vanish identically. This leads to the so-called secularity or integrability or solubility condition of Poincaré,

$$\theta_T = 0 \qquad \text{and} \qquad R_T = \tfrac{1}{8}R\left(4 - R^2\right)$$

Using the initial conditions on R and θ, we therefore have

$$\theta \equiv 0 \qquad \text{and} \qquad R = 2\left(1 + 3e^{-T}\right)^{-1/2}$$

Thus eventually the amplitude of the oscillator drifts to 2. Note that the amplitude and phase of the leading order term are fully determined in the next order problem by asking that the correction term does not break the asymptoticness; it is not necessary to find the correction term.

The correction term can however be found, and is

$$x_1 = -\tfrac{1}{32}R^3 \sin 3\tau + S(T)\sin(\tau + \varphi(T))$$

with new unknown amplitude and phase functions, $S(T)$ and $\varphi(T)$, which satisfy the initial conditions

$$\varphi(0) = 0 \qquad \text{and} \qquad S(0) = -\tfrac{9}{32}$$

These new amplitude and phase functions will become determined by the secularity condition in the x_2 problem.

At higher orders one can find that a resonant forcing is unavoidable: there can be insufficient freedom in the undetermined functions. The asymptoticness is then lost. This is in fact the situation with our van

der Pol oscillator when proceeding to find x to $\text{ord}(\epsilon)$ for t of $\text{ord}(\epsilon^{-1})$. This difficulty can be overcome by introducing an additional slow time scale $T_2 = \epsilon^2 t$.

A simple example which illustrates the need for such a super slow time scale is a linearly damped oscillator

$$\ddot{x} + 2\epsilon\dot{x} + x = 0$$

with solution

$$x = e^{-\epsilon t}\cos\left(\sqrt{1 - \epsilon^2}\, t\right)$$

The amplitude drifts on the time scale ϵ^{-1}, while the phase drifts on the longer time scale ϵ^{-2}. Of course in this example there is not much amplitude left by the time that the phase has slipped significantly.

In general when working to $\text{ord}(\epsilon^k)$ on a time scale $\text{ord}(\epsilon^{k-n})$ one must expect to have a hierarchy of n slow time scales. Some may be essential, representing genuinely different processes. Some however may simply be adjustments to a previous process, e.g. adjustments in the frequency, which are better tackled with something like a co-ordinate straining.

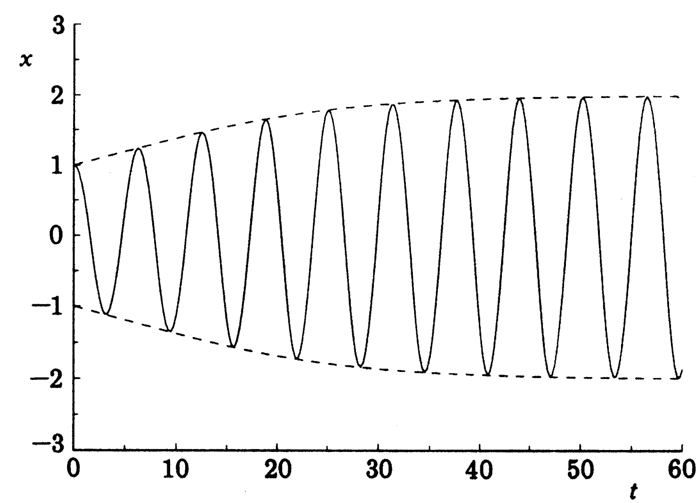

Fig. 7.1 The continuous curve is a numerical solution $x(t;\epsilon)$ of the initial value problem for a van der Pol oscillator with $\epsilon = 0.1$. The dashed curves give the asymptotic predictions for the amplitude, i.e. $\pm R(T)$. The agreement is good at even larger values of the small parameter, as is typical of this class of asymptotic analysis.

Exercise 7.1. Obtain an asymptotic approximation for x to ord(1) which is valid for $t = \text{ord}(\epsilon^{-1})$ to the solution of

$$\ddot{x} + \epsilon\dot{x}^3 + x = 0 \qquad \text{for } t \geq 0$$
$$\text{with} \quad x = 1 \quad \text{and} \quad \dot{x} = 0 \quad \text{at } t = 0$$

Exercise 7.2. Obtain equations for the drift in the amplitude and phase in the solution to

$$\ddot{x} + \epsilon\dot{x}(x^2 - 1) + (1 + \epsilon k)x = \epsilon \cos t$$

with $k = \text{ord}(1)$ as $\epsilon \searrow 0$. [The tough part is then to show that a slave oscillator will lock onto the forcing from a master if the slave is not detuned too much, i.e. if $|k| < k_c$ then R tends to an equilibrium, while if $|k| > k_c$ then R oscillates (the free oscillations beating with the forced response).]

Exercise 7.3. Find the leading order approximation to the general solution for $x(t;\epsilon)$ and $y(t;\epsilon)$ satisfying

$$\frac{d^2x}{dt^2} + 2\epsilon y\frac{dx}{dt} + x = 0$$
$$\frac{dy}{dt} = \tfrac{1}{2}\epsilon \ln x^2$$

which is valid for $t = \text{ord}(1/\epsilon)$ as $\epsilon \to 0$. You may quote the result

$$\frac{1}{2\pi}\int_0^{2\pi} \ln\cos^2\theta \, d\theta = -\ln 4$$

7.2 Instability of the Mathieu equation

This is a simple example where the slow time is not ϵt. Now the Mathieu equation describes the small amplitude oscillations of a pendulum whose length changes slightly in time. If the length changes at a frequency which is near a multiple of half the oscillator's natural frequency, then the amplitude of the pendulum will grow in time, a phenomenon called parametric excitation. [This is how a child's swing works, with the length of the pendulum shortening as one raises one's feet at the lowest point of each half cycle.] We study the case of the length changing with roughly the same frequency of the natural oscillations:

$$\ddot{x} + (1 + k\epsilon^2 + \epsilon\cos t)x = 0$$

with $k = \text{ord}(1)$ as $\epsilon \to 0$.

Now in the leading order approximation, x will just contain the first harmonic. Iterating, this will force a correction with zero and second harmonics. Iterating again, we would find a second correction forced by first and third harmonics. The resonant forcing has to be removed by a slow drift in the amplitude and phase, which therefore occurs on a time scale $\text{ord}(\epsilon^{-2})$. Hence we introduce a slow time scale $T = \epsilon^2 t$ (with the fast time $\tau = t$ unchanged) and pose an expansion

$$x(t;\epsilon) \sim x_0(\tau, T) + \epsilon x_1(\tau, T) + \epsilon^2 x_2(\tau, T)$$

to be asymptotic when $T = \text{ord}(1)$. Substituting into the governing equation and comparing coefficients of ϵ^n, we find in the usual way a sequence of problems.

At ϵ^0:

$$x_{0\tau\tau} + x_0 = 0$$

with a general solution

$$x_0 = A(T)\cos\tau + B(T)\sin\tau$$

At ϵ^1:

$$\begin{aligned} x_{1\tau\tau} + x_1 &= -x_0\cos\tau \\ &= -\tfrac{1}{2}A - \tfrac{1}{2}A\cos 2\tau - \tfrac{1}{2}B\sin 2\tau \end{aligned}$$

This forces a response in x_1,

$$x_1 = -\tfrac{1}{2}A + \tfrac{1}{6}A\cos 2\tau + \tfrac{1}{6}B\sin 2\tau$$

The homogeneous solution can be omitted as we are not tackling a particular initial value problem.

