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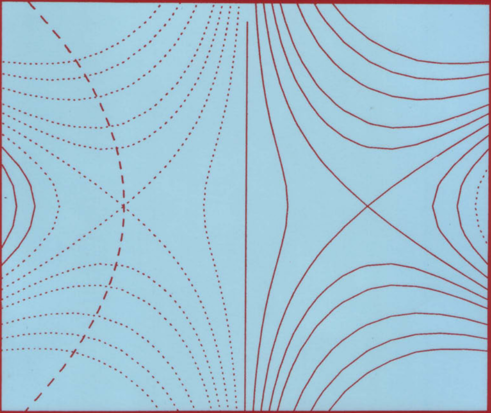

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Preface

Making precise approximations to solve equations is an occupation of applied mathematicians which distinguishes them from pure mathematicians, physicists and engineers. A precise approximation is not a contradiction in terms but rather an approximation with an error which is understood and controllable; in particular the error could be made smaller by some rational procedure. There are two methods for obtaining precise approximations to the solutions of an equation, numerical methods and analytic methods, and this book is about the latter. The analytic approximations are obtained when some parameter of the problem is small, and hence the name *perturbation methods.* The perturbation and numerical methods are not however in competition but rather complement one another as the following example illustrates.

The van der Pol oscillator is governed by the equation

$$\ddot{x} + k\dot{x}(x^2 - 1) + x = 0$$

In time the solution tends to an oscillation with a particular amplitude which does not depend on the initial conditions. The period of this limit oscillation is of interest and is plotted in figure 1 as a function of the strength of the nonlinear friction, k. The circles give the numerical results obtained by a Runge–Kutta method. The dashed curves give the first and second order perturbation approximations

$$\text{Period} = \begin{cases} 2\pi\left(1 + \frac{1}{16}k^2 + O(k^4)\right) & \text{as } k \to 0 \\ k(3 - 2\ln 2) + 7.0143k^{-1/3} + O(k^{-1}\ln k) & \text{as } k \to \infty \end{cases}$$

At intermediate values of the parameter k, from 2 to 6, the numerical method is most useful. At extreme values however the numerical method loses its accuracy rapidly, for example by $k = 10$ the time-step must be reduced to 0.01 in order to obtain 5 figure accuracy. The analytic approximations take over in the extreme conditions. Further they give an explicit dependence on the parameter k rather than the isolated results

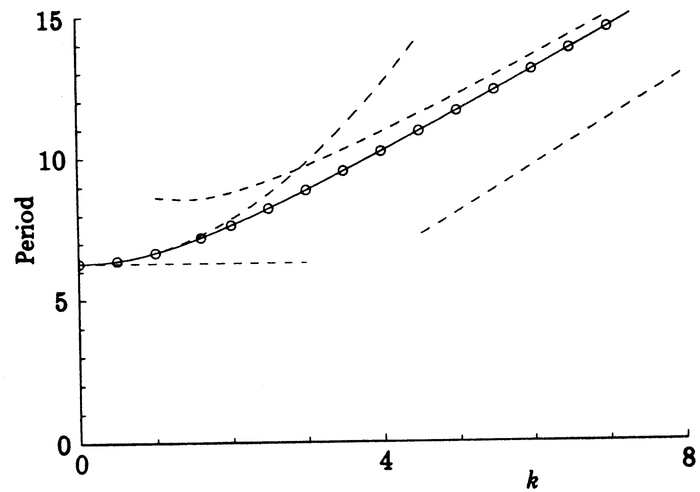

Fig. 1 The period of the limit oscillation of the van der Pol oscillator as a function of the strength of the nonlinear friction k.

at particular values from the numerical method. But the most important feature of the figure is the satisfying agreement between the numerical approximation and the two independent perturbation approximations – such checks are essential in research.

Obtaining good numerical values for the solution is not the only quest of a perturbation approximation. One can hope that the analysis will reveal some physical insights through the simplified physics of the limiting problem. In this book I will however suppress the physics in the problems discussed.

Finding perturbation approximations is an art rather than a science. In research it is useful to be responsive to suggestions from the physics. There is certainly no routine method appropriate to all problems, or even classes of problems. Instead one needs a determination to exploit the smallness of the parameter. This book attempts to present many of the weapons which have been found useful, but they should not be viewed as exhaustive.

While this book is mathematical, no attempt has been made to make the arguments fully rigorous. In general I have tried to explain why the results are correct. Often these reasons can be turned into strict theorems, albeit with some difficulty in the case of singular problems. My own opinion is that such superficial rigour rarely adds to the understanding of the problem, and that of greater use is a numerical statement about the range of applicability achieving some specified accuracy.

This book is based on a course of lectures which I gave for a number of years to first year graduate students in the University of Cambridge. In its turn it was based on my own education from a course of lectures by L. E. Fraenkel and from the book on the subject by M. Van Dyke. These two inspiring teachers asked many interesting questions which I have attempted to answer in this book; questions such as why are some results convergent whilst others only asymptotic, why is matching possible, what selection criterion should be used with strained co-ordinates, and what characterises problems to be tackled by multiple scales.

While no previous knowledge of perturbation methods is assumed, some previous experience is probable. The students who attended my lecture course would have seen several examples (small friction on projectiles, perturbed energy levels in quantum mechanics, adiabatic invariants in Hamiltonian systems, Watson's lemma, and viscous boundary layers in fluid mechanics) usually presented in an informal way relying heavily on physical insight. They would not however have seen a formal and organised approach to a perturbation problem.

The eventual goal of this book is to present the method of matched asymptotic expansions and the method of multiple scales, progressing to an advanced level in considering the more difficult issues such as the occurrence of logarithms and the occurrence of more than two scales. Tackling differential equations with such singular perturbation problems is certainly not easy. Fortunately many of the essential concepts can be presented in the simpler context of algebraic equations and later with integrals. Thus issues such as iterations and expansions, singular problems and rescaling, non-integral powers and logarithms will be presented well before the difficult singular differential equations are encountered. Finally I should observe that most of the chapters follow the basic method with an advanced application whose understanding is not essential to the following chapters – thus §§ 1.6, 3.5, 5.3, 5.4, 5.5, 5.6, 6.3, 7.3, 7.4 and 7.6 should be viewed as optional.

E.J. Hinch
Cambridge, 1990

5

Matched Asymptotic Expansions

We now tackle some singular differential equations. There are two distinct types of singular behaviour, which will be studied separately in this chapter and chapter 7. That considered in this chapter typically (but not always – see §5.2) involves a small parameter multiplying the highest derivative. The highest derivative can thus be ignored, so leading to a singular reduction of the order of the equation, except in thin regions of rapid change where the high value of the derivative cancels the effect of the multiplying small parameter. Often these regions of rapid change occur near to the boundary of the domain, and for this reason they are known as boundary layers.

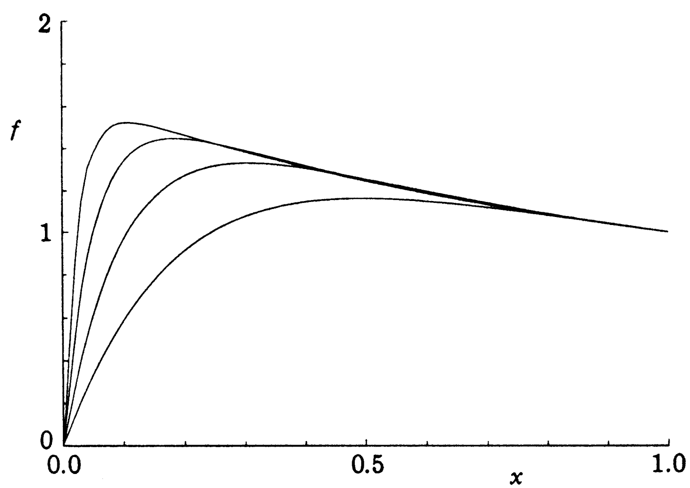

Fig. 5.1 The solution of the problem in §5.1 with $h(x) = e^{-x}$. As ϵ decreases through the values 0.2, 0.1, 0.05 and 0.025 a thin boundary layer of rapid change develops near $x = 0$.

5.1 A linear problem

We start with a simple linear ordinary differential equation, which can be solved exactly. This enables us to probe the structure of the solution and so develop the method of solving asymptotically such singular equations. This model problem is due to Friedrichs. Let $f(x, \epsilon)$ satisfy

$$\begin{aligned} \epsilon f_{xx} + f_x = h_x \qquad & \text{in } 0 < x < 1 \\ \text{with} \quad f(0,\epsilon) = 0 \qquad & \text{and } f(1,\epsilon) = 1 \end{aligned}$$

where h is a given function with N continuous derivatives on $[0, 1]$. We are interested in the behaviour as ϵ tends to zero through positive values, i.e. $\epsilon \searrow 0$.

5.1.1 The exact solution

By direct integration of the differential equation, we have

$$\begin{aligned} f(x,\epsilon) \quad = \quad & \frac{1}{\epsilon}\int_0^x e^{(t-x)/\epsilon} h(t)\,dt \quad + \\ & \left(1 - \frac{1}{\epsilon}\int_0^1 e^{(t-1)/\epsilon} h(t)\,dt\right) \frac{1 - e^{-x/\epsilon}}{1 - e^{-1/\epsilon}} \end{aligned}$$

We can obtain an asymptotic expansion by integrating by parts M ($< N$) times,

$$= \quad 1 \;+\; \sum_0^{M-1} (-\epsilon)^n \left(h^{(n)}(x) - h^{(n)}(1)\right) \;+\; R_I \;+\; R_{II}$$

where

$$\begin{aligned} R_I \quad &= \quad -\left(1 + \sum_0^{M-1}(-\epsilon)^n \left(h^{(n)}(0) - h^{(n)}(1)\right)\right) \frac{e^{-x/\epsilon} - e^{-1/\epsilon}}{1 - e^{-1/\epsilon}} \\ R_{II} \quad &= \quad -(-\epsilon)^{M-1}\int_0^x e^{(t-x)/\epsilon} h^{(M)}(t)\,dt \quad + \\ & \qquad (-\epsilon)^{M-1}\int_0^1 e^{(t-x)/\epsilon} h^{(M)}(t)\,dt \frac{1 - e^{-x/\epsilon}}{1 - e^{-1/\epsilon}} \end{aligned}$$

For $\epsilon \searrow 0$ *with* $x \neq 0$ *fixed*, R_{II} is $O(\epsilon^M)$ by using a bound on $h^{(M)}$, while R_I is exponentially small and so certainly $O(\epsilon^M)$. Thus we have

$$f \quad \sim \quad 1 + \sum_0^{M-1} (-\epsilon)^n \left(h^{(n)}(x) - h^{(n)}(1)\right)$$

an asymptotic series which might diverge if $M = \infty$.

The expansion above is not uniformly asymptotic in x, because the $e^{-x/\epsilon}$ in R_I is not $o(\epsilon^{M-1})$ on the whole interval $0 < x < 1$. Near $x = 0$ we need to have ϵ very small $[< x/((M+)\ln(1/x))]$, i.e. a disaster at $x = 0$. Some singular behaviour of the boundary value problem could have been anticipated, because the equation reduces from second order for $\epsilon > 0$ to first order for $\epsilon = 0$. Thus in the limit we must abandon one boundary condition.

While the above expansion breaks down at $x = 0$, there is an alternative expression which is asymptotic there. For $\epsilon \searrow 0$ *with* $\xi = x/\epsilon$ *fixed* (ξ is sometimes known as the boundary layer co-ordinate)

$$f(\epsilon\xi,\epsilon) \sim 1 - e^{-\xi} + \sum_{n=0}^{M-1}(-\epsilon)^n \left(h^{(n)}(0)\left[\sum_{k=0}^{n}\frac{(-\xi)^k}{k!} - e^{-\xi}\right] + h^{(n)}(1)\left(e^{-\xi}-1\right)\right)$$

obtained by retaining the appropriate parts of R_I, expanding $h^{(n)}(\epsilon\xi)$ and rearranging. This new asymptotic expansion also breaks down, this time when $\epsilon\xi = \text{ord}(1)$ due to the $\sum \xi^k$.

Thus we see that one function can be represented by two asymptotic expansions. Each expansion is limited by some non-uniformity in the asymptoticness, as $x \to 0$ and $\xi \to \infty$ respectively. The two expansions do, however, take a common form in the common ground where x is small but not too small and where ξ is large but not too large, i.e. $\epsilon \ll x = \epsilon\xi \ll 1$, as they must because they represent the same function.

We now try to obtain the above solution for $f(x,\epsilon)$ by solving exactly some approximate problems, rather than approximating the exact solution.

5.1.2 The outer approximation

We start by naively treating the problem as a regular perturbation problem, even though we know it is not. In this way we produce what is known as the outer approximation, which will often be referred to as the 'outer'. Thus we formally pose a Poincaré expansion with $P < N$,

$$f(x,\epsilon) \sim \sum_{n=0}^{P}\epsilon^n f_n(x)$$

Substituting into the governing equation and boundary conditions and comparing coefficients of ϵ^n yields

$$\text{at } \epsilon^0: \quad f_0' = h', \qquad f_0(0) = 0 \ \& \ f_0(1) = 1$$
$$\text{at } \epsilon^n: \quad f_n' = -f_{n-1}'', \qquad f_n(0) = 0 \ \& \ f_n(1) = 0 \text{ for } n > 1$$

Both boundary conditions cannot be satisfied in general, because the problem is not really regular. In order to satisfy the boundary conditions we will need one or more boundary layers in which the above outer approximation is not appropriate. We postpone until §5.1.7 a discussion of where the boundary layers might be. Here we use our knowledge of the exact solution to decide that there will be a boundary layer at $x = 0$ and none at $x = 1$. Thus the above outer must satisfy just the boundary condition at $x = 1$. Hence

$$f_0(x) = h(x) - h(1) + 1$$
$$f_n(x) = (-)^n\left(h^{(n)}(x) - h^{(n)}(1)\right) \qquad \text{for } n > 1$$

A special feature of our model problem is that the outer is now completely determined. Normally one would have some undetermined constants of integration in the outer solution at this stage.

5.1.3 The inner approximation (or boundary layer solution)

The singular reduction of the equation from second order to first order is not appropriate if there are large gradients in thin regions. From the exact solution we know that there is such a thin region near $x = 0$ with a width ϵ. In §5.1.6 we examine how the width of the thin region can be determined. The thin region is studied by introducing a rescaling with a stretched co-ordinate

$$\xi = x/\epsilon$$

The governing equation then becomes when $M < N$

$$\frac{1}{\epsilon}f_{\xi\xi} + \frac{1}{\epsilon}f_\xi = h_x(\epsilon\xi)$$
$$= \sum_{n=1}^{M}\epsilon^{n-1}h^{(n)}(0)\frac{\xi^{n-1}}{(n-1)!} + o(\epsilon^{M-1}\xi^{M-1})$$

We seek a formal expansion of the Poincaré form for $\epsilon \searrow 0$ at fixed ξ with $Q < N$,

$$f(x,\epsilon) \sim \sum_{0}^{Q}\epsilon^n g_n(\xi)$$

We substitute into the governing equation and boundary condition at $x = 0$. Note that the boundary condition at $x = 1$ cannot be applied, because it is not an accessible point for fixed ξ as $\epsilon \searrow 0$. Comparing coefficients of ϵ^n yields

$$\begin{array}{lll} \text{at } \epsilon^{-1}: & g_0'' + g_0' = 0, & g_0(0) = 0 \\ \text{at } \epsilon^{-1+n}: & g_n'' + g_n' = h^{(n)}(0)\frac{\xi^{n-1}}{(n-1)!}, & g_n(0) = 0 \text{ for } n > 1 \end{array}$$

with solutions

$$\begin{aligned} g_0 &= A_0(1 - e^{-\xi}) \\ g_n &= A_n(1 - e^{-\xi}) + (-)^n h^{(n)}(0) \sum_{k=1}^{n} \frac{(-\xi)^k}{k!} \qquad \text{for } n > 1 \end{aligned}$$

with constants of integration A_n. These constants will be determined in the next subsection by applying in some way the other boundary condition at $x = 1$, which is not immediately accessible to this thin inner region.

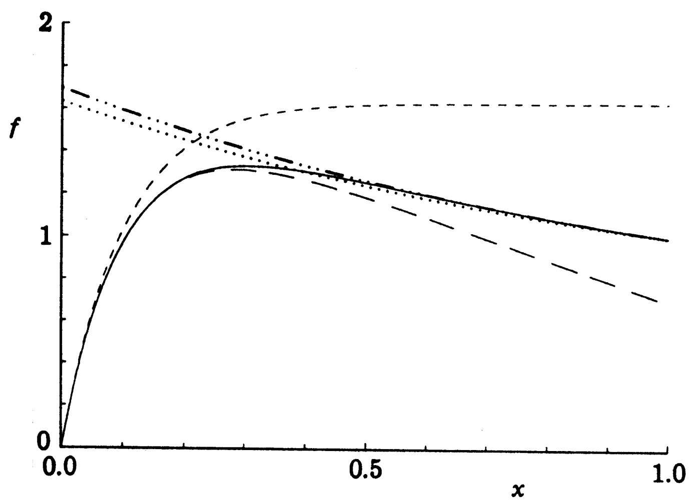

Fig. 5.2 The outer and inner approximations for $h(x) = e^{-x}$ and $\epsilon = 0.1$. The continuous curve is the exact solution. The leading order outer approximation is given by the dotted curve, while the dashed–dotted curve includes the first correction. The leading order inner approximation is given by the short dashed curve, while the long dashed curve includes the first correction.

5.1.4 Matching

We now have two asymptotic expansions for the solution, one for fixed x and one for fixed ξ. We now will see that these two expansions are of a similar form in an overlap region which has both x small and ξ large, i.e. $\epsilon \ll x = \epsilon\xi \ll 1$. Forcing the two expansions to be equal in the overlap determines the unknowns – here just the A_n. This process is called matching.

We express both the outer and the inner in terms of an intermediate variable

$$\eta = x/\epsilon^\alpha = \xi\epsilon^{1-\alpha} \qquad \text{with } 0 < \alpha < 1$$

We then take the limit $\epsilon \searrow 0$ with η fixed, which makes $x \to 0$ and $\xi \to \infty$. To help organise the terms of differing sizes, it is useful to think of some particular value for α, say $\frac{1}{2}$. In the overlap region the outer becomes

$$\begin{array}{l} [h(0) - h(1) + 1] + \epsilon^\alpha \eta h'(0) + \frac{1}{2}\epsilon^{2\alpha}\eta^2 h''(0) + \frac{1}{6}\epsilon^{3\alpha}\eta^3 h'''(0) + \cdots \\ \quad + \epsilon[h'(1) - h'(0)] - \epsilon^{1+\alpha}\eta h''(0) - \epsilon^{1+2\alpha}\eta^2 h'''(0) + \cdots \\ \qquad + \epsilon^2[h''(0) - h''(1)] + \cdots \\ \qquad\quad + \cdots \end{array}$$

where the successive rows come from the successive f_n. In the overlap region the inner becomes

$$\begin{array}{lr} A_0 & + \ E.S.T. \\ \quad + \epsilon^\alpha \eta h'(0) + \epsilon A_1 & + \ E.S.T. \\ \qquad + \frac{1}{2}\epsilon^{2\alpha}\eta^2 h''(0) - \epsilon^{1+\alpha}\eta h''(0) + \epsilon^2 A_2 & + \ E.S.T. \\ \qquad\quad + \cdots & \end{array}$$

where the successive rows come from the successive g_n, and $E.S.T.$ stands for *Exponentially Small Terms*, e.g. $A_0 \exp(-\eta/\epsilon^{1-\alpha})$. Comparing the two expressions, we see that they are identical if we set

$$\begin{aligned} A_0 &= h(0) - h(1) + 1 \\ A_1 &= h'(1) - h'(0) \\ \text{and } A_2 &= h''(0) - h''(1) \end{aligned}$$

which fully determines the solution.

Note that some terms have jumped their order, e.g. the term $\epsilon^\alpha \eta h'(0)$ comes out of the ord(ϵ) term ϵg_1. To stop the first ignored term in the inner $O(\epsilon^{Q+1}\xi^{Q+1})$ being larger than the last retained unknown term $\epsilon^Q A_Q$, it is necessary to take $\alpha > Q/(Q+1)$. This is permissible because we only need $0 < \alpha < 1$. Thus in the above displays, $Q = 2$, and so we should have chosen $\alpha > \frac{2}{3}$. However from the structure of the α and η dependencies of the terms, it is clear that nothing has been overlooked by our book-keeping with $\alpha \approx \frac{1}{2}$.

The order jumping term $\epsilon\xi h'(0)$ in ϵg_1 becomes as large as g_0, i.e. the inner loses its asymptoticness, when $\epsilon\xi = \text{ord}(1)$. This tells us that $h(x)$, which was considered a small correction in the equation governing the inner problem, can no longer be considered a small correction at $x = \text{ord}(1)$, i.e. there we need a new balance – the outer.

Exercise 5.1. The function $y(x;\epsilon)$ satisfies

$$\epsilon y'' + (1+\epsilon)y' + y = 0 \qquad \text{in } 0 \le x \le 1$$

and is subject to boundary conditions $y = 0$ at $x = 0$ and $y = e^{-1}$ at $x = 1$. Find two terms in the outer approximation, applying only the boundary condition at $x = 1$. Next find two terms in the inner approximation for the boundary layer near to $x = 0$, which can be assumed to have a width $\text{ord}(\epsilon)$, and applying only the boundary condition at $x = 0$. Finally determine the constants of integration in the inner approximation by matching.

5.1.5 Van Dyke's matching rule

Matching with an intermediate variable η can be tiresome. Van Dyke's rule usually works and is more convenient.

First we introduce some notation for our two limit operations.

$$\begin{aligned} E_P f &= \text{outer limit } (x \text{ fixed}, \epsilon \searrow 0) \text{ retaining } P+1 \text{ terms} \\ &= \sum_0^P \epsilon^n f_n(x) \\ H_Q f &= \text{inner limit } (\xi \text{ fixed}, \epsilon \searrow 0) \text{ retaining } Q+1 \text{ terms} \\ &= \sum_0^Q \epsilon^n g_n(\xi) \end{aligned}$$

With this notation Van Dyke's matching rule is

$$E_P H_Q f = H_Q E_P f$$

In words this means the following. First one takes the inner solution to $Q+1$ terms and substitutes x/ϵ for ξ. The outer limit of x fixed as

$\epsilon \searrow 0$ is then taken retaining $P+1$ terms. This produces the left hand side of the above equation. A similar process with the inner and outer interchanged produces the right hand side. Requiring that the left and right hand sides of the equation are identical then determines some of the constants of integration.

- For example for $P = Q = 0$

$$\begin{aligned} E_0 H_0 f &= E_0\left\{A_0(1-e^{-\xi})\right\} \\ &= E_0\left\{A_0(1-e^{-x/\epsilon})\right\} \\ &= A_0 \\ H_0 E_0 f &= H_0\left\{h(x) - h(1) + 1\right\} \\ &= H_0\left\{h(\epsilon\xi) - h(1) + 1\right\} \\ &= h(0) - h(1) + 1 \end{aligned}$$

And so the constant A_0 is determined to be $h(0) - h(1) + 1$.

- For example for $P = Q = 1$

$$\begin{aligned} E_1 H_1 f &= E_1\left\{A_0(1-e^{-\xi}) + \epsilon[A_1(1-e^{-\xi}) + h'(0)\xi]\right\} \\ &= E_1\left\{A_0(1-e^{-x/\epsilon}) + \epsilon[A_1(1-e^{-x/\epsilon}) + h'(0)x/\epsilon]\right\} \\ &= A_0 + xh'(0) + \epsilon A_1 \\ H_1 E_1 f &= H_1\left\{h(x) - h(1) + 1 - \epsilon[h'(x) - h'(1)]\right\} \\ &= H_1\left\{h(\epsilon\xi) - h(1) + 1 - \epsilon[h'(\epsilon\xi) - h'(1)]\right\} \\ &= h(0) - h(1) + 1 + \epsilon\xi h'(0) - \epsilon h'(0) + \epsilon h'(1) \end{aligned}$$

Because the $xh'(0)$ in E_1H_1f is equal to $\epsilon\xi h'(0)$ in H_1E_1f, the matching rule is successful and the constants are correctly determined as

$$A_0 = h(0) - h(1) + 1 \qquad \text{and} \qquad A_1 = h'(1) - h'(0)$$

Van Dyke's matching rule does not always work – see §5.2.5. Moreover the rule does not show that the inner and outer are identical in an overlap region.

We have determined the integration constants by matching. A cruder method is to *patch*, in which the value of the inner is set equal to the value of the outer at some particular x. If there is more than one constant to be determined, several x can be used or several derivatives of f at one x. In numerical work it can be difficult to implement a proper matching, and then one uses the unsatisfactory patching method.

Exercise 5.2. Try $P = Q = 2$ and the off diagonal case $P = 0$ with $Q = 1$.

Exercise 5.3. Show that Van Dyke's matching rule will work for general P and Q with the above outer and inner solutions of the particular model equation.

5.1.6 Choice of stretching

In §5.1.3 we used the exact solution to decide the appropriate stretching of the thin region near $x = 0$. In general one has to examine all the possible stretching transformations, as in the rescaling of the singular algebraic equation in §1.2.

Consider the transformation

$$x = \epsilon^\alpha \eta$$

This will expand the region near $x = 0$ if $\alpha > 0$. If α is not an integer, the expansion sequence for f will include non-integer powers of ϵ. The iterative method is therefore more appropriate until the expansion sequence becomes clear. Sometimes a more general stretching than ϵ^α is needed. In a nonlinear problem, one may also have to stretch the dependent variable, here f, possibly differently in different regions.

Substituting into the governing equation, we have for $M < N$

$$\begin{aligned} \epsilon^{1-2\alpha} f_{\eta\eta} + \epsilon^{-\alpha} f_\eta &= h_x(\epsilon^\alpha \eta) \\ &= \sum_{n=0}^{M-1} \epsilon^{n\alpha} h^{(n+1)}(0) \frac{\eta^n}{n!} + o(\epsilon^{\alpha(M-1)}) \end{aligned}$$

We now scan all possible $\alpha > 0$, starting with large α and then decreasing.

- If $\alpha > 1$, i.e. a finer scaling than the known boundary layer, the above equation can be rearranged placing all the small correction terms on the right hand side of the equation:

$$f_{\eta\eta} = -\epsilon^{\alpha-1} f_\eta + \epsilon^{2\alpha-1} h_x(\epsilon^\alpha \eta)$$

Solving iteratively, we find

$$\begin{aligned} f \;=\; & A + B\left(\eta - \tfrac{1}{2}\epsilon^{\alpha-1}\eta^2 + \tfrac{1}{6}\epsilon^{2(\alpha-1)}\eta^3 - \tfrac{1}{24}\epsilon^{3(\alpha-1)}\eta^4 + \cdots\right) \\ & + \epsilon^{2\alpha-1}h'(0)\left(\tfrac{1}{2}\eta^2 - \tfrac{1}{6}\epsilon^{\alpha-1}\eta^3 + \tfrac{1}{24}\epsilon^{2(\alpha-1)}\eta^4 + \cdots\right) \\ & + \epsilon^{3\alpha-1}h''(0)\left(\tfrac{1}{6}\eta^3 - \tfrac{1}{24}\epsilon^{\alpha-1}\eta^4 + \cdots\right) \\ & + \cdots \end{aligned}$$

Further terms horizontally come from iterating again through the small correction term on the right hand side of the governing equation $-\epsilon^{\alpha-1} f_\eta$, while further rows come from further terms in the expansion of h_x about $x = 0$. The constants A and B are available at every order.

Applying the boundary condition $f = 0$ at $x = 0$ yields $A = 0$ at all orders. The constant(s) B are not determined by this boundary condition.

We now find that the solution cannot be matched to the outer solution by any method. Applying Van Dyke's rule at the leading order, i.e. with $P = Q = 0$, one cannot match the inner's linear term $B\eta$ to the outer's approximately constant term $h(x) - h(1) + 1$ (unless by chance $h(0) - h(1)+1 = 0$). The mismatch is worse at higher orders. When attempting to match in an overlap region with an intermediate variable, it becomes clear that one cannot break through the barrier $\eta \ll \epsilon^{1-\alpha}$, because at $\epsilon^{\alpha-1}\eta = \text{ord}(1)$ an infinite number of terms – all those on a horizontal line – become the same size.

The solution for $\alpha > 1$ is not entirely spurious. It can be matched to our inner solution, because the above expression can be written

$$\begin{aligned} &(A = 0) + B\epsilon^{1-\alpha}\left(1 - e^{-\epsilon^{\alpha-1}\eta}\right) + h'(0)\epsilon\left(e^{-\epsilon^{\alpha-1}\eta} - 1 + \epsilon^{\alpha-1}\eta\right) \\ &\quad + h''(0)\epsilon^2\left(-e^{-\epsilon^{\alpha-1}\eta} + 1 - \epsilon^{\alpha-1}\eta + \tfrac{1}{2}\epsilon^{2(\alpha-1)}\eta^2\right) + \cdots \end{aligned}$$

which is of course the inner. Hence the solution for $\alpha > 1$ is simply an expansion of the inner on the finer length scale.

- If $\alpha = 1$, then we have the inner scaling of §5.1.3 which we know works.

- If $0 < \alpha < 1$, i.e. a coarser scaling than the known boundary layer, the governing equation can be rearranged placing all the small correction terms on the right hand side of the equation:

$$f_\eta = \epsilon^\alpha h_x(\epsilon^\alpha \eta) - \epsilon^{1-\alpha} f_{\eta\eta}$$

Solving iteratively again, we find

$$\begin{aligned} f \;=\; & A + \epsilon^\alpha \eta h'(0) \\ & + \tfrac{1}{2}\epsilon^{2\alpha}\eta^2 h''(0) - \epsilon^{a+1}\eta h''(0) \\ & + \tfrac{1}{6}\epsilon^{3\alpha}\eta^3 h'''(0) - \tfrac{1}{2}\epsilon^{2\alpha+1}\eta^2 h'''(0) + \epsilon^{\alpha+2}\eta h'''(0) \\ & + \cdots \end{aligned}$$

The terms on each horizontal line come from iterating through the small correction term $\epsilon^{1-\alpha} f_{\eta\eta}$ on the right hand side. Note that this produces only a finite number of terms for each row. Further rows come from further terms in the expansion of h_x about $x = 0$. The constant A is available at every order.

Applying the boundary condition at $x = 0$ yields $A = 0$ at all orders. We now find that the solution cannot be matched to the outer solution by any method. The leading order terms $\epsilon^\alpha\eta$ and $h(x) - h(1) + 1$ cannot be matched either by Van Dyke's rule or by an intermediate variable in an overlap region.

The solution for $0 < \alpha < 1$ is also not entirely spurious. It can be matched to the outer if we do not take $A = 0$, i.e. we forgo applying the boundary condition which was the very reason for having a boundary layer at $x = 0$. If instead of $A = 0$, we take

$$A \sim [h(0) - h(1) + 1] + \epsilon[h'(1) - h'(0)] + \epsilon^2[h''(0) - h''(1)]$$

then we recover the form of f found earlier in §5.1.4 in the overlap region.

From examining all the possible values of $\alpha > 0$, we conclude that it is only possible to match with the outer if $\alpha \le 1$, while it is only possible to include the ϵf_{xx} as a main term and so satisfy the boundary condition if $\alpha \ge 1$. As the inner must have the two properties, we conclude that the inner must have the scaling of $\alpha = 1$.