At ϵ^2:

$$\begin{aligned} x_{2\tau\tau} + x_2 &= -2x_{0\tau T} - kx_0 - x_1\cos\tau \\ &= -\left(2B_T + (k - \tfrac{5}{12})A\right)\cos\tau + \left(2A_T - (k + \tfrac{1}{12})B\right)\sin\tau \\ &\qquad - \tfrac{1}{12}A\cos 3\tau - \tfrac{1}{12}B\sin 3\tau \end{aligned}$$

As anticipated, the drift in amplitude and phase is determined at $\text{ord}(\epsilon^2)$ by the secularity condition that the asymptoticness is not lost when $T = \text{ord}(1)$,

$$B_T = \tfrac{1}{2}\left(\tfrac{5}{12} - k\right)A \qquad \text{and} \qquad A_T = \tfrac{1}{2}\left(\tfrac{1}{12} + k\right)B$$

The solution of this pair of linear equations has either exponentially growing (and decaying) or stable oscillatory solutions according to whether

$$\left(\tfrac{5}{12} - k\right)\left(\tfrac{1}{12} + k\right) \quad > \text{ or } < \quad 0$$

i.e. the oscillator is unstable if $-\frac{1}{12} < k < \frac{5}{12}$.

Exercise 7.4. The equations governing a satellite orbiting the Earth and experiencing a small frictional force proportional to the square of its velocity may be written in the form

$$\begin{aligned} u_{\theta\theta} + u &= h^2 \\ h_\theta &= \epsilon h \frac{\sqrt{u_\theta^2 + u^2}}{u^2} \end{aligned}$$

Writing the leading order solution

$$u(\theta, \epsilon) \sim h^2(\epsilon\theta)\left[1 + e(\epsilon\theta)\cos(\theta - \alpha(\epsilon\theta))\right]$$

with $e < 1$, obtain the drift equation

$$\begin{aligned} h' &= \langle f \rangle / h \\ e' &= -2\left\langle (e + \cos\varphi) f \right\rangle / h^2 \\ \alpha' &= -2\left\langle \sin\varphi\, f \right\rangle / h^2 e \end{aligned}$$

where $f = \sqrt{1 + 2e\cos\varphi + e^2}/(1 + e\cos\varphi)^2$ and the angle brackets denote an average over $0 \le \varphi \le 2\pi$.

Deduce that as the satellite falls to the Earth ($r = 1/u$), its angular momentum h increases, its eccentricity e decreases, and the direction of the perihelion α does not drift at this order.

If the eccentricity e is small initially, find an approximate solution for the drift in h and e.

[You may assume that $\langle (e + \cos\varphi) f \rangle > 0$ for $0 < e < 1$.]

Exercise 7.5. Find the leading order approximation valid for times t of order ϵ^{-1} as $\epsilon \to 0$, to the solution $x(t; \epsilon)$ and $y(t; \epsilon)$ satisfying

$$\begin{aligned} \ddot{x} + \epsilon y \dot{x} + x &= y^2 \\ \dot{y} = \epsilon(1 + x - y - y^2) \end{aligned}$$

subject to $x = 1$, $\dot{x} = 0$ and $y = 0$ at $t = 0$.

Exercise 7.6. Find the leading order approximation which is valid for times $t = \text{ord}(\epsilon^{-1})$ as $\epsilon \to 0$, to the solution $x(t; \epsilon)$ and $y(t; \epsilon)$ satisfying

$$\begin{aligned} \frac{dx}{dt} + x^2 y \cos t &= \epsilon(x - 2x^2) \\ \frac{dy}{dt} &= \epsilon\left(1 - \frac{\sin t}{x}\right) \end{aligned}$$

with $x = 1$ and $y = 0$ at $t = 0$.

7.3 A diffusion–advection equation

An essential feature of problems requiring the method of multiple scales is that there is some quantity which is preserved at leading order, but which can drift through the accumulation of small disturbances. In the first two sections, we had a conservative oscillator which preserved the amplitude at leading order. Over a long time, however, the amplitude drifted through respectively the accumulated effect of the ϵ damping term and the ϵ work done on the oscillator by changing the length of the pendulum. Our next application of the method of multiple scales is to a partial differential equation describing advection around a periodic domain $0 < \theta < 2\pi$ with some small diffusion. Thus at leading order information is preserved as it is advected around. While over a long time this information changes from the initial data through the accumulated effect of weak diffusion.

We consider the initial value problem for $f(\theta, t; \epsilon)$ with $\epsilon \searrow 0$

$$\begin{aligned} \frac{\partial f}{\partial t} + \frac{\partial}{\partial \theta}(\omega(\theta) f) &= \epsilon \frac{\partial^2 f}{\partial \theta^2} \qquad \text{in } t \ge 0 \text{ and } 0 \le \theta \le 2\pi \\ \text{with} \quad f = F(\theta) & \qquad \text{at } t = 0 \end{aligned}$$

where $\omega(\theta)$ is given over $0 \le \theta \le 2\pi$, periodic and positive.

The two processes acting simultaneously are advection on the fast time scale $\tau = t$ and diffusion on the slow time scale $T = \epsilon t$. Thus we pose an asymptotic expansion

$$f(\theta, t; \epsilon) \sim f_0(\theta, \tau, T) + \epsilon f_1(\theta, \tau, T)$$

to be asymptotic at $T = \text{ord}(1)$. Substituting into the governing equation and comparing coefficients of ϵ^n, we obtain in the usual way a sequence of problems.

At ϵ^0:

$$\frac{\partial f_0}{\partial \tau} + \frac{\partial}{\partial \theta}(\omega f_0) = 0$$

i.e.

$$\frac{1}{\omega}\left(\frac{\partial}{\partial \tau} + \omega\frac{\partial}{\partial \theta}\right)(\omega f_0) = 0$$

This equation says that an observer moving at a speed ω will see the quantity ωf_0 remain constant (on the fast time scale, only). It is therefore necessary to find out where an observer moving at ω has progressed to after a time τ. Let $\Theta(t)$ be the solution of the initial value problem

$$\dot{\theta} = \omega(\theta) \qquad \text{in } t \geq 0 \ \text{ with } \ \theta = 0 \text{ at } t = 0$$

Because $\omega(\theta)$ is positive and 2π-periodic, $\Theta(t)$ modulo 2π must be periodic, say with period P.

We now transform from the co-ordinates θ and τ to the Lagrangian co-ordinates s and τ with

$$\theta(s,\tau) = \Theta(\tau - s)$$

The variable s is the time delay since θ was zero, and this is more useful than the usual Lagrangian variable of the initial angle. The variable is a periodic variable, with period P. Restricted to this period, there is an inverse

$$s = S(\theta,\tau)$$

To recast the differential equation in terms of the new co-ordinates s and τ, we first note

$$\delta\theta = \omega(\theta)(\delta\tau - \delta s)$$

so that

$$\left(\frac{\partial \theta}{\partial \tau}\right)_s = \omega(\theta) \qquad \text{and} \qquad \left(\frac{\partial s}{\partial \theta}\right)_\tau = -\frac{1}{\omega(\theta)}$$

Thence

$$\left(\frac{\partial}{\partial \tau}\right)_s = \left(\frac{\partial \tau}{\partial \tau}\right)_s\left(\frac{\partial}{\partial \tau}\right)_\theta + \left(\frac{\partial \theta}{\partial \tau}\right)_s\left(\frac{\partial}{\partial \theta}\right)_\tau = \left(\frac{\partial}{\partial \tau}\right)_\theta + \omega\left(\frac{\partial}{\partial \theta}\right)_\tau$$

$$\left(\frac{\partial}{\partial \theta}\right)_\tau = \left(\frac{\partial \tau}{\partial \theta}\right)_\tau\left(\frac{\partial}{\partial \tau}\right)_s + \left(\frac{\partial s}{\partial \theta}\right)_\tau\left(\frac{\partial}{\partial s}\right)_\tau = -\frac{1}{\omega}\left(\frac{\partial}{\partial s}\right)_\tau$$

Thus the controlling differential equation at ϵ^0 becomes

$$\left(\frac{\partial}{\partial \tau}\right)_s(\omega f_0) = 0$$

with solution

$$f_0(\theta,\tau,T) = \frac{A_0(s,T)}{\omega(\theta)}$$

This expresses formally the idea that on the fast time scale ωf_0 remains constant, moving with an observer at velocity ω, the observer being labelled by his value of s. This constant is, however, allowed to drift on the slow time scale. The initial value of the 'constant' is given from the initial conditions

$$A_0(s,0) = F(\Theta(-s))\,\omega(\Theta(-s))$$

At ϵ^1:

$$\frac{\partial f_0}{\partial T} + \frac{\partial f_1}{\partial \tau} + \frac{\partial}{\partial \theta}(\omega f_1) = \frac{\partial^2 f_0}{\partial \theta^2}$$

In the transformed co-ordinates, and with the partially determined f_0, this becomes

$$\frac{1}{\omega}\frac{\partial A_0}{\partial T} + \frac{1}{\omega}\left(\frac{\partial}{\partial \tau}\right)_s(\omega f_1) = \frac{1}{\omega}\left(\frac{\partial}{\partial s}\right)_\tau\frac{1}{\omega}\left(\frac{\partial}{\partial s}\right)_\tau\frac{A_0}{\omega}$$

with now $\omega = \omega(\Theta(\tau - s))$, a P-periodic function of both s and τ. Thus

$$\frac{\partial A_0}{\partial T} + \left(\frac{\partial}{\partial \tau}\right)_s(\omega f_1) =$$
$$\frac{1}{\omega^2}\cdot\frac{\partial^2 A_0}{\partial s^2} + \tfrac{3}{2}\frac{\partial}{\partial s}\left(\frac{1}{\omega^2}\right)\cdot\frac{\partial A_0}{\partial s} + \tfrac{1}{2}\frac{\partial^2}{\partial s^2}\left(\frac{1}{\omega^2}\right)\cdot A_0$$

Now the right hand side of the equation is P-periodic in τ with a non-zero average. In order to maintain the asymptoticness of the expansion of f to $T = \text{ord}(1)$, we must keep ωf_1 bounded as τ increases. Thus the average with respect to τ of the right hand side must be removed by setting it equal to $\partial A_0/\partial T$. Note that the second and third terms on the right hand side have zero averages, because they are derivatives with respect to s of functions of ω and so derivatives with respect to τ by $\omega = \omega(\Theta(\tau - s))$. Hence

$$\frac{\partial A_0}{\partial T} = \frac{1}{P}\int_0^P \frac{d\tau}{\omega^2(\Theta(\tau - s))}\,\frac{\partial^2 A_0}{\partial s^2}$$

Our result is that the amplitude function $A_0(s,T)$ satisfies a simple diffusion equation with an effective diffusivity $< 1/\omega^2 >$. As this diffusivity is constant, the equation for A_0 can be readily solved for any particular initial condition F using Fourier Series over the P-periodic variable s.

The result $< 1/\omega^2 >$ for the effective diffusivity can be explained as follows. At lowest order f is conserved moving with speed ω. Thus where ω slows down, adjacent moving points crowd together with separations proportional to ω. This leads to the density of the conserved f increasing like $1/\omega$ as seen in the result for f_0. Now the steepening of the spatial gradients like $1/\omega$ enhances the diffusion like $1/\omega^2$. This enhanced diffusion acts ϵ slowly while f is being advected rapidly around the θ space. We thus need an average of the enhanced diffusivity, weighting with the time spent at each location.

7.4 Homogenised media

So far the differential equation has generated the two scales of interest. In this and the following sections the two scales are specified by the geometry. In this section we are concerned with the effective properties of a medium with some fine scale structure, e.g. the effective elastic moduli of a composite material with carbon fibre strengthening. Rather than the vector problem for the elastic displacement, we look at the simpler scalar problem of heat conduction,

$$\nabla \cdot k \cdot \nabla T = Q$$

with k and Q given functions which have a fine scale structure described by the short scale variable $\xi = x/\epsilon$. We pose an expansion which is to be asymptotic on the long scale $x = \mathrm{ord}(1)$:

$$T(x,\epsilon) \sim T_0(\xi,x) + \epsilon T_1(\xi,x) + \epsilon^2 T_2(\xi,x)$$

Substituting into the governing equation and comparing coefficients of ϵ^n produces a sequence of problems.

At ϵ^{-2}:

$$\frac{\partial}{\partial\xi}\cdot k\cdot\frac{\partial T_0}{\partial\xi} = 0$$

with a solution

$$T_0 = T_0(x)$$

Thus at leading order, the temperature does not vary on the microscale.

At ϵ^{-1}:

$$\frac{\partial}{\partial\xi}\cdot k\cdot\frac{\partial T_1}{\partial\xi} = -\frac{\partial}{\partial\xi}\cdot k\cdot\frac{\partial T_0}{\partial x}$$

Thus T_1 will be linear in the forcing which is proportional to $\partial T_0/\partial x$, with a coefficient of linearity which depends on the details of $k(\xi)$, i.e.

$$T_1(\xi,x) = A(\xi)\cdot\frac{\partial T_0}{\partial x}$$

At ϵ^0:

$$\frac{\partial}{\partial\xi}\cdot k\cdot\frac{\partial T_2}{\partial\xi} = Q(\xi,x) - \frac{\partial}{\partial x}\cdot k\cdot\frac{\partial T_0}{\partial x} - \frac{\partial}{\partial\xi}\cdot k\cdot\frac{\partial T_1}{\partial x} - \frac{\partial}{\partial x}\cdot k\cdot\frac{\partial T_1}{\partial\xi}$$

To ensure that the expansion is asymptotic on $x = \mathrm{ord}(1)$, it is necessary to insist that the right hand side of the equation has a zero average over the fine scale details, otherwise T_2 would grow $\mathrm{ord}(\xi^2)$. Thus the secularity condition is

$$\frac{\partial}{\partial x}\cdot k^*\cdot\frac{\partial T_0}{\partial x} = Q^*$$

with

$$k^* = \left\langle k(\xi) + k(\xi)\cdot\frac{\partial A}{\partial\xi}\right\rangle \quad \text{and} \quad Q^* = \langle Q(\xi,x)\rangle$$

where the angled brackets denote an average over the ξ-microscale.

Note that our asymptotic analysis has shown that at leading order the temperature is constant over the local microstructure, and that to leading order the temperature satisfies a standard heat conduction equation with an effective heat conductivity k^* and heat source strength Q^*. Higher order corrections will find that there is a correction to the heat flux which is non-local, i.e. the heat flux at one point depends on the value of the temperature gradient in the neighbourhood of that point.