It is interesting to watch the relative importance of the terms in the equation as α varies.

	ϵf_{xx}	+	f_x	=	h_x	
$\alpha = 0$			...... balance			the outer
$0 < \alpha < 1$			dominant			the overlap
$\alpha = 1$	...... balance					the inner
$1 < \alpha$	dominant					the sub-inner

The reason that the inner expansion (governed by one equation) can be matched to the outer expansion (governed by a different equation at leading order), is that there exists an intermediate expansion in an overlap region. The intermediate expansion is governed by an intermediate equation which at leading order is the common terms of the leading order of the equations for the inner and the outer. It is the lack of any common terms between the sub-inner (with $\alpha > 1$) and the outer which makes it impossible to match them with an intermediate variable.

The potentially interesting scalings of the equation (rescaling both dependent and independent variables in a nonlinear problem) are those which produce a balance between two or more terms in the equation. Such scalings are sometimes called *distinguished limits.*

Exercise 5.4. Find the rescaling of x near $x = 0$ for

$$\epsilon x^m y' + y = 1 \qquad \text{in } 0 < x < 1 \qquad \text{with } y(0) = 0$$

when $0 < m < 1$

Exercise 5.5. (Stone) The function $y(x;\epsilon)$ satisfies

$$\epsilon y'' + x^{1/2}y' + y = 0 \qquad \text{in } 0 \le x \le 1$$

and is subject to boundary conditions $y = 0$ at $x = 0$ and $y = 1$ at $x = 1$. First find the rescaling for the boundary layer near $x = 0$, and obtain the leading order inner approximation. Then find the leading order outer approximation and match the two approximations.

5.1.7 Where is the boundary layer?

We simplified §5.1.2 by assuming that the boundary condition at $x = 1$ could be applied to the outer and that no boundary layer was needed there. We now see what would have happened if we had tried to put an unnecessary boundary layer near $x = 1$. From this study, we shall learn how to anticipate where boundary layers will be needed.

We expand the region near $x = 1$ by using a stretched co-ordinate $x = 1 - \epsilon^\alpha\eta$, with $\alpha > 0$ for a stretching and $\eta > 0$. The governing equation then becomes

$$\epsilon^{1-2\alpha} f_{\eta\eta} - \epsilon^{-\alpha} f_\eta = h_x(1 - \epsilon^\alpha\eta)$$

As in §5.1.6, the choices $0 < \alpha < 1$ and $\alpha > 1$ do not give useful balances in the equation. So we look at $\alpha = 1$:

$$f_{\eta\eta} - f_\eta = \epsilon \sum_{n=0}^{M-1} \frac{(-\epsilon\eta)^n}{n!} h^{(n+1)}(1) + o(\epsilon^M)$$

with $M < N$.

Solving iteratively, and applying the boundary equation at $x = 1$, i.e. $\eta = 0$,

$$f \sim 1 + A(e^\eta - 1) - \epsilon\eta h'(1) + \epsilon^2\left(\tfrac{1}{2}\eta^2 + \eta)\right)h''(1)$$

with the constant A available at all orders.

The above inner solution near $x = 1$ has to be matched to the outer solution to which no boundary condition has been applied, i.e.

$$f \sim B + h(x) - \epsilon h'(x) + \epsilon^2 h''(x)$$

with the constant B available at all orders.

Matching by Van Dyke's rule or by the intermediate variable is successful and determines the integration constants

$$A = 0 \qquad \text{and} \qquad B \sim 1 - h(1) + \epsilon h'(1) - \epsilon^2 h''(1)$$

With $A = 0$, the inner solution becomes merely a re-expression of the outer in terms of the stretched co-ordinate. This is quite different to the boundary layer at $x = 0$ which through terms like $[h(0) - h(1) + 1]\, e^{-\xi}$ deviates from the outer (re-expressed in terms of the stretched co-ordinate there).

The difficulty in trying to place a boundary layer at $x = 1$ is that the additional solution of the equation, which enters when the stretched co-ordinates restore the order of the governing equation to 2, blows up exponentially away from $x = 1$ (and so must be zero), while it decays away from $x = 0$. To have a non-trivial boundary layer one needs some extra solutions for the inner which decay into the outer region. They do not, however, have to decay exponentially.

- Example 1. Consider

$$\epsilon^2 f'' - f = -1 \quad \text{in } 0 < x < 1 \quad \text{with} \quad f = 0 \quad \text{at } x = 0 \text{ and } 1$$

For the stretchings ϵ^α with $\alpha = 1$, this equation has exponentially decaying solutions for x increasing and for x decreasing. Thus boundary layers are possible both near $x = 0$ and near $x = 1$, and both layers are needed.

$$f \quad \sim \quad 1 - e^{-x/\epsilon} - e^{(x-1)/\epsilon}$$

- Example 2. Consider

$$\epsilon^2 f'' + f = 1 \quad \text{in } 0 < x < 1 \quad \text{with} \quad f = 0 \quad \text{at } x = 0 \text{ and } 1$$

While the same stretching ϵ^α with $\alpha = 1$ produces a possible equation for the inner, there are however no decaying solutions in either direction. Hence it is not possible to add any boundary layers at $x = 0$ and $x = 1$ to help the candidate for the outer $f \sim 1$ satisfy the boundary conditions. The exact solution is

$$f \quad = \quad 1 - \frac{\sin(x/\epsilon) + \sin((1-x)/\epsilon)}{\sin(1/\epsilon)}$$

which shows that the boundary scaling is applicable all the way across the domain.

- Example 3. Consider

$$\epsilon^2 f'' + 2f(1 - f^2) = 0 \quad \text{in } -1 < x < 1$$
$$\text{with} \quad f = -1 \quad \text{at } x = -1 \quad \text{and} \quad f = 1 \quad \text{at } x = 1$$

The solution is

$$f \quad \sim \quad \tanh(x/\epsilon)$$

which has a thin region of high gradients in the middle of the range near $x = 0$. The example demonstrates that the boundary layers are not always near to boundaries.

Exercise 5.6. Find two terms in ϵ in the outer region, having matched to the inner solutions at both boundaries for

$$\epsilon^2 y'''' - y'' = -1 \qquad \text{in } -1 < x < 1$$
$$\text{with } y = y' = 0 \quad \text{at } x = -1 \text{ and } 1$$

Exercise 5.7. (Cole) The function $y(x, \epsilon)$ satisfies

$$\epsilon y'' + yy' - y \; = \; 0 \qquad \text{in } 0 \le x \le 1$$

and is subject to the boundary condition $y = 0$ at $x = 0$ and $y = 3$ at $x = 1$. Assuming that there is a boundary layer only near $x = 0$, find the leading order terms in the outer and inner approximations and match them.

Exercise 5.8. (Cole) Reconsider the equation of exercise 5.7 but now apply boundary conditions $y = -\frac{3}{4}$ at $x = 0$ and $y = \frac{5}{4}$ at $x = 1$. The boundary layer has moved to an intermediate position which is determined by the property of the inner that y jumps within the boundary layer from $-M$ to M, for some value M. Find the leading order matched asymptotic expansions.

5.1.8 Composite approximation

Now the outer expansion breaks down as $x \to 0$ because it lacks the $e^{-\xi}$ terms, while the inner expansion breaks down because it expresses $h^{(n)}(\epsilon\xi)$ as a power series in ϵ. Thus by correcting either of these two faults we can construct a uniformly valid asymptotic approximation

$$f(x, \epsilon) \quad \sim \quad 1 - e^{-x/\epsilon} +$$
$$\sum_0^M (-\epsilon)^n \left\{ h^{(n)}(x) - h^{(n)}(1) - \left[h^{(n)}(0) - h^{(n)}(1) \right] e^{-x/\epsilon} \right\}$$

This is called a composite approximation. Because it is not of the Poincaré form, it is not unique.

In general composite approximations can be formed by adding the inner and the outer and then subtracting their common form in the overlap. So with Van Dyke's matching rule we can form a composite limiting operator

$$C_{P,Q}f \quad = \quad E_P f \; + \; H_Q f \; - \; E_P H_Q f$$

Note that, if either the outer or the inner operators are applied to the composite, then we recover the outer or inner respectively:

$$E_P C_{P,Q} f = E_P f \qquad \text{and} \qquad H_Q C_{P,Q} f = H_Q f$$

When using Van Dyke's rule to do the matching, it takes little extra effort to form the composite. Thus using §5.1.5 we have for $P = Q = 0$

$$\begin{aligned} C_{0,0}f &= [h(x) - h(1) + 1] + [h(0) - h(1) + 1]\left(1 - e^{-\xi}\right) \\ &\quad - [h(0) - h(1) + 1] \\ &= h(x) - h(1) + 1 - [h(0) - h(1) + 1]\, e^{-x/\epsilon} \end{aligned}$$

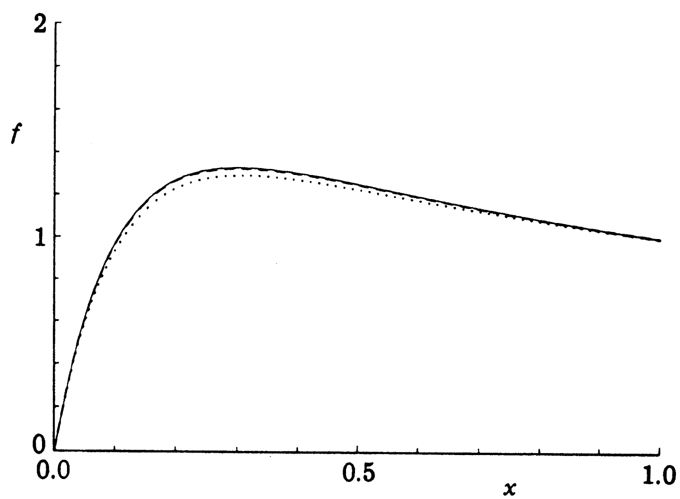

Fig. 5.3 The composite approximations for $h(x) = e^{-x}$ and $\epsilon = 0.1$. The continuous curve is the exact solution. The leading order composite approximation $C_{0,0}f$ is given by the dotted curve. The dashed curve, which is virtually superimposed on the exact solution, gives the higher composite approximation $C_{1,1}f$.

Composite approximations are often quantitatively better, i.e. with numerical values for x and ϵ, than either the inner or the outer. Composite approximations are also useful when proving the asymptoticness, because they should satisfy the governing equation and boundary conditions to ord$[\epsilon^{\min(P,Q)}]$. Some variations of the methods of matched asymptotic expansions seek from the outset a composite approximation. As some way is needed to avoid the non-uniqueness of the non-Poincaré form, these variant methods are restricted to special classes of problem.

Exercise 5.9. Find the composite $C_{1,1}f$.

Exercise 5.10. Show that $E_P C_{P,Q} f = E_P f$ and $H_Q C_{P,Q} f = H_Q f$.

Exercise 5.11. (Van Dyke) Calculate three terms of the outer solution of

$$(1+\epsilon)x^2 y' \; = \; \epsilon\left((1-\epsilon)xy^2 - (1+\epsilon)x + y^3 + 2\epsilon y^2\right) \quad \text{in } 0 < x < 1$$

with $y(1) = 1$. Locate the non-uniformity of the asymptoticness, and hence the rescaling for an inner region. Thence find two terms for this inner solution.

Exercise 5.12. (Van Dyke) Consider the following problem which has an outer, an inner and an inner-inner inside the inner (called a triple deck problem)

$$x^3 y' \quad = \quad \epsilon\left((1+\epsilon)x + 2\epsilon^2\right) y^2 \qquad \text{in } 0 < x < 1$$

with $y(1) = 1 - \epsilon$. Calculate two terms of the outer, then two of the inner, and finally one for the inner-inner. At each stage find the rescaling required for the next layer by examining the non-uniformity of the asymptoticness in the current layer.

5.2 Logarithms

We now progress to a more advanced topic in matched asymptotic expansions. This involves logarithms. While we will find that there are two regions, each with its own asymptotic approximation which must be matched together, the governing equation no longer gives an immediate hint of the existence of the two regions, because the small parameter does not multiply the highest derivative.

5.2.1 The problem and initial observations

We consider a model problem which looks like a problem for heat conduction outside a sphere with a small nonlinear heat source. With ϵ small and positive, let $f(r,\epsilon)$ be governed by

$$f_{rr} + \tfrac{2}{r}f_r + \epsilon f f_r = 0 \qquad \text{in } r>1$$
$$\text{with } f=0 \text{ at } r=1$$
$$\text{and } f\to 1 \text{ as } r\to\infty$$

First we try naively a regular perturbation expansion for f with r fixed as $\epsilon \searrow 0$. We therefore pose formally

$$f(r,\epsilon) \sim f_0(r) + \epsilon f_2(r)$$

Note that f_1 has been omitted, because I know that this problem is not straightforward. Substituting into the governing equation and comparing coefficients of ϵ^n yields a sequence of problems.

At ϵ^0: $f_0'' + \frac{2}{r}f_0' = 0$ with $f_0(0)=0$ and $f_0\to 1$ as $r\to\infty$ with solution

$$f_0 = 1 - \frac{1}{r}$$

At ϵ^1: $f_2'' + \frac{2}{r}f_2' = -f_0 f_0'$ with $f_2(0)=0$ and $f_2\to 0$ as $r\to\infty$

The governing equation for f_2 can be rearranged to

$$\frac{1}{r^2}\left(r^2 f_2'\right)' = -\frac{1}{r^2} + \frac{1}{r^3}$$

with a solution satisfying the boundary condition at $r=1$,

$$f_2 = -\ln r - \frac{\ln r}{r} + A_2\left(1-\frac{1}{r}\right)$$

There is clearly trouble here, because the condition at infinity cannot be satisfied by any choice of the free constant A_2.

At this stage one might doubt that the problem has a solution: although the linearised problem is known to be well-posed, there is no supporting general theory which says that the nonlinear problem must have a solution. In a totally new problem one would probably retreat to a numerical solution of the problem, and only proceed with an asymptotic analysis when there is some evidence that a solution does exist.

The trouble with our naive expansion is that it is not uniformly asymptotic at large r. When r is large, we note that the main term in the equation $f_0'' \sim 2/r^3$ while the small nonlinear $\epsilon f_0 f_0' \sim \epsilon/r^2$. Thus the nonlinear term cannot be viewed as a small correction at $r = \text{ord}(\epsilon^{-1})$.

The difficulty at large r can be examined by introducing a rescaling $\rho = \epsilon r$. Further when $r = \text{ord}(\epsilon^{-1})$, our naive expansion suggests that $f = 1 + \text{ord}(\epsilon \ln \frac{1}{\epsilon}) + \text{ord}(\epsilon)$, and so we try the asymptotic sequence 1, $\epsilon \ln\frac{1}{\epsilon}$ and ϵ.

5.2.2 Approximation for r fixed as $\epsilon \searrow 0$

It is tempting to call this the outer approximation, because it is the solution for the unstretched variable. Unfortunately this region is inside the region with the stretched variable $\rho = \epsilon r$ fixed as $\epsilon \searrow 0$. The names outer and inner will therefore not be used in this problem.

We formally pose a Poincaré expansion in the unstretched variable,

$$f(r,\epsilon) \sim \left(1-\frac{1}{r}\right) + \epsilon \ln\tfrac{1}{\epsilon} f_1(r) + \epsilon f_2(r)$$

where the obvious leading order term from §5.2.1 has been substituted.

The next function, f_1, satisfies the linearised equation. The solution satisfying the boundary condition at $r=1$ is therefore

$$f_1 = A_1\left(1-\frac{1}{r}\right)$$

The constant A_1 cannot be determined by applying the condition at infinity directly, because that is outside the region of r fixed as $\epsilon \searrow 0$. Instead this unknown will be determined by matching with an expansion valid in the region ρ fixed as $\epsilon \searrow 0$ to which the condition at infinity can be applied.