7.5 The WKBJ approximation

While everyone agrees that Messrs W, K, B and J did not invent this method, there is little agreement over who did. Certainly the following were all involved with important developments: Liouville 1837, Green 1837, Horn 1899, Rayleigh 1912, Gans 1915, Jeffrey 1923, Wentzel 1926, Kramers 1926, Brillouin 1926, Langer 1931, Olver 1961 and Meyer 1973. The problem is to obtain an asymptotic solution to the equation

$$\ddot{x} + f(\epsilon t)x = 0$$

We will tackle the problem with the method of multiple scales. Note that a more general equation

$$\ddot{y} + \epsilon a(\epsilon t)\dot{y} + b(\epsilon t)y = 0$$

can be transformed to our canonical form by the substitution

$$y = x \exp\left(\tfrac{1}{2}\epsilon \int^t a(\epsilon t')\, dt'\right)$$

resulting in

$$\ddot{x} + \left(b - \tfrac{1}{2}\epsilon^2 a'' - \tfrac{1}{4}\epsilon^2 a^2\right) x = 0$$

7.5.1 Solution by multiple scales

First we consider the case of $f > 0$, so that we may put $f = \omega^2$ with ω real and positive. Then the obvious solution is a fast oscillation with a local frequency ω and a slowly drifting amplitude and phase. In the notation of the method of multiple scales, with fast time $\tau = t$ and slow time $T = \epsilon t$, we expect a solution at leading order of the form

$$R(T) \cos\left[\omega(T)\tau + \theta(T)\right]$$

Unfortunately the secularity condition, which is found in the problem for x_1

$$\theta_T = -\omega_T \tau$$

is unacceptable because of the occurrence of the fast variable in the drift equation, with the other factors depending only on the slow time. The failure of our attempted solution does, however, suggest a cure. If the phase θ had been much larger, $\mathrm{ord}(\epsilon^{-1})$ rather than $\mathrm{ord}(1)$, then we could have multiplied the above equation by ϵ to produce an acceptable equation, which only involved the slow time T.

We thus start again with a leading order solution of the form

$$x_0(\tau, T) = R(T)\cos\theta \qquad \text{with} \qquad \theta = \epsilon^{-1}\Theta_0(T) + \Theta_1(T)$$

It is not immediately apparent that this solution does oscillate on the fast time scale. Note however that the time derivative of the phase is

$$\theta_t = \Theta_{0T} + \epsilon\Theta_{1T}$$

which is order 1. Thus for an order 1 change in time t, there is a small relative change in the slowly varying $\Theta_0(T)$, but this small relative change of a large object results in an order 1 absolute change, and for the arguments of trigonometric functions it is the magnitude of the absolute change which is relevant. It was this consideration which also required the third unknown function Θ_1 term to be included in the leading order term for $x(t, \epsilon)$.

Evaluating the time derivatives of the above x_0 we have

$$\begin{aligned} \dot{x}_0 &= -R\Theta_{0T}\sin\theta + \epsilon\left[R_T\cos\theta - R\Theta_{1T}\sin\theta\right] \\ \ddot{x}_0 &= -R\Theta_{0T}^2\cos\theta + \epsilon\big[-(2R_T\Theta_{0T} + R\Theta_{0TT})\sin\theta \\ &\qquad - 2R\Theta_{0T}\Theta_{1T}\cos\theta\big] + O(\epsilon^2) \end{aligned}$$

Substituting into the governing equation yields at leading order

$$\Theta_{0T} = \omega$$

At the next order, the secularity conditions associated with the equation for x_1 are

$$\begin{aligned} 2R_T\Theta_{0T} + R\Theta_{0TT} &= 0 \\ 2R\Theta_{0T}\Theta_{1T} &= 0 \end{aligned}$$

with solutions

$$\begin{aligned} R^2\omega &= \text{constant} \\ \Theta_1 &= \text{constant} \end{aligned}$$

Note that it is not the energy $E = \frac{1}{2}R^2\omega^2$ which is conserved over the long time scales, but rather the action E/ω (sometimes called an adiabatic invariant).

Note that once the secularity conditions are satisfied, then there is no forcing in the equation for x_1, although there is some forcing for x_2. Thus if one were to satisfy the boundary conditions with x_0 to order ϵ instead of just to order 1, then the error of x_0 would be only $O(\epsilon^2)$.

Returning to the original canonical equation, we now have a solution when $f > 0$ for x which is asymptotic to $t = \mathrm{ord}(\epsilon^{-1})$. This solution is often expressed as

$$x(t, \epsilon) \sim [f(\epsilon t)]^{-1/4}(a\cos\theta + b\sin\theta) \quad \text{with } \theta = \int_0^t [f(\epsilon t')]^{1/2}\, dt'$$

with constants a and b. Similarly when $f < 0$, the solution takes the form

$$x(t, \epsilon) \sim [-f(\epsilon t)]^{-1/4}\left(Ae^{\varphi} + Be^{-\varphi}\right) \quad \text{with } \varphi = \int_0^t [-f(\epsilon t')]^{1/2}\, dt'$$

with constants A and B.

7.5.2 Higher approximations

To obtain higher approximations it would be necessary to introduce a hierarchy of super slow time scales $T_n = \epsilon^n t$. Here we avoid these and instead use a method particular to the WKBJ problem. The first order asymptotic theory suggests a transformation

$$x(t,\epsilon) \equiv \mathrm{Re}\left\{ r(\epsilon t,\epsilon) \exp\left[i \int^t \sigma(\epsilon t',\epsilon)\, dt' \right] \right\}$$

with r and σ required to be real quantities. Then dropping the real part sign

$$\begin{aligned} \dot{x} &= ir\sigma \exp + \epsilon r_T \exp \\ \ddot{x} &= -r\sigma^2 \exp + \epsilon i (2r_T\sigma + r\sigma_T) \exp + \epsilon^2 r_{TT} \exp \end{aligned}$$

Substituting into the governing equation and comparing real and imaginary parts, we find

$$\begin{aligned} 2r_T\sigma + r\sigma_T &= 0 \\ \epsilon^2 r_{TT} - r\left(f - \sigma^2\right) &= 0 \end{aligned}$$

The first equation can be integrated to give the general result

$$r^2\sigma = \text{constant}$$

The second equation is then a nonlinear differential equation for the amplitude r. To solve it we expand in powers of ϵ^2:

$$\begin{aligned} \sigma(T,\epsilon) &\sim \sigma_0(T) + \epsilon^2\sigma_1(T) \\ r(T,\epsilon) &\sim r_0(T) + \epsilon^2 r_1(T) \end{aligned}$$

Substituting into the nonlinear differential equation and comparing coefficients of ϵ^n, we obtain

$$\begin{aligned} \sigma_0 &= f^{1/2} \\ r_0 &= kf^{-1/4} \quad \text{with } k \text{ a constant} \\ \sigma_1 &= \frac{r_{0TT}}{2\sigma_0 r_0} = f^{1/2}\left[\frac{5f'^2}{32f^3} - \frac{f''}{8f^2}\right] \\ r_1 &= -\frac{r_0\sigma_1}{2\sigma_0} \end{aligned}$$

Exercise 7.7. Use the transformation at the beginning of §7.5.2 to obtain solutions of the WKBJ type to the fourth order equation

$$\ddot{\ddot{x}} + f(\epsilon t)x = 0$$

7.5.3 Turning points

Our solutions in §7.5.1 work well while $f > 0$ or $f < 0$, but there is trouble at $f = 0$ where the frequency vanishes and the amplitude becomes infinite. Without loss of generality we can move the point where f vanishes, the so-called turning point, to the origin. Thus we have $f(0) = 0$. We start with the case $f'(0) > 0$, and leave other cases to the end of this section.

Now far from the origin, where $\epsilon t = \mathrm{ord}(1)$, we have the solutions given at the end of §7.5.1; the trigonometric solutions being applicable in $t > 0$, and the exponential solutions in $t < 0$. The aim of this section is to produce a solution which is valid near $t = 0$, so that we can provide a connection between the constants a and b in the trigonometric region to the constants A and B in the exponential region. We are thus involved in a problem of matched asymptotic expansions.

Now near to the origin, where $|\epsilon t| \ll 1$, we can approximate the governing equation by

$$\ddot{x} + \epsilon t f'(0) x = 0$$

If we introduce the rescaling

$$\tau = -t\left(\epsilon f'(0)\right)^{1/3}$$

we recover Airy's equation

$$x_{\tau\tau} - \tau x = 0$$

with a general solution

$$x = \alpha \mathrm{Ai}(\tau) + \beta \mathrm{Bi}(\tau)$$

in which α and β are constants and Ai and Bi are Airy functions. The asymptotic behaviour of Ai at large (positive and negative) arguments was evaluated in §3.3 by the method of steepest descents. The second Airy function Bi can be treated similarly.

Matching for negative times as $\tau \nearrow +\infty$ in the inner and as $t \nearrow 0$ in the outer

$$\begin{aligned} \text{inner} &= \frac{1}{\tau^{1/4}\sqrt{\pi}}\left(\tfrac{1}{2}\alpha \exp(-\tfrac{2}{3}\tau^{3/2}) + \beta \exp(\tfrac{2}{3}\tau^{3/2})\right) \\ \text{outer} &= \frac{1}{[-\epsilon t f'(0)]^{1/4}}\left(A\exp(\varphi) + B\exp(-\varphi)\right) \\ &\qquad \text{where } \varphi = -\tfrac{2}{3}\left[\epsilon f'(0)\right]^{1/2}(-t)^{3/2} \end{aligned}$$

The matching is successful if we take for the constants

$$\alpha = \frac{2\sqrt{\pi}}{[\epsilon f'(0)]^{1/6}} A \qquad \text{and} \qquad \beta = \frac{\sqrt{\pi}}{[\epsilon f'(0)]^{1/6}} B$$

Now matching for positive times as $\tau \searrow -\infty$ in the inner and as $t \searrow 0$ in the outer

$$\text{inner} = \frac{1}{(-\tau)^{1/4}\sqrt{\pi}}(\alpha\sin\Theta + \beta\cos\Theta)$$

$$\text{where } \Theta = \tfrac{2}{3}(-\tau)^{3/2} + \tfrac{1}{4}\pi$$

$$\text{outer} = \frac{1}{[\epsilon t f'(0)]^{1/4}}(a\cos\theta + b\sin\theta)$$

$$\text{where } \theta = \tfrac{2}{3}[\epsilon f'(0)]^{1/2} t^{3/2}$$

The matching is again successful if we take for the constants

$$a = \frac{[\epsilon f'(0)]^{1/6}}{(2\pi)^{1/2}}(\alpha + \beta) \qquad \text{and} \qquad b = \frac{[\epsilon f'(0)]^{1/6}}{(2\pi)^{1/2}}(\alpha - \beta)$$

Thus we have obtained the important *connection formulae*

$$A = \frac{a+b}{2\sqrt{2}} \qquad \text{and} \qquad B = \frac{a-b}{\sqrt{2}}$$

So if x is exponentially small in $t < 0$, i.e. $B = 0$ (because $\varphi < 0$), we emerge into $t > 0$ with $a \sim b$, i.e. with a solution

$$2Af^{-1/4}\sin\left(\int_0^t f^{1/2} + \frac{\pi}{4}\right)$$

showing a phase shift of $\pi/4$ for coming through the turning point.

We have studied above the case with $f'(0) > 0$. The case with $f'(0) < 0$ just requires a reversal of the t co-ordinate. For higher order turning points with $f \sim k^2 t^n$ as $t \to 0$, one has a governing equation in the inner region

$$\ddot{x} + k^2 t^n x = 0$$

which has solutions in terms of Bessel functions $t^{1/2} J_{\pm\nu}(2k\nu t^{1/2\nu})$ where $\nu = 1/(2+n)$.

If the details of the solution in the neighbourhood of the turning point are not required, there is an alternative way to derive the connection formulae. It is possible to go from the region where $f < 0$ to the region where $f > 0$ avoiding the point where $f = 0$ by making a detour on the complex t-plane. Caution is needed because of a Stokes phenomenon in which different asymptotic expansions are restricted to different sectors

in the complex t-plane. The origin of the $\pi/4$ phase shift is however seen as the analytic continuation of $(-f)^{-1/4}e^{\varphi}$ to $(f)^{-1/4}e^{-i(\theta-\pi/4)}$.

7.5.4 Examples

Example 1 is to find the high energy eigensolutions of Schrödinger's equation for a simple harmonic oscillator. The problem is to solve

$$\psi'' + (E - x^2)\psi = 0$$

$$\text{with } \psi \to 0 \quad \text{as} \quad x \to \pm\infty$$

when the eigenvalue E is large.

Now when E is large, there will be an oscillatory solution in $x^2 < E$ with a wavelength $\text{ord}(E^{-1/2})$ which is short compared with the scale on which this wavelength changes, $\text{ord}(E^{1/2})$. Thus we have a WKBJ problem. There are turning points at $x = \pm E^{1/2}$, and we want the exponentially decaying solution beyond these turning points.

Then by our WKBJ theory we have in $x^2 < E$

$$\psi \sim \frac{1}{(E-x^2)^{1/4}}\sin\left(\int_{-\sqrt{E}}^{x}(E-x^2)^{1/2}\,dx + \frac{\pi}{4}\right)$$

using the turning point connection formula for the decaying solution in $x < -\sqrt{E}$. Requiring the solution to decay also in $x > \sqrt{E}$ via a connection formula at $x = \sqrt{E}$, we have

$$\int_{-\sqrt{E}}^{\sqrt{E}}(E-x^2)^{1/2}\,dx + \frac{\pi}{4} = n\pi - \frac{\pi}{4}$$

the minus sign coming from the reversing of the x-coordinate at the second turning point. The above equation determines the eigenvalues to be

$$E = 2n - 1$$

which happens to be exact for all n, rather than just asymptotically true for large E. Our WKBJ solution also easily gives the behaviour

$$\psi \sim x^{n-1}e^{-x^2/2} \qquad \text{as } x \to \pm\infty$$

as well as the behaviour in $x^2 < E$.

Example 2 is to find the large eigenvalue solutions of the Legendre's equation. The standard form of Legendre's equation (to be solved for the solution which is regular at $x=-1$ and 1)

$$\left[(1-x^2)\,y'\right]' + \lambda y \;=\; 0$$

can be transformed into our canonical WKBJ form by setting

$$y \;=\; Y\exp\left(\int \frac{x}{(1-x^2)}\right) \;=\; Y(1-x^2)^{-1/2}$$

to give

$$Y'' + \left[\frac{\lambda}{1-x^2} + \frac{1}{(1-x^2)^2}\right] Y \;=\; 0$$

As in the first example, the solution has a short scale oscillation when λ is large. Thus in the oscillating range $-1<x<1$ our WKBJ solution gives

$$Y \;\sim\; k(1-x^2)^{1/4}\sin\left[\lambda^{1/2}\int_{-1}^{x}(1-x^2)^{-1/2}\,dx + \theta\right]$$

with the phase θ to be found from a turning point analysis.