The second correction f_2 is the same as that found in §5.2.1 with the constant A_2 to be determined now by the matching.

5.2.3 Approximation for $\rho = \epsilon r$ fixed as $\epsilon \searrow 0$

With this stretched variable the governing equation becomes

$$f_{\rho\rho} + \tfrac{2}{\rho}f_\rho + f f_\rho = 0$$

This strictly nonlinear equation is tractable only because f is very near to 1 at large r. Thus we can formally pose a Poincaré expansion

$$f(r,\epsilon) \sim 1 + \epsilon \ln\tfrac{1}{\epsilon} g_1(\rho) + \epsilon g_2(\rho)$$

Both g_1 and g_2 satisfy the same equation

$$g'' + \left(\tfrac{2}{\rho} + 1\right) g' = 0$$

i.e.

$$(\rho^2 e^\rho g')' = 0$$

Thus applying the condition that $g \to 0$ as $\rho \to \infty$, we find

$$g_i(\rho) = B_i \int_\rho^\infty \frac{e^{-\tau}}{\tau^2}\, d\tau$$

with constants B_1 and B_2 to be determined by matching. The integral above can be expressed in terms of the exponential integral; it is $E_2(\rho)/\rho$.

In preparation for matching, we need to know the behaviour of the integral in $g_i(\rho)$ for small ρ:

$$\int_\rho^\infty \frac{e^{-\tau}}{\tau^2}\, d\tau \sim \frac{1}{\rho} + (\ln\rho + \gamma - 1) - \tfrac{1}{2}\rho + o(\rho) \quad \text{as } \rho \to 0$$

in which γ is the Euler constant 0.57722.

5.2.4 Matching by intermediate variable

Introducing the intermediate variable $\eta = \epsilon^\alpha r = \rho/\epsilon^{1-\alpha}$ with $0 < \alpha < 1$, we re-express the r-approximation and the ρ-approximation in terms of η and then take the intermediate limit of η fixed as $\epsilon \searrow 0$.

$$\begin{aligned} r\text{-approximation} &= \left(1 - \frac{\epsilon^\alpha}{\eta}\right) \\ &\quad + \epsilon \ln\tfrac{1}{\epsilon} A_1 \left(1 - \frac{\epsilon^\alpha}{\eta}\right) \\ &\quad + \epsilon\left[-\alpha\ln\tfrac{1}{\epsilon} - \ln\eta + A_2 - \alpha\ln\tfrac{1}{\epsilon}\frac{\epsilon^\alpha}{\eta} - \epsilon^\alpha \frac{\ln\eta + A_2}{\eta}\right] + \cdots \\ \rho\text{-approximation} &= 1 \\ &\quad + \epsilon\ln\tfrac{1}{\epsilon} B_1 \left[\frac{\epsilon^{\alpha-1}}{\eta} + (\alpha-1)\ln\tfrac{1}{\epsilon} + \ln\eta + \gamma - 1 + \cdots\right] \\ &\quad + \epsilon B_2 \left[\frac{\epsilon^{\alpha-1}}{\eta} + (\alpha-1)\ln\tfrac{1}{\epsilon} + \ln\eta + \gamma - 1 + \cdots\right] + \cdots \end{aligned}$$

These two expressions have the same form, and by forcing them to be identical we determine the constants of integration. Matching at sequential orders we find

at ϵ^0:	$1 = 1$, i.e. we started the r-approximation correctly	
at $\epsilon^\alpha \ln\frac{1}{\epsilon}$:	$0 = B_1 \frac{1}{\eta}$,	i.e. $B_1 = 0$
at ϵ^α:	$-\frac{1}{\eta} = B_2 \frac{1}{\eta}$,	i.e. $B_2 = -1$
at $\epsilon\ln\frac{1}{\epsilon}$:	$A_1 - \alpha = B_2(\alpha - 1)$,	i.e. $A_1 = 1$
at ϵ:	$-\ln\eta + A_2 = B_2(\ln\eta + \gamma - 1)$,	i.e. $A_2 = 1 - \gamma$

Note in the last but one equation that once the constants A_1 and B_2 are determined for one value of α then the equation becomes true for all α, i.e. true for all intermediate limits.

We have now determined the solution. For r fixed

$$f \sim \left(1 - \frac{1}{r}\right) + \epsilon\ln\tfrac{1}{\epsilon}\left(1 - \frac{1}{r}\right) + \epsilon\left[-\ln r - \frac{\ln r}{r} + (1-\gamma)\left(1 - \frac{1}{r}\right)\right]$$

while for ρ fixed

$$f \sim 1 + 0\epsilon\ln\tfrac{1}{\epsilon} - \epsilon\int_\rho^\infty \frac{e^{-\tau}}{\tau^2}\, d\tau$$

5.2.5 Further terms

To understand the correction terms to the above solution, it is necessary to review where the present terms have come from. The leading order, ord(1), term in the r-region forces through the small nonlinear term in the governing equation the correction ord(ϵ). This term contained an unmatchable $\ln r$, which called for the introduction of the ρ-region. In the ρ-region, the forced ord(ϵ) term behaved like $\ln\rho$ as $\rho \to 0$. Matching then required the introduction of an ord($\epsilon\ln\frac{1}{\epsilon}$) term in the r-region. Note that this unexpected ord($\epsilon\ln\frac{1}{\epsilon}$) term in the r-region is not directly forced by the field equation there – it is a homogeneous or eigensolution of the linearised equation. Such a term which is forced by the matching is sometimes called a *switchback*. These logarithmic terms naturally occur in particular integrals of differential equations,

$$\int_{\text{ord}(1)}^{\text{ord}(\frac{1}{\epsilon})} \frac{dr}{r}$$

Now turning to the correction terms. The $\epsilon\ln\frac{1}{\epsilon}$ term in the r-region will force through the small nonlinear term in the governing equation a correction $\epsilon^2\ln\frac{1}{\epsilon}$ which must have a $\ln r$ behaviour at large r, just like the ord(1) term of which it is a copy. The process of matching will then require an ord($\epsilon^2\ln\frac{1}{\epsilon}$) term in the ρ-region with a $\ln\rho$ behaviour at small ρ and thence an $\epsilon^2[\ln\frac{1}{\epsilon}]^2$ term in the r-region. Thus we can expect corrections ord($\epsilon^2[\ln\frac{1}{\epsilon}]^2$), ord($\epsilon^2\ln\frac{1}{\epsilon}$) and ord($\epsilon^2$).

5.2.6 Failure of Van Dyke's matching rule

If we take the E operator for the r-limit and the H operator for the ρ-limit, then Van Dyke's matching rule works at $P = Q = 0$ and at $P = Q = 2$. It fails, however, at $P = Q = 1$, i.e. when retaining the terms ord(1) and ord($\epsilon\ln\frac{1}{\epsilon}$).

$$\begin{aligned} H_1E_1f &= H_1\left\{\left(1-\frac{1}{r}\right)+\epsilon\ln\tfrac{1}{\epsilon}\left(1-\frac{1}{r}\right)\right\} \\ &= H_1\left\{\left(1-\frac{\epsilon}{\rho}\right)+\epsilon\ln\tfrac{1}{\epsilon}\left(1-\frac{\epsilon}{\rho}\right)\right\} \\ &= 1+\epsilon\ln\tfrac{1}{\epsilon} \\ E_1H_1f &= E_1\left[1+\epsilon\ln\tfrac{1}{\epsilon}0\right] \\ &= 1 \end{aligned}$$

The trouble is that the $\epsilon\ln r$ term changes its order. Consider a $\ln r$ term on its own. Let

$$\varphi(r,\epsilon) \equiv 0+\frac{\ln r}{\ln\frac{1}{\epsilon}} \equiv 1+\frac{\ln\rho}{\ln\frac{1}{\epsilon}}$$

Then Van Dyke's matching rule with the asymptotic sequence 1, $[\ln\frac{1}{\epsilon}]^{-1}$ will work for this function for $(P,Q) = (0,1)$, $(1,0)$ and $(1,1)$, and it fails for $(0,0)$. A more general term $(\ln r)^n$ will lead to *some* failures near to the diagonal where $|P-Q| < n$. A term like $1/\ln r$ or $\ln\ln r$ leads to more serious trouble.

When applying Van Dyke's rule, good advice is to match only at a break where the power of ϵ changes, if that is possible. Thus in our problem with the asymptotic sequence

$$1;\quad \epsilon\ln\tfrac{1}{\epsilon},\ \epsilon;\quad \epsilon^2[\ln\tfrac{1}{\epsilon}]^2,\ \epsilon^2\ln\tfrac{1}{\epsilon},\ \epsilon^2;\quad \ldots$$

it is wisest to match with P and Q corresponding to the semicolons. Moreover, because $\ln\frac{1}{\epsilon}$ is rarely large for typical small values of ϵ, it is always necessary to calculate all the terms which only differ by a factor of $\ln\frac{1}{\epsilon}$.

5.2.7 Composite approximation

Because Van Dyke's rule fails for $P = Q = 1$, there is no composite $C_{1,1}f$. But the other composites exist.

$$\begin{aligned} C_{0,0}f &= 1-\frac{1}{r} \\ C_{2,2}f &= 1-\epsilon\left[\int_{\epsilon r}^{\infty}\frac{e^{-\tau}}{\tau^2}\,d\tau+\frac{1}{r}\left(\ln\tfrac{1}{\epsilon}+1-\gamma+\ln r\right)\right] \end{aligned}$$

Exercise 5.13. The function $f(r,\epsilon)$ satisfies the equation

$$f_{rr}+\tfrac{2}{r}f_r+\tfrac{1}{2}\epsilon^2(1-f^2) = 0 \qquad \text{in } r>1$$

and is subject to the boundary conditions

$$f=0 \quad\text{at}\quad r=1 \qquad\text{and}\qquad f\to 1 \quad\text{as}\quad r\to\infty$$

Using the asymptotic sequence 1, ϵ, $\epsilon^2\ln\frac{1}{\epsilon}$, ϵ^2, obtain asymptotic expansions for f at fixed r as $\epsilon\searrow 0$ and at fixed $\rho=\epsilon r$ as $\epsilon\searrow 0$. Match the expansions using the intermediate variable $\eta=\epsilon^\alpha r$ with $0<\alpha<1$.

You may quote that the solution to the equation

$$y_{xx}+\tfrac{2}{x}y_x-y = \frac{e^{-2x}}{x^2}$$

subject to the condition $y\to 0$ as $x\to\infty$ is

$$y = A\frac{e^{-x}}{x}+\frac{1}{2x}\int_x^{\infty}\frac{e^{-x-t}-e^{x-3t}}{t}\,dt$$

with A a constant. Further as $x\to 0$

$$y \sim \frac{2A+\ln 3}{2x}+\ln x-A+\gamma+\tfrac{1}{2}\ln 3-1$$

Exercise 5.14. The function $f(r,\epsilon)$ satisfies the equation

$$f_{rr}+\tfrac{3}{2r}f_r+\epsilon f f_r = 0 \qquad \text{in } r>1$$

and is subject to the boundary conditions

$$f=0 \quad\text{at}\quad r=1 \qquad\text{and}\qquad f\to 1 \quad\text{as}\quad r\to\infty$$

and with $\epsilon>0$. Obtain an asymptotic expansion for f at fixed r as $\epsilon\to 0$ in the asymptotic sequence 1, $\epsilon^{1/2}$, $\epsilon\ln\frac{1}{\epsilon}$, ϵ; and an asymptotic expansion for f at fixed $\rho=\epsilon r$ as $\epsilon\to 0$ in the sequence 1, $\epsilon^{1/2}$, ϵ. Match these expansions.

The general solution $y(x)$ with $y \to 0$ as $x \to \infty$ of

$$\left(x^{3/2}e^x y'\right)' \quad = \quad E_{3/2}(x) \quad = \quad \int_x^\infty t^{-3/2}e^{-t}\,dt$$

is $y(x) = BE_{3/2} + F(x)$ with B an arbitrary constant, and that as $x \to 0$,

$$y \;=\; B\left(2x^{-1/2} - 2\sqrt{\pi} + 2x^{1/2} + O(x^{3/2})\right) + \left(4\ln x + C + O(x^{1/2})\right)$$

where C is a numerical constant.

5.2.8 A worse problem

This second model problem is like heat conduction outside a cylinder with a small nonlinear heat source. With ϵ small and positive, let $f(r,\epsilon)$ be governed by

$$f_{rr} + \tfrac{1}{r}f_r + \epsilon f f_r \;=\; 0 \qquad \text{in } r > 1$$
$$\text{with } f = 0 \text{ at } r = 1$$
$$\text{and } f \to 1 \text{ as } r \to \infty$$

First we try a regular perturbation expansion for r fixed as $\epsilon \searrow 0$, i.e. we pose formally

$$f(r,\epsilon) \;\sim\; f_0(r) + \epsilon f_1(r)$$

At ord(ϵ^0) we find that

$$f_0(r) \;=\; A_0 \ln r$$

satisfying the boundary condition at $r = 1$. But the condition at infinity cannot be satisfied with any choice of the free constant A_0. If we were to continue to ord(ϵ^1), we would have a worse problem satisfying the condition at infinity with

$$f_1(r) \;=\; -A_0^2\,(r\ln r - 2r + 2) + A_1 \ln r$$

At this stage one might wonder whether the problem for f is well-posed and a solution exists. The linearised problem is certainly ill-posed. It happens that the nonlinear problem does have a solution; see figure 5.4.

First we note that the rescaling $\rho = \epsilon r$ is applicable to the new equation because it only differs from the previous equation in a numerical factor. The troublesome condition at infinity is then in the region ρ fixed as $\epsilon \searrow 0$. Now as f must be something like the above f_0 in

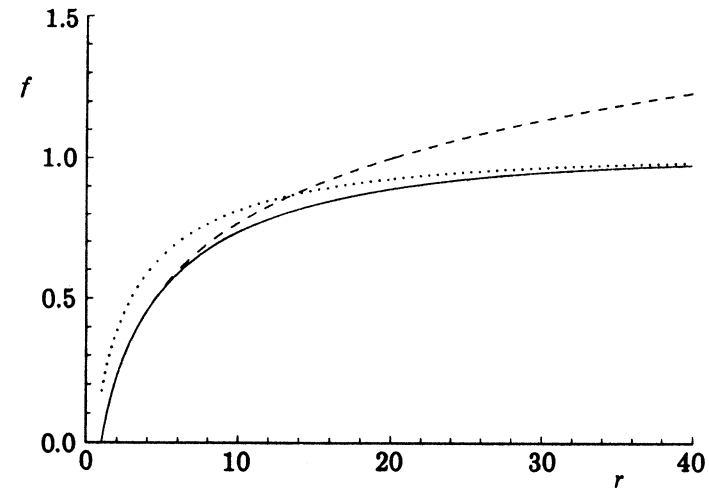

Fig. 5.4 The solution of the problem of §5.2.8 with $\epsilon = 0.05$. The continuous curve gives the exact solution obtained numerically. The dashed curve is the leading order approximation of the r-fixed region, while the dotted curve gives the approximation for the ρ-fixed region of the leading order unity plus one correction term.

the r-region and it must also be ord(1) in the ρ-region, we see that A_0 should be ord($[\ln\frac{1}{\epsilon}]^{-1}$). This suggests an asymptotic sequence 1, $[\ln\frac{1}{\epsilon}]^{-1}$, $[\ln\frac{1}{\epsilon}]^{-2}, \ldots$ with no leading order term in the r-region.