Note that the wavenumber $[\lambda/(1-x^2)]^{1/2}$ is singular at the ends $x=\pm 1$, although in an integrable way. This singular behaviour, plus the more singular correction $(1-x^2)^{-2}$ in the wavenumber squared, calls for a modified turning point analysis. The regular solution with $y(-1)=1$ has a solution near $x=-1$

$$Y \;\sim\; \sqrt{2}(1+x)^{1/2}\mathrm{J}_0\left([2\lambda(1+x)]^{1/2}\right)$$

Matching this to our solution away from the end $x=-1$, we find the ubiquitous phase shift $\theta=\pi/4$ and also the constant $k=2(2\pi\lambda)^{-1/4}$. Applying a similar analysis at $x=1$, we obtain the eigenvalue condition

$$\lambda^{1/2}\pi + \tfrac{1}{4}\pi \;=\; (n+1)\pi - \tfrac{1}{4}\pi$$
$$\text{i.e. } \lambda \;=\; n(n+1)+\tfrac{1}{4}$$

i.e differing from the exact result through the error of 1/4.

Exercise 7.8. Find the large eigenvalue solutions of the equation

$$y'' + \lambda(1-x^2)^2 y \;=\; 0$$

subject to $y=0$ at $x=\pm 1$.

At the ends $x=\pm 1$ you will need to use turning point solutions like $(1-x^2)^{1/2}\mathrm{J}_{1/4}(\lambda^{1/2}(1-x^2)^2/4)$, and then use $\mathrm{J}_{1/4}(z)\sim(2/\pi z)^{1/2}\cos(z-3\pi/8)$ as $z\to\infty$.

7.5.5 Use of the WKBJ approximation to study an exponentially small term

Consider the matched asymptotic expansion problem for $y(x;\epsilon)$

$$\epsilon y'' - a(x)y' + b(x)y \;=\; 0 \quad\text{in}\quad -1\le x\le 1$$
$$\text{with } y(-1)=A \text{ and } y(1)=B$$

with the function $a(x)$ such that $a>0$ near $x=1$ and $a<0$ near $x=-1$. This restriction on the behaviour of a is crucial to the structure of this problem. To simplify the analysis we require additionally that $a>0$ in $x>0$ and $a<0$ in $x<0$, and that at $a(0+)=-a(0-)$ and $b(0+)=-b(0-)$.

Simple use of the method of matched asymptotic expansions produces an outer solution for the interior $-1<x<1$

$$y \;\sim\; k\exp\int_0^x \frac{b(x)}{a(x)}\,dx$$

with corrections ord(ϵ). Note that the additional restrictions on a and b make y and y' continuous at $x=0$. In order to satisfy the two boundary conditions, inner solutions are required, and these are possible near both boundaries because of the restriction on the sign of a. The inner near $x=-1$ is

$$y \;\sim\; A + \left(k\exp\int_0^{-1}\frac{b}{a}\,dx - A\right)\left[1-\exp\frac{a(-1)(1+x)}{\epsilon}\right]$$

while the inner near $x=1$ is

$$y \;\sim\; B + \left(k\exp\int_0^{1}\frac{b}{a}\,dx - B\right)\left[1-\exp\frac{a(1)(1-x)}{\epsilon}\right]$$

There now appears to be a paradox: the above matched asymptotic solution appears to be valid for all values of k, whereas the original equation had a unique solution. Proceeding to higher order corrections does not help to determine k.

The resolution of the paradox comes from realising that the second, rapidly decaying solution of the differential equation near to $x=-1$ is related to the second, rapidly decaying solution near to $x=1$. One is therefore not at liberty to pick the amplitude as $(k\exp -A)$ at one end and as $(k\exp -B)$ at the other end. To find out how the amplitude of this second, rapidly decaying solution is related from one end to the other, we use the WKBJ approximation.

The rapidly varying (decaying) second solution of the differential equation is for $x \neq 0$

$$y \sim \left(\exp \int_0^x \frac{a(x')}{\epsilon}\, dx'\right) \frac{1}{|a|} \left(\exp - \int_0^x \frac{b(x')}{a(x')}\, dx'\right)$$

This needs modifying in the neighbourhood of $x = 0$ in order to make y' continuous there.

We can now conclude that there is an inner solution only near $x = -1$ or only near $x = 1$ according to whether

$$\int_{-1}^{1} a(x')\, dx' \lessgtr 0$$

7.5.6 The small reflected wave in the WKBJ approximation

Consider waves propagating in one dimension through a medium whose properties vary slowly with position, i.e.

$$y_{tt} = \left(c(\epsilon x)^2 y_x\right)_x$$

Then the WKBJ solutions are

$$y \sim Ac^{-1/2}e^{it-i\theta} + Bc^{-1/2}e^{it+i\theta}$$
$$\text{where} \quad \theta = \int_0^x \frac{dx'}{c(\epsilon x')}$$

The A-term and B-term represent waves travelling respectively to the right and to the left. The variation of the amplitude like $c^{-1/2}$ means that the flux of energy for each wave, $c^2 y_x y_t$, is constant. Thus when a single right-moving wave is incident on a region where the medium varies, $c(\epsilon x)$, all of its energy is transmitted and there is no reflected wave, to leading order.

Now if c has a discontinuity (where it is not slowly varying), then equating y and $c^2 y_x$ on the two sides of the discontinuity yields a reflected wave with a relative magnitude ord$(\Delta c/c)$.

Now if c is continuous but c_x is discontinuous, then equating y and $c^2 y_x$ on the two sides of the discontinuity yields a smaller reflected wave ord$(\epsilon \Delta c')$.

Using the higher order solutions of §7.5.2, one can show that if $c(\epsilon x)$ has a discontinuity in its n^{th} derivative, then there will be a reflected wave of order ϵ^n.

If all the derivatives of $c(\epsilon x)$ are continuous (a C^∞ function), then the reflected wave is exponentially small. This exponentially small reflected wave can be calculated by making a transformation

$$y(x,t;\epsilon) \equiv A(x)c^{-1/2}e^{it-i\theta} + B(x)c^{-1/2}e^{it+i\theta}$$

Following the method of variations of parameters, this transformation yields equations for A and B, the energy amplitudes of the transmitted and reflected waves,

$$\begin{aligned} A_x &= i\epsilon^2 g\left(A + Be^{2i\theta}\right) \\ B_x &= -i\epsilon^2 g\left(B + Ae^{-2i\theta}\right) \\ \text{with } g &= (c'' + c'^2/2c)/4 \end{aligned}$$

A further transformation $A = a(x)e^{i\epsilon^2\varphi}$ and $B = b(x)e^{-i\epsilon^2\varphi}$ with $\varphi = \int_0^x g(\epsilon x')\, dx'$ yields

$$b_x = -i\epsilon^2 g a e^{-2i\theta + i2\epsilon^2\varphi}$$

In this equation one may take a to be a constant when the reflected wave b is small ($b \ll a$). The integral for the change in b can then be evaluated by deforming the contour on the complex x-plane. The major contribution will come from the complex singularity of $c(\epsilon x)$ at $\epsilon x = X_*$ which has the smallest real part of

$$I(X_*) = \frac{1}{i}\int_0^{X_*} \frac{dX}{c(X)}$$

If the nearest singularity is a pole of c, the reflected wave is found to be

$$b(-\infty) = -i\epsilon a e^{-2I(X_*)/\epsilon}\left[1.41 + O(\epsilon^{1/2})\right]$$

7.6 Slowly varying waves

7.6.1 A model problem

We consider a model problem due to Bretherton which describes waves propagating with some dispersion, with a small nonlinearity, and in a slowly varying medium.

$$\varphi_{tt} + (\alpha\varphi_{xx})_{xx} + (\beta\varphi_x)_x + \gamma\varphi = \epsilon\varphi^3$$

where α, β and γ are given functions of a slow space variable $X = \epsilon x$ and a slow time $T = \epsilon t$. The solution will clearly be a wave with a local frequency and a local wavenumber, and with a slowly varying amplitude. Now the basic oscillation of the wave has an order 1 change in the basic phase. In order to produce this order 1 change and also to have derivatives which vary only slowly in space and time (the slowly varying wave number and frequency), we rescale the phase to be very large and allow this large phase to have only slow variations. Thus we pose

$$\varphi(x,t;\epsilon) \sim a(X,T)\cos\theta \quad \text{with} \quad \theta = \epsilon^{-1}\Theta_0(X,T) + \Theta_1(X,T)$$

The *local wavenumber* is then $k(X,T) \equiv \theta_x \sim \Theta_{0X}$ and the *local frequency* is $\omega(X,T) \equiv -\theta_t \sim -\Theta_{0t}$. Note that this is the appropriate generalisation of the integral $\int^t \omega(\epsilon t')\,dt'$ in the WKBJ method.

With the assumed asymptotic form of the solution for φ, various time and space derivatives can be evaluated, e.g.

$$\begin{aligned}\varphi_{0xxxx} &= ak^4\cos\theta + \epsilon\big[(4a_Xk^3 + 6ak^2k_X)\sin\theta \\ &\qquad + 4ak^3\Theta_{1X}\cos\theta\big] + O(\epsilon^2)\end{aligned}$$

Substituting these into the governing equation yields at ϵ^0 just the *dispersion relation*

$$\omega^2 = \alpha k^4 - \beta k^2 + \gamma$$

At next order, the need to avoid a resonating forcing like $\sin\theta$ requires

$$\begin{aligned}(2a_T\omega + a\omega_T) + \alpha(4a_Xk^3 + 6ak^2k_X) + 2\alpha_X ak^3 \\ -\beta(2a_Xk + ak_X) - \beta_X ak = 0\end{aligned}$$

And avoiding a resonating forcing like $\cos\theta$ requires

$$2a\omega\Theta_{1T} + \alpha 4ak^3\Theta_{1X} - \beta 2ak\Theta_X = \tfrac{3}{4}a^3$$

We have at this stage to assume that the third harmonic $\cos 3(kx-\omega t)$ is not also resonant, which it can be for some dispersion relations – see the exercise below.

To make some sense out of our secularity conditions, we introduce the *group velocity* $c = \partial\omega/\partial k$. Differentiating the dispersion relation with respect to k, we have

$$2\omega c = 4\alpha k^3 - 2\beta k$$

We can now recognise the $\cos\theta$ secularity condition as

$$\left(\frac{\partial}{\partial T} + c\frac{\partial}{\partial X}\right)\Theta_1 = \frac{3a^2}{8\omega}$$

i.e. the extra phase Θ_1 drifts as in Duffing's equation, as seen by an observer moving with the group velocity.

Multiplying the $\sin\theta$ secularity condition by a and regrouping, we have

$$\frac{\partial}{\partial T}(a^2\omega) + \frac{\partial}{\partial X}(ca^2\omega) = 0$$

i.e. moving with the group velocity the quantity $a^2\omega$ is conserved. Note that this quantity is (energy $= a^2\omega^2)/\omega$ and is known as *wave action*.

We now have relations for the drift in the amplitude and extra phase. To complete the description we need how the wave number and frequency vary as they obey the dispersion relation. If we differentiate the dispersion relation with respect to time, we find

$$2\omega\omega_T = (4\alpha k^3 - 2\beta k)k_T + \alpha_T k^4 - \beta_T k^2 + \gamma_T$$

Now $k \equiv \Theta_X$, so that $k_T = \Theta_{XT} = -\omega_X$. Thus we have

$$\left(\frac{\partial}{\partial T} + c\frac{\partial}{\partial X}\right)\omega = (\alpha_T k^4 - \beta_T k^2 + \gamma_T)/2\omega$$

i.e. moving with the group velocity the frequency changes due to slow time changes in the medium. Similarly differentiating the dispersion relation with respect to space and using again $\omega_X \equiv -k_T$, we find

$$\left(\frac{\partial}{\partial T} + c\frac{\partial}{\partial X}\right)k = -(\alpha_X k^4 - \beta_X k^2 + \gamma_X)/2\omega$$

i.e. moving with the group velocity the wavenumber changes due to slow spatial variations in the medium.

Our model can be generalised to a moving medium with a slowly varying velocity $U(X,T)$, by replacing the partial double derivative with respect to time ∂_t^2 with the self-adjoint advected double derivative $(\partial_t + \partial_x U)(\partial_t + U\partial_x)$. Thus in the governing equation the term φ_{tt} becomes

$$\varphi_{tt} + 2U\varphi_{xt} + U^2\varphi_{xx} + \epsilon\left[U_X\varphi_t + (U_T + 2UU_X)\varphi_x\right] + O(\epsilon^2)$$

The effect of this modification on the dispersion relation is to turn ω^2 into $\omega^2 - 2U\omega k + U^2k^2$, i.e. $(\omega - Uk)^2$. Hence we find in the moving medium

$$\omega = Uk + \omega^+(k)$$

where ω^+ is the intrinsic frequency of the stationary medium. Thus the group velocity becomes $c = U + c^+$, where c^+ is the intrinsic group velocity of the stationary medium. The moving medium changes the term $2a_T\omega + a\omega_T$ in the $\sin\theta$ secularity condition to

$$[2a_T\omega + a\omega_T] + 2U\left[-a_Tk + a_X\omega + \tfrac{9}{2}(\omega_X - k_T)\right]$$

$$+ \; U^2\left[-2a_X k - ak_X\right] + U_X a\omega - \left[U_T + 2UU_X\right]ak$$
$$= \; \frac{1}{a}\left[\left[a^2(\omega - Uk)\right]_T + \left(a^2(\omega - Uk)\right]_X\right]$$

Adding this to the remaining unmodified term $\left(a^2\omega^+c^+\right)_X/a$ yields

$$\frac{\partial}{\partial T}\left(a^2\omega^+\right) + \frac{\partial}{\partial X}\left(ca^2\omega^+\right) \;=\; 0$$

i.e. the wave action $a^2\omega^+$ is conserved moving with the group velocity. Similarly one can show that the drift equation for the extra phase Θ_1 is modified by replacing ω by ω^+ on the right hand side, and that the wave number and frequency drift equations are modified by a similar replacement of ω by ω^+ together with additional terms $-U_X k$ and $U_T k$ added to the right hand sides.

Exercise 7.9 on two waves resonantly interacting. Two possible solutions of the partial differential equations

$$5\psi_{tt} + \psi_{xxxx} + 4\psi \;=\; 0$$

are the waves $\cos(x-t)$ and $\cos(2x-2t)$.

(i) Obtain the first order partial differential equations which govern the slow drift in the amplitudes of these two waves on the space and time scales of order ϵ^{-1} for the weak interaction between wave packets governed by

$$5\psi_{tt} + \psi_{xxxx} + 4\psi \;=\; \epsilon\psi\psi_x$$

You may neglect the slower drift in the phases.

(ii) Look for the steady periodic solutions of the equation

$$5\psi_{tt} + \psi_{xxxx} + 4\psi \;=\; \epsilon\psi^2$$

which take the form

$$A\cos\left[(1+\epsilon k)x - t\right] + B\cos 2\left[(1+\epsilon k)x - t\right] + O(\epsilon)$$

obtaining relationships between A, B and k.

7.6.2 Ray theory

We now move from the particular model problem with its unpleasant algebraic details to a general theory of waves propagating through a slowly varying medium. The first part which describes how the local

wavenumber and frequency change is variously called ray theory, wave kinematics or geometric optics.