Approximation for r fixed as $\epsilon \searrow 0$

We now start again with the new, less obvious asymptotic sequence. Thus we pose formally a Poincaré expansion

$$f(r,\epsilon) \quad \sim \quad \frac{1}{\ln\frac{1}{\epsilon}}f_1(r) \;+\; \frac{1}{[\ln\frac{1}{\epsilon}]^2}f_2(r)$$

Then each $f_i(r)$ satisfies the same linearised equation, with solution satisfying the boundary condition at $r = 1$

$$f_i(r) \;=\; A_i \ln r$$

Approximation for ρ fixed as $\epsilon \searrow 0$

In terms of the stretched variable $\rho = \epsilon r$ the governing equation becomes

$$f_{\rho\rho} + \tfrac{1}{\rho}f_\rho + f f_\rho \quad = \quad 0$$

Posing formally a Poincaré expansion

$$f(r,\epsilon) \quad \sim \quad 1 + \frac{1}{\ln\frac{1}{\epsilon}} g_1(\rho) + \frac{1}{[\ln\frac{1}{\epsilon}]^2} g_2(\rho)$$

we find at ord($[\ln\frac{1}{\epsilon}]^{-1}$) that g_1 is governed by

$$g_1'' + \left(\tfrac{1}{\rho}+1\right) g_1' \quad = \quad 0$$

with a solution satisfying the condition at infinity $g_1 \to 0$ as $\rho \to \infty$,

$$g_1 \quad = \quad B_1 \int_\rho^\infty \frac{e^{-\tau}}{\tau}\,d\tau \quad = \quad B_1 E_1(\rho)$$

in which B_1 is a constant and E_1 is the exponential integral. At ord($[\ln\frac{1}{\epsilon}]^{-2}$) we find that g_2 is governed by

$$(\rho e^\rho g_2')' = B_1^2 \int_\rho^\infty \frac{e^{-\tau}}{\tau}\,d\tau$$

with solution satisfying the condition $g_2 \to 0$ as $\rho \to \infty$,

$$g_2 \;=\; B_2 E_1(\rho) + B_1^2\left(2E_1(2\rho) - e^{-\rho}E_1(\rho)\right)$$

In preparation for matching, we need the behaviour of the g_i as $\rho \to 0$. Now

$$\begin{aligned} E_1(\rho) \quad &\sim \quad -\ln\rho - \gamma + \rho \\ 2E_1(2\rho) - e^{-\rho}E_1(\rho) \quad &\sim \quad -\ln\rho - \gamma - \ln 4 - \rho\ln\rho + (3-\gamma)\rho \end{aligned}$$

Matching by intermediate variable

With $\eta = \epsilon^\alpha r = \rho/\epsilon^{1-\alpha}$ with $0 < \alpha < 1$, our r-approximation becomes

$$0 + \frac{1}{\ln\frac{1}{\epsilon}} A_1\left(\alpha\ln\tfrac{1}{\epsilon} + \ln\eta\right) + \frac{1}{[\ln\frac{1}{\epsilon}]^2} A_2\left(\alpha\ln\tfrac{1}{\epsilon} + \ln\eta\right) + \cdots$$

while our ρ-approximation becomes

$$\begin{aligned} 1 &+ \frac{1}{\ln\frac{1}{\epsilon}} B_1\left[-(\alpha-1)\ln\tfrac{1}{\epsilon} - \ln\eta - \gamma + \cdots\right] \\ &+ \frac{1}{[\ln\frac{1}{\epsilon}]^2} B_2\left[-(\alpha-1)\ln\tfrac{1}{\epsilon} - \ln\eta - \gamma + \cdots\right] \\ &+ \frac{1}{[\ln\frac{1}{\epsilon}]^2} B_1^2\left[-(\alpha-1)\ln\tfrac{1}{\epsilon} - \ln\eta - \gamma - \ln 4 + \cdots\right] + \cdots \end{aligned}$$

Comparing terms of sequential order

at $\ln\frac{1}{\epsilon}$: $\alpha A_1 = 1 - B_1(\alpha - 1)$

This is true for all α, i.e. for all intermediate limits which means the entire overlap region, if

$$B_1 = -1 \qquad \text{and} \qquad A_1 = 1$$

At $[\ln\frac{1}{\epsilon}]^{-1}$: $A_1\ln\eta + \alpha A_2 = -B_1\ln\eta - B_1\gamma - (\alpha-1)B_2 - (\alpha-1)B_1^2$

Substituting the known A_1 and B_1, and again requiring the matching to work for all intermediate limits gives

$$A_2 = \gamma \qquad \text{and} \qquad B_2 = -1-\gamma$$

Thus all the unknowns have been determined without proceeding to ord($[\ln\frac{1}{\epsilon}]^{-2}$).

Matching by Van Dyke's rule

This fails when $P = Q$, as explained in §5.2.5. Thus if we take E for the r-limit and H for the ρ-limit, we find

$$\begin{aligned} &E_0H_0f = 1 && \text{and } H_0E_0f = 0, && \text{which is impossible} \\ &E_0H_1f = 1 + B_1 && \text{and } H_1E_0f = 0, && \text{i.e. } B_1 = -1 \text{ correctly} \\ &E_1H_0f = 1 && \text{and } H_0E_1f = A_1, && \text{i.e. } A_1 = 1 \text{ correctly} \\ &E_1H_1f = 1 + B_1 + [\ln\tfrac{1}{\epsilon}]^{-1}B_1(-\ln r - \gamma) \\ \text{and}\quad &H_1E_1f = A_1 + [\ln\tfrac{1}{\epsilon}]^{-1}A_1\ln\rho = A_1 - 1 + [\ln\tfrac{1}{\epsilon}]^{-1}\ln r \end{aligned}$$

In this last case if we put $A_1 = 1$ and $B_1 = -1$ then the ord(1) terms match and the $\ln r$ dependence of the ord($[\ln\frac{1}{\epsilon}]^{-1}$) matches, but the constant term at this order does not match.

Exercise 5.15. Check that Van Dyke's rule works for $(P, Q) = (0, 2)$, $(1, 2)$, $(2, 1)$ and $(2, 0)$, but fails for $(2, 2)$.

5.2.9 A terrible problem

This third model equation is unusually difficult. The function is something like $\ln(1 + \ln r/\ln\frac{1}{\epsilon})$. For this function Van Dyke's rule fails for all values of P and Q. The intermediate variable method of matching also struggles, because the leading order part of an infinite number of terms must be calculated before the matching can be made successfully. It is unusual to find such a difficult problem in practice.

With ϵ small and positive, let $f(r,\epsilon)$ be governed by

$$f_{rr} + \tfrac{1}{r}f_r + f_r^2 + \epsilon f f_r = 0 \quad \text{in } r > 1$$
$$\text{with } f = 0 \text{ at } r = 1$$
$$\text{and } f \to 1 \text{ as } r \to \infty$$

The new extra nonlinear term appears to be an ord(1) disruption to the equation studied in §5.2.8. In the r-region, however, f was small, ord($[\ln\frac{1}{\epsilon}]^{-1}$), and so this quadratic term would be smaller than the first two linear terms. And in the ρ-region f was very nearly 1 with a deviation, and hence gradient, of order $[\ln\frac{1}{\epsilon}]^{-1}$. Thus we can anticipate that f has the same asymptotic scalings in this new problem despite the new term.

Approximation for r fixed as $\epsilon \searrow 0$

We start by posing a Poincaré expansion in inverse powers of $\ln\frac{1}{\epsilon}$ which starts with the first power

$$f(r,\epsilon) \sim \frac{1}{\ln\frac{1}{\epsilon}}f_1(r) + \frac{1}{[\ln\frac{1}{\epsilon}]^2}f_2(r) + \frac{1}{[\ln\frac{1}{\epsilon}]^3}f_3(r)$$

Substituting into the governing equation and the boundary condition at $r=1$, and comparing coefficients of $[\ln\frac{1}{\epsilon}]^{-n}$, we find

at $[\ln\frac{1}{\epsilon}]^{-1}$: $f_1'' + \frac{1}{r}f_1' = 0$ with $f_1 = 0$ at $r = 1$

with solution $f_1 = A_1 \ln r$

at $[\ln\frac{1}{\epsilon}]^{-2}$: $f_2'' + \frac{1}{r}f_2' = -f_1'^2 = \frac{1}{r}(rf_2')' = -A_1^2\frac{1}{r^2}$

with solution $f_2 = A_2 \ln r - \frac{1}{2}A_1^2 \ln^2 r$

at $[\ln\frac{1}{\epsilon}]^{-3}$: $f_3'' + \frac{1}{r}f_3' = -2f_1'f_2' = -A_1\frac{2}{r}\left(-A_1^2\frac{\ln r}{r} + A_2\frac{1}{r}\right)$

with solution $f_3 = A_3 \ln r + \frac{1}{3}A_1^3 \ln^3 r - A_1A_2 \ln^2 r$.

From the structure of the above problems we can see that the general term f_n will have leading order behaviour as $r \to \infty$

$$f_n \sim (-)^n\left(-\tfrac{1}{n}A_1^n \ln^n r + A_1^{n-2}A_2 \ln^{n-1} r\right)$$

This can be checked with an induction argument. Note that these leading order parts of the f_n can be summed to

$$\ln\left[1 + \left(\frac{A_1}{\ln\frac{1}{\epsilon}} + \frac{A_2}{[\ln\frac{1}{\epsilon}]^2} + \cdots\right)\ln r\right]$$

which satisfies

$$f_{rr} + \tfrac{1}{r}f_r + f_r^2 = 0$$

Approximation for ρ fixed as $\epsilon \searrow 0$

In terms of the ρ variable the governing equation becomes

$$f_{\rho\rho} + \tfrac{1}{\rho}f_\rho + f_\rho^2 + ff_\rho = 0$$

We formally pose a Poincaré expansion

$$f(r,\epsilon) \sim 1 + \frac{1}{\ln\frac{1}{\epsilon}}g_1(\rho) + \frac{1}{[\ln\frac{1}{\epsilon}]^2}g_2(\rho) + \frac{1}{[\ln\frac{1}{\epsilon}]^3}g_3(\rho)$$

Substituting into the above stretched form of the governing equation, and comparing coefficients of $[\ln\frac{1}{\epsilon}]^{-n}$, we find

at $[\ln\frac{1}{\epsilon}]^{-1}$: $g_1'' + \frac{1}{\rho}g_1' + g_1' = \frac{1}{\rho}e^{-\rho}(\rho e^\rho g_1')' = 0$

Integrating and imposing the condition that $g_1 \to 0$ as $\rho \to \infty$ yields

$$g_1 = B_1\int_\rho^\infty \frac{e^{-\tau}}{\tau}\,d\tau = B_1E_1(\rho)$$

At $[\ln\frac{1}{\epsilon}]^{-2}$: $g_2'' + \frac{1}{\rho}g_2' + g_2' = -g_1'^2 - g_1g_1'$, i.e.

$$(\rho e^\rho g_2')' = -B_1^2\left(\frac{e^{-\rho}}{\rho} - \int_\rho^\infty \frac{e^{-\tau}}{\tau}\,d\tau\right)$$

with solution satisfying the condition at infinity

$$g_2 = B_2E_1(\rho) + B_1^2\left(2E_1(2\rho) - \tfrac{1}{2}E_1^2(\rho) - e^{-\rho}E_1(\rho)\right)$$

In preparation for matching, we need the behaviour as $\rho \to 0$,

$$g_1 \sim B_1(-\ln\rho - \gamma)$$
$$g_2 \sim B_2(-\ln\rho - \gamma) + B_1^2\left(-\tfrac{1}{2}\ln^2\rho - (\gamma+1)\ln\rho - \tfrac{1}{2}\gamma^2 - \gamma - \ln 4\right)$$

We note that the leading order behaviour in g_2 comes from

$$g_2'' + \tfrac{1}{\rho}g_2' \sim -g_1'^2 \sim -\frac{B_1^2}{\rho^2}$$

Hence the leading order behaviour in the following g_3 will come from

$$g_3'' + \tfrac{1}{\rho}g_3' \sim -2g_1'g_2' \sim -2B_1^3\frac{\ln\rho}{\rho^2} - \left(2B_1^3(\gamma+1) + 2B_1B_2\right)\frac{1}{\rho^2}$$

and so

$$g_3 \sim -\tfrac{1}{3}B_1^3\ln^3\rho - \left(B_1^3(\gamma+1) + B_1B_2\right)\ln^2\rho$$

Again it can be shown that the leading order behaviour as $\rho \to 0$ of the general term $g_n(\rho)$ is

$$g_n(\rho) \sim -\tfrac{1}{n}B_1^n \ln^n \rho - \left(B_1^n(\gamma+1) + B_1^{n-2}B_2\right)\ln^{n-1}\rho$$

Matching by intermediate variable

With $\eta = \epsilon^\alpha r = \rho/\epsilon^{1-\alpha}$ fixed as $\epsilon \searrow 0$, we have

$$\begin{aligned}
r\text{-approximation} \sim\ & \frac{1}{\ln\frac{1}{\epsilon}}A_1\left[\alpha\ln\tfrac{1}{\epsilon}+\ln\eta\right] \\
+\ & \frac{1}{[\ln\frac{1}{\epsilon}]^2}\left\{-\tfrac{1}{2}A_1^2\left[\alpha\ln\tfrac{1}{\epsilon}+\ln\eta\right]^2 + A_2\left[\alpha\ln\tfrac{1}{\epsilon}+\ln\eta\right]\right\} \\
+\ & \frac{1}{[\ln\frac{1}{\epsilon}]^3}\left\{\tfrac{1}{3}A_1^3\left[\alpha\ln\tfrac{1}{\epsilon}+\ln\eta\right]^3 - A_1A_2\left[\alpha\ln\tfrac{1}{\epsilon}+\ln\eta\right]^2\right. \\
& \qquad\qquad \left. + A_3\left[\alpha\ln\tfrac{1}{\epsilon}+\ln\eta\right]\right\} \\
+\ & \frac{1}{[\ln\frac{1}{\epsilon}]^4}\left\{-\tfrac{1}{4}A_1^4\left[\alpha\ln\tfrac{1}{\epsilon}+\ln\eta\right]^4 + A_1^2A_2\left[\alpha\ln\tfrac{1}{\epsilon}+\ln\eta\right]^3+\cdots\right\} \\
+\ & \cdots
\end{aligned}$$

and

$$\begin{aligned}
\rho\text{-approximation} \sim\ & 1 + \frac{1}{\ln\frac{1}{\epsilon}}B_1\left[-(\alpha-1)\ln\tfrac{1}{\epsilon}-\ln\eta-\gamma+\cdots\right] \\
+\ & \frac{1}{[\ln\frac{1}{\epsilon}]^2}\left\{-B_1^2\left(\tfrac{1}{2}\left[(\alpha-1)\ln\tfrac{1}{\epsilon}+\ln\eta\right]^2-(\gamma+1)\left[(\alpha-1)\ln\tfrac{1}{\epsilon}+\ln\eta\right]\right.\right. \\
& \left.\left. + \tfrac{1}{2}\gamma^2+\gamma+\ln 4+\cdots\right) + B_2\left[-(\alpha-1)\ln\tfrac{1}{\epsilon}-\ln\eta-\gamma+\cdots\right]\right\} \\
+\ & \frac{1}{[\ln\frac{1}{\epsilon}]^3}\left\{-B_1^3\tfrac{1}{3}\left[(\alpha-1)\ln\tfrac{1}{\epsilon}+\ln\eta\right]^3 - B_1^3(\gamma+1)\times\right. \\
& \left. \left[(\alpha-1)\ln\tfrac{1}{\epsilon}+\ln\eta\right]^2 - B_1B_2\left[(\alpha-1)\ln\tfrac{1}{\epsilon}+\ln\eta\right]^2+\cdots\right\} \\
+\ & \cdots
\end{aligned}$$