We consider a scalar wave field $\varphi(\mathbf{x},t)$ in three-dimensional space. We assume that asymptotically it takes the form of a single wave propagating with a slowly varying amplitude and slowly varying wavenumber and frequency,

$$\varphi(\mathbf{x},t;\epsilon) \;\sim\; a(\mathbf{X},T)\cos\theta \quad\text{with}\quad \theta \;=\; \epsilon^{-1}\Theta(\mathbf{X},T)$$

This wave has a local frequency $\omega = -\theta_t = -\Theta_T$ and a local wavenumber $\mathbf{k} = \partial\theta/\partial\mathbf{x} = \partial\Theta/\partial\mathbf{X}$. Immediately from these definitions we have two consistency relations – known as the *conservation of wave crests* in space and time:

$$\frac{\partial}{\partial\mathbf{X}}\wedge\mathbf{k} \;=\; 0 \quad\text{and}\quad \frac{\partial\omega}{\partial\mathbf{X}} + \frac{\partial\mathbf{k}}{\partial T} \;=\; 0$$

Now the local dynamics will produce a dispersion relation

$$\omega \;=\; \Omega(\mathbf{k}, \alpha(\mathbf{X},T))$$

where the dependence on the medium is shown by a dependence on the parameter α. (Note that α could also represent the amplitude of the wave in a nonlinear problem.) The group velocity is defined by $\mathbf{c} = \partial\Omega/\partial\mathbf{k}$. Differentiating the dispersion relation with respect to time, we find

$$\frac{\partial\omega}{\partial T} \;=\; \frac{\partial\Omega}{\partial\mathbf{k}}\frac{\partial\mathbf{k}}{\partial T} + \frac{\partial\Omega}{\partial\alpha}\frac{\partial\alpha}{\partial T}$$

Using the consistency relation for $\mathbf{k}_T$, this becomes

$$\frac{\partial\omega}{\partial T} + \mathbf{c}\cdot\frac{\partial\omega}{\partial\mathbf{X}} \;=\; \frac{\partial\Omega}{\partial\alpha}\frac{\partial\alpha}{\partial T}$$

Exercise 7.10. Derive the similar result

$$\frac{\partial\mathbf{k}}{\partial T} + \left(\mathbf{c}\cdot\frac{\partial}{\partial\mathbf{X}}\right)\mathbf{k} \;=\; -\frac{\partial\Omega}{\partial\alpha}\frac{\partial\alpha}{\partial\mathbf{X}}$$

7.6.3 Averaged Lagrangian

The second part of the general behaviour of waves propagating in a slowly varying medium concerns the conservation of wave action, which gives how the amplitude varies.

In a continuum, the Lagrangian L is expressed in terms of a Lagrangian density $\mathcal{L}$, so $L = \int\mathcal{L}\,dxdt$. In this subsection we work in just one space dimension, three dimensions being a trivial extension.

The Lagrangian density will depend on the scalar wave field φ and its derivatives, and parametrically on the slow space and time as the medium varies, i.e. $\mathcal{L} = \mathcal{L}(\varphi, \varphi_t, \varphi_x, \varphi_{xx}, \dots ; X, T)$. For linear waves $\mathcal{L}$ is quadratic in φ. In our model problem,

$$\mathcal{L} \;=\; \tfrac{1}{2}\left((\varphi_t + U\varphi_x)^2 - \alpha\varphi_{xx}^2 + \beta\varphi_x^2 - \gamma\varphi^2 + \tfrac{1}{2}\epsilon\varphi^4\right)$$

The field equations for the wave field φ follow as the Euler–Lagrange equations, corresponding to the vanishing of the first variation of L with respect to φ, i.e.

$$\frac{\partial \mathcal{L}}{\partial \varphi} - \frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial \varphi_t}\right) - \frac{\partial}{\partial x}\left(\frac{\partial \mathcal{L}}{\partial \varphi_x}\right) + \frac{\partial^2}{\partial x^2}\left(\frac{\partial \mathcal{L}}{\partial \varphi_{xx}}\right) + \cdots \;=\; 0$$

We now substitute a solution of a single slowly varying wave

$$\varphi(x, t; \epsilon) \sim a(X, T)\cos\theta \quad \text{with} \quad \theta = \epsilon^{-1}\Theta(X, T)$$

with local wavenumber $k = \theta_x = \Theta_X$ and local frequency $\omega = -\theta_t = -\Theta_T$. Our Lagrangian density becomes $\mathcal{L}(a\cos\theta, a\omega\sin\theta, -ak\sin\theta, -ak^2\cos\theta, \dots ; X, T)$, at least asymptotically.

Now the full Lagrangian L is evaluated by integrating the density $\mathcal{L}$ over all x and t. In this integration there is a slow variation with X and T, and a fast variation in θ. The integration therefore averages over the rapid θ variations, producing an effective averaged Lagrangian density $\overline{\mathcal{L}}$

$$L \;=\; \epsilon^{-2}\int \overline{\mathcal{L}}\, dX\, dT \quad \text{with} \quad \overline{\mathcal{L}} \;=\; \frac{1}{2\pi}\int \mathcal{L}(a, \theta; X, T)\, d\theta$$

Thus in our model problem

$$\overline{\mathcal{L}} \;=\; \tfrac{1}{2}a^2\left((\omega - Uk)^2 - \alpha k^4 + \beta k^2 - \gamma - \tfrac{3}{16}\epsilon a^2\right)$$

The dynamics for the wave are derived from Lagrange's principle that the first variations of L with respect to the generalised co-ordinates a and Θ must vanish. (Note that Θ enters only through its derivatives ω and k.) The variation with respect to the amplitude a,

$$\frac{\partial \overline{\mathcal{L}}}{\partial a} = 0$$

is the dispersion relation. In our model problem this is

$$(\omega - Uk)^2 \;=\; \alpha k^4 - \beta k^2 + \gamma + \tfrac{3}{8}\epsilon a^2$$

In a linear wave theory $\overline{\mathcal{L}} = \frac{1}{2}a^2F(\omega, k)$, with F the dispersion relation. Hence the first variation with respect to the amplitude a gives $F = 0$.

This therefore implies that there is an equipartion of energy between the average potential and average kinetic energies.

The first variation with respect to the phase Θ gives

$$-\frac{\partial}{\partial T}\left(-\frac{\partial \overline{\mathcal{L}}}{\partial \omega}\right) - \frac{\partial}{\partial X}\left(\frac{\partial \overline{\mathcal{L}}}{\partial k}\right) \;=\; 0$$

This is a conservation equation which says that the density $-\partial\overline{\mathcal{L}}/\partial\omega$ changes due to a divergence in the flux $\partial\overline{\mathcal{L}}/\partial k$. Now for linear waves

$$\frac{\partial \overline{\mathcal{L}}}{\partial k} \;=\; \tfrac{1}{2}a^2\frac{\partial F}{\partial k} \;=\; -\tfrac{1}{2}a^2\frac{\partial F}{\partial \omega}\left(\frac{\partial \omega}{\partial k}\right)_{F=0} \;=\; -\frac{\partial \overline{\mathcal{L}}}{\partial \omega}c$$

Thus the variation with respect to Θ for linear waves becomes

$$\frac{\partial A}{\partial T} + \frac{\partial}{\partial X}(cA) \;=\; 0$$

i.e. the wave action $A = \partial\overline{\mathcal{L}}/\partial\omega$ is conserved moving with the group velocity $c = \partial\omega/\partial k$ evaluated on the dispersion relation $F = 0$.