Comparing coefficients of $[\ln\frac{1}{\epsilon}]^0$, we find

$$\begin{aligned}
A_1\alpha - \tfrac{1}{2}A_1^2\alpha^2 + \tfrac{1}{3}A_1^3\alpha^3 - \tfrac{1}{4}A_1^4\alpha^4+\cdots\ &= \\
1 - B_1(\alpha-1) - \tfrac{1}{2}B_1^2(\alpha-1)^2 - \tfrac{1}{3}B_1^3(\alpha-1)^3+\cdots
\end{aligned}$$

The two sides of the equation are well known infinite series which converge for $0 < \alpha < 1$, yielding

$$\ln(1+A_1\alpha) = 1+\ln[1-B_1(\alpha-1)] = \ln(e[1-B_1(\alpha-1)])$$

Requiring the matching to work throughout the overlap region, i.e. for all intermediate limits, i.e. all values of α, gives

$$A_1 = e-1 \qquad\text{and}\qquad B_1 = -(e-1)/e$$

Comparing coefficients of $[\ln\frac{1}{\epsilon}]^{-1}$ in the r- and ρ-approximations in the overlap region, we find

$$\begin{aligned}
& \ln\eta\left(A_1 - \tfrac{1}{2}A_1^2 2\alpha + \tfrac{1}{3}A_1^3 3\alpha^2 - \tfrac{1}{4}A_1^4 4\alpha^3+\cdots\right) \\
& \quad + \left(A_2\alpha - A_1A_2\alpha^2 + A_1^2A_2\alpha^3+\cdots\right) \\
= \ & \ln\eta\left(-B_1 - \tfrac{1}{2}B_1^2 2(\alpha-1) - \tfrac{1}{3}B_1^3 3(\alpha-1)^2+\cdots\right) \\
& + \left(-B_1\gamma - B_1^2(\gamma+1)(\alpha-1) - B_1^3(\gamma+1)(\alpha-1)^2+\cdots\right) \\
& + \left(-B_2(\alpha-1) - B_1B_2(\alpha-1)^2+\cdots\right)
\end{aligned}$$

Again summing, this is

$$\begin{aligned}
\frac{A_1}{1+A_1\alpha}\ln\eta + \frac{A_2\alpha}{1+A_1\alpha} \ &= \ \frac{-B_1}{1-B_1(\alpha-1)}\ln\eta \\
& - \frac{B_1\gamma}{1-B_1(\alpha-1)} + \frac{B_1^2(\alpha-1)}{1-B_1(\alpha-1)} - \frac{B_2(\alpha-1)}{1-B_1(\alpha-1)}
\end{aligned}$$

The $\ln\eta$ terms balance with the previously determined A_1 and B_1. The constant terms give

$$\frac{A_2}{A_1}\alpha = \gamma - B_1(\alpha-1) + \frac{B_2}{B_1}(\alpha-1)$$

Requiring this to be true for all α, we find

$$A_2 = \gamma(e-1) \qquad\text{and}\qquad B_2 = (e-1)(e-1-\gamma e)/e^2$$

Note that in the matching an infinite number of terms jumped their order. It was therefore necessary to have obtained the leading order behaviour of the general terms f_n and g_n.

5.3 Slow viscous flow

The model problems of §5.2 contain the mathematically difficulty in finding the small inertial corrections to the viscous flow past a sphere and past a cylinder. These two problems are known as the Stokes–Whitehead paradoxes, and their resolution was influential in the development of the method of matched asymptotic expansions.

5.3.1 Past a sphere

Axisymmetric flow past a sphere can be described by a Stokes stream-function ψ which satisfies the Navier–Stokes equation in the form

$$\frac{\epsilon}{r^2 \sin\theta}\left(\frac{\partial\psi}{\partial\theta}\frac{\partial}{\partial r} - \frac{\partial\psi}{\partial r}\frac{\partial}{\partial\theta} + 2\cot\theta\frac{\partial\psi}{\partial r} - \frac{2}{r}\frac{\partial\psi}{\partial\theta}\right) D^2\psi = D^2 D^2 \psi \quad \text{in } r \geq 1$$

with $\psi = \dfrac{\partial\psi}{\partial r} = 0$ on $r = 1$

and $\psi \to \frac{1}{2}r^2 \sin^2\theta$ as $r \to \infty$

in which $D^2 = \dfrac{\partial^2}{\partial r^2} + \dfrac{\sin\theta}{r^2}\dfrac{\partial}{\partial\theta}\left(\dfrac{1}{\sin\theta}\dfrac{\partial}{\partial\theta}\right)$

Approximation for r fixed as $\epsilon \searrow 0$

We formally pose an expansion

$$\psi(r,\theta;\epsilon) \sim \psi_0(r,\theta) + \epsilon\psi_1(r,\theta)$$

The lowest term is governed by the equation

$$D^2 D^2 \psi_0 = 0$$

and is forced by the condition at infinity. Looking for a solution proportional to $\sin^2\theta$, we find possible radial dependencies r^4, r^2, r and r^{-1}. Satisfying the boundary conditions on $r = 1$,

$$\psi_0 = \tfrac{1}{4}\left(2r^2 - 3r + \frac{1}{r}\right)\sin^2\theta$$

Substituting this into the left hand side of the governing equation produces

$$-\epsilon\tfrac{9}{4}\left(\frac{2}{r^2} - \frac{3}{r^3} + \frac{1}{r^5}\right)\sin^2\theta\cos\theta$$

to force ψ_1. Again we look for a solution proportional to $\sin^2\theta\cos\theta$. To the particular integral (which is made to satisfy the boundary condition on $r = 1$) we must add a homogeneous solution, which turns out just to be a multiple of ψ_0. (Other homogeneous solutions can be added, but

during the matching they will be found to have zero coefficients.)

$$\psi_1 = -\tfrac{3}{32}\left(2r^2 - 3r + 1 - \frac{1}{r} + \frac{1}{r^2}\right)\sin^2\theta\cos\theta + A_1\left(2r^2 - 3r + \frac{1}{r}\right)\sin^2\theta$$

But no choice of the free constant A_1 enables the condition at infinity to be satisfied at all θ.

Approximation for $\rho = \epsilon r$ fixed as $\epsilon \searrow 0$

Now in the far field at large r the above r-approximation has

$$\psi = \tfrac{1}{2}r^2\sin^2\theta + \operatorname{ord}(r, \epsilon r^2)$$

corresponding to a uniform flow plus the disturbance from a point force together with some inertial corrections. This suggests an expansion in the ρ-region

$$\psi(r,\theta;\epsilon) \sim \frac{1}{\epsilon^2}\tfrac{1}{2}\rho^2\sin^2\theta + \frac{1}{\epsilon}\Psi_1(\rho,\theta)$$

The equation governing Ψ_1 is the Oseen equation in the form

$$\left(D_\rho^2 - \cos\theta\frac{\partial}{\partial\rho} + \frac{\sin\theta}{\rho}\frac{\partial}{\partial\theta}\right)D_\rho^2\Psi_1 = 0$$

where D^2 has been modified with ρ replacing r. The homogeneous solution we need turns out to be that corresponding to a point force

$$\Psi_1 = B_1(1+\cos\theta)\left(1 - e^{-\frac{1}{2}\rho(1-\cos\theta)}\right)$$

which can most easily be obtained in Cartesian co-ordinates by Fourier transforming the linear Oseen equation. This derivation also gives an immediate connection between B_1 and the drag. In Ψ_1 the $(1+\cos\theta)$ factor corresponds to a source flow, while the exponential describes a wake concentrated in a region $\rho\theta^2 = \operatorname{ord}(1)$; the source being needed to remove a mass defect in the wake.

Matching by intermediate variable

With $\eta = \epsilon^\alpha r = \rho/\epsilon^{1-\alpha}$ fixed as $\epsilon \searrow 0$

$$\begin{aligned} r\text{-approximation} &= \tfrac{1}{4}\left(2\epsilon^{-2\alpha}\eta^2 - 3\epsilon^{-\alpha}\eta + \cdots\right)\sin^2\theta \\ &\quad - \epsilon\tfrac{3}{32}\left(2\epsilon^{-2\alpha}\eta^2 + \cdots\right)\sin^2\theta\cos\theta + \epsilon A_1\left(2\epsilon^{-2\alpha}\eta^2 + \cdots\right)\sin^2\theta \\ &\quad + \cdots \end{aligned}$$

and

$$\rho\text{-approximation} = \frac{1}{\epsilon^2}\tfrac{1}{2}\epsilon^{2-2\alpha}\eta^2\sin^2\theta$$
$$+ \frac{1}{\epsilon}B_1(1+\cos\theta)\left(\tfrac{1}{2}\epsilon^{1-\alpha}\eta(1-\cos\theta) - \tfrac{1}{8}\epsilon^{2-2\alpha}\eta^2(1-\cos\theta)^2 + \cdots\right)$$
$$+ \cdots$$

Comparing terms of sequential order we find

at $\epsilon^{-2\alpha}$: $\tfrac{1}{2}\eta^2\sin^2\theta = \tfrac{1}{2}\eta^2\sin^2\theta$ – just uniform flow

at $\epsilon^{-\alpha}$: $-\tfrac{3}{4}\eta\sin^2\theta = \tfrac{1}{2}B_1\eta(1-\cos^2\theta)$, i.e. $B_1 = -\tfrac{3}{2}$

at $\epsilon^{1-2\alpha}$: $-\tfrac{3}{16}\eta^2\sin^2\theta\cos\theta + 2A_1\eta^2\sin^2\theta$

$= -\tfrac{1}{8}B_1\eta^2\sin^2\theta(1-\cos\theta)$, i.e. $A_1 = \tfrac{3}{32}$

The fact that the value of A_1 is $\tfrac{3}{8}$ times the $\tfrac{1}{4}$ in ψ_0 leads to an enhancement in the drag on the sphere by a factor $(1+\tfrac{3}{8}\epsilon)$. In effect at this order the r-region sees a uniform flow $(1+\tfrac{3}{8}\epsilon)$. The next order terms are ord $\left(\epsilon^2\ln\tfrac{1}{\epsilon}\right)$ and ord(ϵ^2).

5.3.2 Past a cylinder

For this two-dimensional flow we use a streamfunction ψ which satisfies

$$\epsilon\left(\frac{1}{r}\frac{\partial\psi}{\partial\theta}\frac{\partial}{\partial r} - \frac{1}{r}\frac{\partial\psi}{\partial r}\frac{\partial}{\partial\theta}\right)\nabla^2\psi = \nabla^2\nabla^2\psi \qquad \text{in } r\geq 1$$

with $\psi = \dfrac{\partial\psi}{\partial r} = 0$ on $r=1$

and $\psi \to r\sin\theta$ as $r\to\infty$

Approximation for r fixed as $\epsilon \searrow 0$

Now the lowest order term must be governed by $\nabla^2\nabla^2\psi_0 = 0$. Looking for a solution proportional to the forcing $\sin\theta$, we find possible radial dependencies r^3, $r\ln r$, r and r^{-1}. It is not possible to satisfy the conditions on $r=1$ and $r\to\infty$: the least unpleasant solution at infinity satisfying the boundary conditions at $r=1$ is

$$f_*(r,\theta) = \left(r\ln r - \tfrac{1}{2}r + \frac{1}{2r}\right)\sin\theta$$

As in §5.2.8, we expect this solution not to apply when $\rho = \epsilon r = \text{ord}(1)$, and so we need to multiply the above f_* by $[\ln\tfrac{1}{\epsilon}]^{-1}$ to reduce the magnitude correctly. This leads to an expansion in powers of $[\ln\tfrac{1}{\epsilon}]^{-1}$. All the terms in the r-region must then be proportional to the above f_*.

$$\psi(r,\theta;\epsilon) \sim \frac{1}{\ln\frac{1}{\epsilon}}A_1 f_*(r,\theta) + \frac{1}{[\ln\frac{1}{\epsilon}]^2}A_2 f_*(r,\theta)$$

Approximation for $\rho = \epsilon r$ fixed as $\epsilon \searrow 0$

In this region the flow is the uniform flow plus a small ord$([\ln\tfrac{1}{\epsilon}]^{-1})$ correction,

$$\psi(r,\theta;\epsilon) \sim \frac{1}{\epsilon}\rho\sin\theta + \frac{1}{\epsilon\ln\frac{1}{\epsilon}}\Psi_1(\rho,\theta)$$

The equation governing Ψ_1 is Oseen's

$$\left(\nabla^2_\rho - \cos\theta\frac{\partial}{\partial\rho} + \frac{\sin\theta}{\rho}\frac{\partial}{\partial\theta}\right)\nabla^2_\rho\Psi_1 = 0$$

Again we need the homogeneous solution corresponding to a point force which can be obtained by Fourier transforming (but see also the hint for the exercise at the end of the section). The inversion for Ψ_1 cannot be expressed in closed form, although the vorticity $\nabla^2\Psi_1$ and the velocity $\nabla\Psi_1$ can be. From the Fourier transform, one can extract the behaviour of Ψ_1 as $\rho\to 0$

$$\Psi_1 \sim B_1\rho(\ln\rho - \ln 4 + \gamma - 1)\sin\theta$$

Matching by intermediate variable

With $\eta = \epsilon^\alpha r = \rho/\epsilon^{1-\alpha}$ fixed as $\epsilon\searrow 0$

$$r\text{-approximation} = 0 + \frac{1}{\ln\frac{1}{\epsilon}}A_1\epsilon^{-\alpha}\eta\sin\theta\left(\alpha\ln\tfrac{1}{\epsilon} + \ln\eta - \tfrac{1}{2} + \cdots\right)$$
$$+ \frac{1}{[\ln\frac{1}{\epsilon}]^2}A_2\epsilon^{-\alpha}\eta\sin\theta\left(\alpha\ln\tfrac{1}{\epsilon} + \ln\eta - \tfrac{1}{2} + \cdots\right)$$
$$+ \cdots$$

and

$$\rho\text{-approximation} = \frac{1}{\epsilon}\epsilon^{1-\alpha}\eta\sin\theta$$
$$+ \frac{1}{\epsilon\ln\frac{1}{\epsilon}}B_1\left(\epsilon^{1-\alpha}\eta\sin\theta\left[(\alpha-1)\ln\tfrac{1}{\epsilon} + \ln\eta - \ln 4 + \gamma - 1\right] + \cdots\right)$$
$$+ \cdots$$

Comparing terms of sequential orders,

at $\epsilon^{-\alpha}$: $\alpha A_1\eta\sin\theta = \eta\sin\theta + B_1(\alpha-1)\eta\sin\theta$

which is true for all intermediate limits if

$$A_1 = B_1 = 1$$

At $\epsilon^{-\alpha}[\ln\frac{1}{\epsilon}]^{-1}$: $A_1\left(\ln\eta - \frac{1}{2}\right)\eta\sin\theta + \alpha A_2\eta\sin\theta$

$$= B_1(\ln\eta - \ln 4 + \gamma - 1)\eta\sin\theta + B_2(\alpha - 1)\eta\sin\theta$$

in which B_2 is the coefficient of a similar homogeneous solution at $\mathrm{ord}(\epsilon^{-1}[\ln\frac{1}{\epsilon}]^{-2})$ in the ρ-region, which happens to dominate the particular integral. Again matching for arbitrary intermediate limit, we find

$$A_2 = B_2 = -\ln 4 + \gamma - \tfrac{1}{2}$$

Note that because $\ln\frac{1}{\epsilon}$ is rarely large numerically, some people combine $[\ln\frac{1}{\epsilon}]^{-1}A_1$ and $[\ln\frac{1}{\epsilon}]^{-2}A_2$ when they occur in the drag to form $1/[\ln(4/\epsilon) - \gamma + \frac{1}{2}]$. At higher orders there are alternative methods of improving the convergence of a series – see chapter 8.

Exercise 5.16. Consider the heat transfer from a cylinder in a weak potential flow. Thus solve for $T(\mathbf{x}, \epsilon)$ which satisfies

$$\epsilon\mathbf{u}\cdot\nabla T = \nabla^2 T \qquad \text{in } r \geq 1$$

with $T = 1$ on $r = 1$

and $T \to 0$ as $r \to \infty$

$$\text{where } \mathbf{u} = \mathbf{U}\left(1 + \frac{1}{r^2}\right) - \mathbf{x}\frac{2\,(\mathbf{U}\cdot\mathbf{x})}{r^4}$$

Then calculate

$$\int_{r=1} \frac{\partial T}{\partial n}\,dA$$

Hint: The substitution $\varphi = \psi e^{x/2}$ turns the Oseen equation governing φ

$$\left(\frac{\partial}{\partial x} - \nabla^2\right)\varphi$$

into the simpler equation

$$\left(\tfrac{1}{4} - \nabla^2\right)\psi$$

Exercise 5.17. Now try the case of weak potential flow past a sphere, with

$$\mathbf{u} = \mathbf{U}\left(1 + \frac{1}{2r^3}\right) - \mathbf{x}\frac{3\,(\mathbf{U}\cdot\mathbf{x})}{2r^5}$$

5.4 Slender body theory

The method of matched asymptotic expansions is used in problems which have two (or more) naturally occurring length scales; asymptotic expansions being made for each of the scales, the expansions then being matched in order to determine some constants of integration. In the problems considered so far, the governing equation generated the second length scale; that of the thin boundary layer in §5.1 and that of the far field in §§5.2 and 5.3. It is also possible for the basic geometry to have more than one natural length scale. Long slender bodies considered in this section have the scales of their width and their length. Thus on the smaller scale the bodies appear to be nearly infinitely long, quasi-uniform, finite diameter cylinders, while on the longer scale they appear to have a finite length, but to be vanishingly thin. Other problems whose geometry has more than one natural length scale include the interaction between greatly separated particles (with the scales of their size and their separation – see exercise 5.18 at the end of this section) and waves scattering off small scale inhomogeneities (with the scales of the inhomogeneities and the wavelength).

For simplicity we will only study slender bodies with straight centrelines, although in §5.4.1 we will have non-circular cross-sections. Let ϵ be the (small) slenderness; then in cylindrical polar co-ordinates the surface of the body may be taken as

$$r = \epsilon R(\theta, z) \qquad \text{in } |z| \leq 1$$

5.4.1 Electrical capacitance

We must solve for the potential $\varphi(r, \theta, z; \epsilon)$ which satisfies

$$\nabla^2\varphi = 0 \qquad \text{outside the body}$$

and is subject to boundary conditions

$$\varphi = 1 \text{ on the body and } \varphi \to 0 \text{ at infinity}$$

The capacitance can then be evaluated as

$$-\int \frac{\partial\varphi}{\partial n}\,dA$$

Approximation for $\rho = r/\epsilon$ fixed as $\epsilon \searrow 0$. In this scaling the governing equation becomes

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\varphi}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\varphi}{\partial\theta^2} + \epsilon^2\frac{\partial^2\varphi}{\partial z^2} = 0 \quad \text{in } \rho \geq R(\theta, z)$$

This suggests an expansion in ϵ^2. Here we look at just the leading order term in such an expansion; a correction term will be calculated for the problem in the following section. For the leading order term, the variable z does not occur in the field equation. We therefore have to solve a two-dimensional potential problem, i.e. at this order and in this scaling the cylinder appears to be infinitely long. The shape of the cylinder is specified by $R(\theta, z)$ in which we must view z just as a parameter. Applying the boundary condition on the surface, we have a general solution

$$\varphi_0(r,\theta,z) = 1 + Q(z)\left(\frac{1}{2\pi}\ln\frac{\rho}{R_*(z)} + \sum_{n=1}^{\infty} B_n(z)\rho^{-n}\begin{matrix}\cos\\ \sin\end{matrix}\, n\theta\right)$$

where $R_*(z)$ and the $B_n(z)$ depend on the local shape $R(\theta, z)$ and where $Q(z)$ has to be found from matching. The matching effectively applies the condition at infinity which cannot be applied in the strictly two-dimensional potential problem.

The quantity R_* is known in two-dimensional potential theory as the effective or equivalent radius of the cross-section.

- For a circular cross-section $R(\theta, z) = R(z)$,

$$R_*(z) = R(z) \qquad \text{and} \qquad B_n = 0.$$

- For an elliptical cross-section with semi-diameters $a(z)$ and $b(z)$,

$$R_*(z) = \tfrac{1}{2}(a+b).$$

This can be derived using the conformal map

$$\zeta = \tfrac{1}{2}\rho e^{i\theta} + \tfrac{1}{2}\sqrt{\rho^2 e^{2i\theta} - a^2 + b^2}$$

which takes the ellipse into the circle $|\zeta| = \frac{1}{2}(a+b)$. Hence the complex potential is

$$1 + \frac{1}{2\pi}Q\ln\frac{2\zeta}{a+b}$$

But as $\rho \to \infty$, $\zeta \sim \rho e^{i\theta}\left\{1 - \frac{1}{4}(a^2-b^2)e^{-2i\theta}\rho^{-2}\right\}$. Substituting this into the potential yields $R_* = \frac{1}{2}(a+b)$ and $B_1 = 0$ and $B_2 = \frac{1}{4}(a^2-b^2)$.

- For a cross-section which is a star composed of n points of zero width and length a,

$$R_*(z) = a(z)2^{-2/n}$$

This can also be derived using a conformal map. This time use $\zeta = \rho^n e^{in\theta}$ to map the star into a degenerate ellipse with $b = 0$.

Approximation for r fixed as $\epsilon \searrow 0$. In this scaling the body has a finite length, but nearly no thickness. We thus see the body as a line of distributed charges $Q(z)$, with weaker higher order poles $B_n(z)$. Thus we try as a first approximation in this r-region

$$\varphi(r,\theta,z;\epsilon) \sim -\int_{-1}^{1}\frac{Q(z';\epsilon)\,dz'}{4\pi\sqrt{r^2+(z-z')^2}}$$

In preparation for matching, our r-approximation takes the form

$$\varphi(r,\theta,z;\epsilon) = -\frac{1}{4\pi}Q(z)2\ln\frac{1}{r} + O(Q) \qquad \text{as } r \to 0$$

Matching with an intermediate variable $\eta = r/\epsilon^\alpha = \epsilon^{1-\alpha}\rho$ fixed as $\epsilon \searrow 0$.

$$\begin{aligned} r\text{-approximation} &= \tfrac{1}{2\pi}Q(z)\left(-\alpha\ln\tfrac{1}{\epsilon} + \ln\eta + O(1)\right) + \cdots \\ \rho\text{-approximation} &= 1 + \tfrac{1}{2\pi}Q(z)\Big((1-\alpha)\ln\tfrac{1}{\epsilon} \\ &\qquad + \ln\frac{\eta}{R_*} + \epsilon^{1-\alpha}B_1\eta^{-1}\cos\theta + \cdots\Big) + \cdots \end{aligned}$$

Thus at leading order $Q(z) = -2\pi[\ln\frac{1}{\epsilon}]^{-1}$ with an error of order $[\ln\frac{1}{\epsilon}]^{-2}$. If we expand Q in a series of inverse powers of $\ln\frac{1}{\epsilon}$

$$Q(z;\epsilon) \sim -\frac{1}{\ln\frac{1}{\epsilon}}2\pi + \frac{1}{[\ln\frac{1}{\epsilon}]^2}Q_2(z)$$

then from the earlier solution of the integral equation in §3.5 we have in the r-region

$$\varphi \sim \frac{1}{\ln\frac{1}{\epsilon}}\frac{1}{2}\ln\frac{4(1-z^2)}{r^2} - \frac{1}{[\ln\frac{1}{\epsilon}]^2}\frac{1}{4\pi}Q_2\left(2\ln\frac{1}{r} + O(1)\right) \qquad \text{as } r \to 0$$

So matching again

$$\begin{aligned} r\text{-approximation} &= -\frac{1}{\ln\frac{1}{\epsilon}}\left(-\alpha\ln\tfrac{1}{\epsilon} + \ln\eta - \tfrac{1}{2}\ln 4(1-z^2)\right) \\ &\quad + \frac{1}{[\ln\frac{1}{\epsilon}]^2}\frac{1}{2\pi}Q_2\left(-\alpha\ln\tfrac{1}{\epsilon} + \cdots\right) + \cdots \\ \rho\text{-approximation} &= 1 \end{aligned}$$

$$+\ \frac{1}{2\pi}\left(-\frac{1}{[\ln\frac{1}{\epsilon}]^2}\pi+\frac{1}{[\ln\frac{1}{\epsilon}]^2}Q_2(z)\right)\left((1-\alpha)\ln\tfrac{1}{\epsilon}+\ln\frac{\eta}{R_*}+\cdots\right)+\cdots$$

we find that at leading order $\alpha = 1 - 1 + \alpha$, and at $[\ln\frac{1}{\epsilon}]^{-1}$ matching is successful if

$$Q_2(z) \quad = \quad 2\pi \ln\frac{2\sqrt{1-z^2}}{R_*(z)}$$

The electrical capacitance of the slender body can now be evaluated:

$$-\int\frac{\partial\varphi}{\partial n}\,dA \quad = \quad -\int_{-1}^{1}Q(z)\,dz$$

$$\sim \quad \frac{4\pi}{\ln\frac{1}{\epsilon}} \;-\; \frac{2\pi}{[\ln\frac{1}{\epsilon}]^2}\int_{-1}^{1}\ln\frac{2\sqrt{1-z^2}}{R_*(z)}\,dz$$

This expression shows that the capacitance depends only weakly on the shape, involving just the logarithm of the slenderness at the leading order, and in the correction just an integral of the cross-section.

We note that prior to the matching the asymptotic theory had error terms ord(ϵ^2). During the matching it was necessary to introduce an expansion in powers of $[\ln\frac{1}{\epsilon}]^{-1}$, which is rarely small in practice. Now the matching is equivalent to solving the integral equation

$$1 \quad = \quad -\int_{-1}^{1}\frac{Q(z')\,dz'}{\sqrt{\epsilon^2R_*(z)^2+(z-z')^2}}$$

i.e. in the r-region the body appears to be a circular cylinder of radius R_*. To avoid the expansion in logarithms, this integral equation can be solved numerically (for particular values of ϵ).

For slender bodies with centre lines which are not straight, the form of the solution in the ρ-region will not be changed by the curvature until ord(ϵ^2). In the r-region the body will be represented by charges $Q(z)$ distributed along the curved centre line. In the evaluation of this r-region integral for matching, the leading order term of the $\ln\frac{1}{\epsilon}$ expansion is found to be unaffected by the curvature, but the correction, $[\ln\frac{1}{\epsilon}]^{-1}$ smaller, does depend on the global shape of the centre line. Thus Q_2 would be different.

5.4.2 Axisymmetric potential flow

For simplicity we now restrict attention to a circular cross-section. Irrotational incompressible flow can be described by a velocity potential

$\varphi(r,z;\epsilon)$ which satisfies

$$\nabla^2\varphi \;=\; 0 \qquad \text{outside the body}$$

$$\text{with}\quad \frac{\partial\varphi}{\partial n}=0 \qquad \text{on the surface of the body}$$

The condition at infinity corresponding to a uniform flow in the axial direction is

$$\varphi \to z$$

Before we start, we need to note that the normal (not a unit normal) to the surface $r = \epsilon R(z)$ is $(1, -\epsilon R')$, where the prime denotes differentiation with respect to z. Thus the boundary condition on the surface of the body becomes

$$\frac{\partial\varphi}{\partial r} \;-\; \epsilon R'\frac{\partial\varphi}{\partial z} \;=\; 0$$

Approximation for $\rho = r/\epsilon$ fixed as $\epsilon \searrow 0$. In this scaling the field equation and boundary condition become

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\varphi}{\partial\rho}\right) \;+\; \epsilon^2\frac{\partial^2\varphi}{\partial z^2} \quad = \quad 0 \qquad \text{in } \rho \geq R(z)$$

$$\text{and}\quad \frac{\partial\varphi}{\partial\rho} - \epsilon^2R'\frac{\partial\varphi}{\partial z} \quad = \quad 0 \qquad \text{on } \rho = R(z)$$

These suggest an expansion in ϵ^2, the first term corresponding to an infinite cylinder, while the second term includes some effect of the tapering ($R' \neq 0$). In the leading order term, there is just a constant (which may depend on the parameter z), because the boundary condition rules out a $\ln\rho$. The ϵ^2 correction term is forced by the field equation and the boundary condition. Thus

$$\varphi(r,z;\epsilon) \quad \sim \quad A_0(z) + \epsilon^2\left(-\tfrac{1}{4}A_0''\rho^2 + A_2(z) + B_2(z)\ln\rho\right)$$

where

$$B_2 \;=\; R\left(A_0'R' + \tfrac{1}{2}A_0''R\right)$$

Approximation for r fixed as $\epsilon \searrow 0$. In this scaling the body appears as a line distribution of sources $2\pi\epsilon^2B_2(z)$, i.e.

$$\varphi(r,z) \quad \sim \quad z \;-\; \int_{-1}^{1}\frac{\epsilon^2B_2(z')\,dz'}{2\sqrt{r^2+(z-z')^2}}$$

Preparing for matching, this r-approximation takes the form

$$\varphi \quad \sim \quad z \;-\; \epsilon^2B_2(z)\left(\ln\frac{1}{r}+O(1)\right) \qquad \text{as } r \to 0$$

Matching with an intermediate variable $\eta = r/\epsilon^a = \rho\epsilon^{1-\alpha}$ *fixed as* $\epsilon \searrow 0$.

$$\begin{aligned} r\text{-approximation} &= z + B_2(z)\left(-\alpha \ln\tfrac{1}{\epsilon} + \ln\eta + O(1)\right) + \cdots \\ \rho\text{-approximation} &= A_0(z) + \epsilon^2\Big(\tfrac{1}{4} - A_0''\epsilon^{2\alpha-2}\eta^2 + A_2(z) \\ &\quad + B_2(z)\left(-(\alpha-1)\ln\tfrac{1}{\epsilon} + \ln\eta\right)\Big) + \cdots \end{aligned}$$

Matching at ord(1) we find

$$A_0 = z \qquad \text{and so} \qquad B_2 = RR'$$

The ord($\epsilon^{2\alpha}$) term therefore evaporates. At ord($\epsilon^2 \ln \frac{1}{\epsilon}$) we find A_2 must be slightly larger than expected, with $A_2 = B_2 \ln \frac{1}{\epsilon}$ plus $O(1)$ corrections. However B_2 remains unchanged up to ord($\epsilon^2 \ln \frac{1}{\epsilon}$).

The value $RR' = (\frac{1}{2}R^2)'$ for B_2 can be interpreted as follows. The volume flux of the undisturbed flow across the cross-sectional area of the body is $\pi\epsilon^2R^2$. The internally distributed sources $2\pi\epsilon^2B_2$ ensure that there is no flow into the body. These sources are needed where the cross-sectional area changes.

Exercise 5.18. Another problem where the geometry presents two length scales is that of the electrical capacitance of a pair of thin parallel wires. Let the wires have radii ϵ and be centred at $x = \pm\frac{1}{2}$, i.e. have surfaces

$$(x \pm \tfrac{1}{2})^2 + y^2 = \epsilon^2$$

Then outside both these surfaces one needs to solve $\nabla^2\varphi = 0$ for the potential φ subject to the boundary conditions that φ is a constant, $V(\epsilon)$, independent of position around the boundary on one of the wires, and $-V$ on the other. If the charge on the wires is taken to be ∓ 1 per unit length, then the capacitance is $2V$.

Now at leading order the electrical charges are uniformly distributed around the surface of each wire, i.e.

$$\varphi \sim -\frac{1}{2\pi}\ln\sqrt{(x-\tfrac{1}{2})^2 + y^2} + \frac{1}{2\pi}\ln\sqrt{(x+\tfrac{1}{2})^2 + y^2}$$

Examine the potential in the neighbourhood of one of the wires by expanding the above potential to ord(ϵ^2) using the stretched variables $x = \frac{1}{2} + \epsilon\xi$ and $y = \epsilon\eta$. Now construct an improved approximation for the potential by adding dipoles (like $(x \pm \frac{1}{2})/[(x \pm \frac{1}{2})^2 + y^2]$) and quadrupoles (like $[y^2 - (x \pm \frac{1}{2})^2]/[(x \pm \frac{1}{2})^2 + y^2]^2$) of appropriate magnitudes to ensure that the potential is constant around the surfaces to ord(ϵ^2). (The magnitude of the monopoles is the total charge which does not vary from unity.) Hence find the capacitance $2V(\epsilon)$ to ord(ϵ^4).

5.5 Moon space ship problem

In the Grand Tour space mission, a space craft passed the major planets in sequence, receiving much of its kinetic energy with respect to the Sun from being redirected as it flew past each planet. Very high accuracy was needed in predicting the path of the space craft in order that it would fly closely past the second and subsequent planets. Integrating the primitive equations of motion for the space craft in the gravitational field of the Sun and its orbiting planets does not produce sufficient accuracy on the largest computers. Lagerstrom and Kevorkian showed in 1963 that the problem could be tackled with matched asymptotic expansions based on the smallness of the mass of the planets compared with that of the Sun, and that the desired accuracy could then be achieved in the calculations using a small computer. The idea is that the space craft moves along a classical elliptical or hyperbolic orbit around the Sun until it comes near to a planet. It then orbits the planet and on leaving the planet it sets off on a new orbit around the Sun. The matching involves extracting from the old orbit the energy and impact parameter for the new orbit.

Here we study very briefly a model problem in which the space craft moves in the gravitational field of the Earth at $(x, y) = (0, 0)$ and an ϵ-mass Moon which is fixed at $(1, 0)$, starting the space craft at $t = 0$ from the Earth with its escape velocity (i.e. potential plus kinetic energy is zero) in nearly the direction of the Moon, dy/dx along the path is ϵk with k a constant. The governing equations are

$$\begin{aligned} \ddot{x} &= -\frac{x}{(x^2+y^2)^{3/2}} - \epsilon\frac{x-1}{((x-1)^2+y^2)^{3/2}} \\ \ddot{y} &= -\frac{y}{(x^2+y^2)^{3/2}} - \epsilon\frac{y}{((x-1)^2+y^2)^{3/2}} \end{aligned}$$

Approximation for orbit about the Earth. In the 'outer' approximation we pose an expansion

$$x(t,\epsilon) \sim x_0(t) + \epsilon x_1(t) \qquad \text{and} \qquad y(t,\epsilon) \sim 0 + \epsilon y_1(t)$$

in which ϵx_1 is the first effect of the Moon, and the ϵy_1 term does not in fact feel the Moon but is small because there is little motion in the

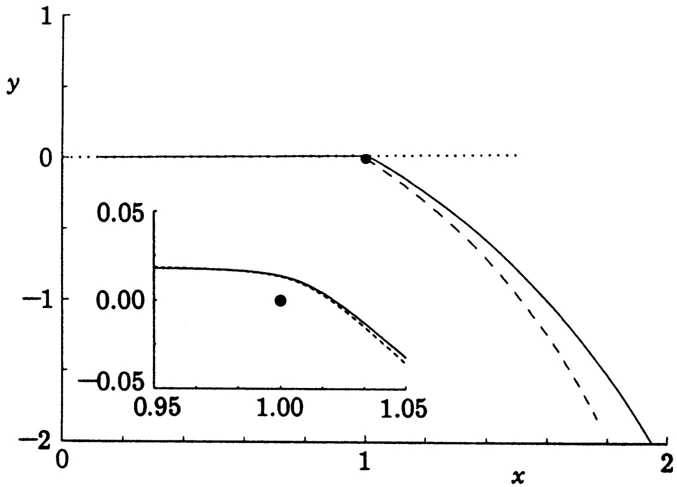

Fig. 5.5 The trajectory of the space craft for $\epsilon = 0.02$ and $k = 1$. The continuous curve is the exact solution obtained numerically. The approximate orbits around the Earth are given by the dotted and the dashed curves. The inset gives the fine details in the neighbourhood of the Moon at $x = 1$ and $y = 0$, with the dashed curve being the approximate orbit around the Moon.

y-direction initially. For the leading order approximation, we have

$$\ddot{x}_0 = -\frac{1}{x_0^2}$$

with the first integral

$$\tfrac{1}{2}\dot{x}_0^2 - \frac{1}{x_0} = 0$$

choosing the constant of integration to be zero by the initial condition that the space craft sets out with precisely its escape velocity. Integrating again

$$x_0 = \left(\tfrac{9}{2}\right)^{1/3} t^{2/3}$$

As claimed above, y_1 is not affected by the Moon since it is governed by

$$\ddot{y}_1 = -\frac{y_1}{x_0^3}$$

with simple solution

$$y_1 = k x_0(t)$$

i.e. the space craft keeps going in the same direction as its initial direction (conservation of angular momentum). The equation for x_1 is

$$\ddot{x}_1 = \frac{2x_1}{x_0^3} + \frac{1}{(1-x_0)^2}$$

As $t \nearrow t_* = \sqrt{2}/3$, $x_0 \nearrow 1$ and the above equation gives

$$x_1 \sim -\tfrac{1}{2}\ln(t_* - t)$$

and so the orbit about the Earth breaks down as the craft approaches the Moon. A complete solution for x_1 can be obtained most easily by recombining x_0 and ϵx_1 into $\tilde{x}$, their recombined equation being twice integrable to

$$\tfrac{2}{3}\tilde{x}^{3/2} - \epsilon\left(\tfrac{1}{2}\ln\frac{1+\tilde{x}^{1/2}}{1-\tilde{x}^{1/2}} - \tilde{x}^{1/2} - \tfrac{1}{3}\tilde{x}^{3/2}\right) = \sqrt{2}t$$

and so

$$x_1 = \tfrac{1}{2}x_0^{-1/2}\ln\frac{1+x_0^{1/2}}{1-x_0^{1/2}} - 1 - \tfrac{1}{3}x_0$$

Approximation for orbit around the Moon. The scaling for this 'inner' region is found from the requirements that the velocity $\Delta x/\Delta t = \mathrm{ord}(1)$ in order to match to the above outer and that the acceleration is due to the ϵ mass $\Delta x/(\Delta t)^2 = \mathrm{ord}(\epsilon/(\Delta x)^2)$. Thus $\Delta x = \Delta y = \Delta t = \epsilon$, so we rescale

$$x = 1 + \epsilon\xi, \qquad y = \epsilon\eta \qquad \text{and} \qquad t = t_* + \epsilon\tau$$

producing governing equations

$$\xi_{\tau\tau} = -\frac{\xi}{(\xi^2+\eta^2)^{3/2}} - \epsilon + O(\epsilon^2)$$

$$\eta_{\tau\tau} = -\frac{\eta}{(\xi^2+\eta^2)^{3/2}} + O(\epsilon^2)$$

i.e. the Moon's attraction dominates and the Earth gives rise to a relatively small (the ϵ term) uniform gravitation acceleration in the neighbourhood of the Moon. At lowest order, we have a classical central orbit problem with solution

$$\left(\xi^2+\eta^2\right)^{1/2} = \frac{b\sin\alpha}{\cos\alpha + \cos(\alpha - \tan^{-1}(\eta/\xi))}$$

with impact parameter b and deflection $\pi - 2\alpha$, which are related to the velocity at infinity v_* by

$$v_*^2 = \frac{\tan\alpha}{b}$$

At higher orders we would find that the constant acceleration from the Earth leads to a switch-back $\epsilon \ln \frac{1}{\epsilon}$ term.

Matching. As we approach the Moon, the orbit around the Earth has

$$\dot{x}_0 \sim \sqrt{2} \quad \text{and} \quad y_1 \sim k \quad \text{as } t \nearrow t_*$$

while as we start the orbit about the Moon

$$\xi_\tau \sim v_* \quad \text{and} \quad \eta \sim b \quad \text{as } \tau \searrow -\infty$$

Hence $b = k$ and $v_* = \sqrt{2}$. The deflection angle for the orbit around the Moon is then $\pi - 2\tan^{-1}(2k)$. At higher orders there is a time delay $\text{ord}(\epsilon \ln \frac{1}{\epsilon})$.

Second orbit around the Earth. This starts from near $x = 1$ and $y = 0$ with a velocity at the end of the orbit around the Moon v_* in the direction 2α. Thus we need a central orbit around the Earth through $x = 1$ and $y = 0$ with energy $\frac{1}{2}v_*^2 - 1 = 0$ and angular momentum $\sqrt{2}\sin 2\alpha = 4\sqrt{2}k/(1+4k^2)$. Note that the energy is unchanged by the encounter but the angular momentum is increased. (The energy with respect to the Earth would have been increased if we had considered a moving Moon.) The desired second orbit about the Earth is then

$$(x^2+y^2)^{1/2} = \frac{32k^2}{(1+4k^2)^2\left[1-\cos\left(\tan^{-1}\frac{y}{x} - 4\tan^{-1}(2k)\right)\right]}$$

Exercise 5.19. Solve for $a(t,\epsilon)$ and $r(t,\epsilon)$ which are governed by

$$\dot{a} = -a\left(\frac{1}{a} - \epsilon r^2\right)$$

$$\dot{r} = r\left(\frac{1}{a} - \epsilon r^2\right) - 1$$

with $a = r = 1$ at $t = 0$.

Find the leading order approximation for when a, r, $t = \text{ord}(1)$. Note that this approximation breaks down when $t = 1 - \text{ord}(\epsilon)$ with $a = \text{ord}(\epsilon)$ and $r = \text{ord}(\epsilon^{-1})$. With the rescaling $a = \text{ord}(\epsilon)$, $r = \text{ord}(\epsilon^{-1})$ and $\Delta t = \text{ord}(\epsilon^{-1})$, find the new leading order approximation and match crudely. [The tough part of this problem is to match properly through a transition region with $a = \text{ord}(\epsilon)$, $r = \text{ord}(\epsilon^{-1})$ and $\Delta t = \text{ord}(\epsilon)$.]

5.6 van der Pol relaxation oscillator

This problem is typical of one of the more complicated applications of the method of matched asymptotic expansions. To make the details look simpler the formal matching with an intermediate variable is abandoned. Along with the less formal approach, however, there is the new idea of examining the way one expansion breaks down in order to find the rescaling appropriate for the next region.

The van der Pol oscillator is governed by the equation

$$\ddot{x} + \mu\dot{x}(x^2-1) + x = 0$$

This oscillator has large nonlinear friction which is negative in $|x| < 1$ and positive in $|x| > 1$. As a result the trivial solution $x \equiv 0$ is unstable, while large amplitudes are damped. Thus all solutions tend to a finite amplitude oscillation, which balances energy losses in $|x| > 1$ with energy gains in $|x| < 1$. We now try to find the form of this so-called relaxation oscillation or limit cycle as $\mu \to \infty$. Setting $\mu = \infty$ in the equation shows that this problem is singular. From computations (see figure 5.6) it is found that the oscillation consists of fast phases with $\Delta t = \text{ord}(\mu^{-1})$ in which the large friction or anti-friction is balanced by inertia and slow phases with $\Delta t = \text{ord}(\mu)$ in which the large friction balances the restoring force. We now briefly construct a solution for this

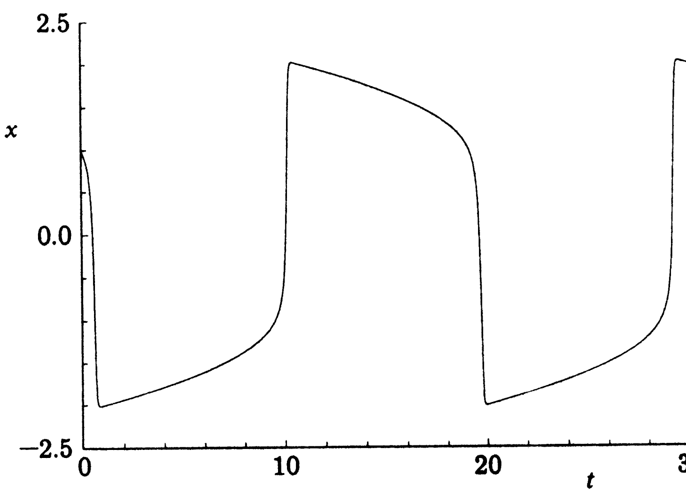

Fig. 5.6 The relaxation oscillation of the van der Pol equation with $\mu = 10$.

relaxation oscillation by matched asymptotic expansions. Without loss of generality we may start at $t = 0$ with $x = 1$, because the relaxation oscillation must pass between the damping $|x| > 1$ and the anti-damping $|x| < 1$.

Slow phase. Rescaling with $t = \mu T$, the governing equations become

$$\mu^{-2} x_{TT} + x_T(x^2 - 1) + x = 0$$

This suggests an expansion

$$x(t,\mu) \sim X_0(T) + \mu^{-2} X_1(T)$$

At μ^0: $\quad X_0'(X_0^2 - 1) + X_0 = 0 \quad$ with $X_0 = 1$ at $T = 0$
with implicit solution

$$T = \ln X_0 - \tfrac{1}{2}(X_0^2 - 1)$$

As $T \nearrow 0$, this solution breaks down because the right hand side is negative, with

$$X_0 \sim 1 + (-T)^{1/2} \qquad \text{as } T \nearrow 0$$

At μ^{-2}: $\quad X_1'(X_0^2 - 1) + 2X_0' X_0 X_1 + X_1 = -X_0''$
This can be solved in the form $X_1 = f(X_0)$, but here we need only the behaviour of the particular integral as $T \nearrow 0$, because it is the particular integral which breaks the asymptoticness of the expansion and calls for another balance in the equation. (This particular integral represents the effect of inertia which was neglected in the first approximation to the slow phase.)

$$X_1 \sim \tfrac{1}{4}(-T)^{-1} \qquad \text{as } T \nearrow 0$$

Matching forwards, we reconstruct our slow phase solution as it approaches its breakdown at $T = 0$ using the original time variable (rather than properly using a general intermediate time variable)

$$x \sim \left(1 + (-\mu^{-1}t)^{1/2} + \cdots\right) + \mu^{-2}\left(\tfrac{1}{4}(-\mu^{-1}t)^{-1} + \cdots\right) + \cdots$$

The correction term is as large as the leading order term minus 1 (which is the important measure) when $t = \mathrm{ord}(\mu^{-1/3})$ where $x = 1 + \mathrm{ord}(\mu^{-2/3})$. The trouble is that as $T \nearrow 0$, $x \searrow 1$, so the coefficient of friction $(x^2 - 1)$ drops, so the velocity increases, and so the inertia is no longer negligible. In fact with the above scaling, all three terms in the governing equation are ord(1):

$$\ddot{x} : \mu\dot{x}(x^2 - 1) : x \;=\; \frac{\mu^{-2/3}}{(\mu^{-1/3})^2} : \mu\frac{\mu^{-2/3}}{\mu^{-1/3}}\left(\mu^{-2/3}\right) : 1$$

Transition phase. With the rescaling suggested above, $t = \mu^{-1/3}s$ and $x = 1 + \mu^{-2/3}z$, the governing equation becomes

$$z_{ss} + 2z_s z + 1 + \mu^{-2/3}(z_s z^2 + z) = 0$$

which suggests an expansion in a revised sequence

$$x(t,\mu) \sim 1 + \mu^{-2/3} z_1(s) + \mu^{-4/3} z_2(s)$$

Matching backwards into the slow phase, we find

$$z_1 \sim (-s)^{1/2} + \tfrac{1}{4}(-s)^{-1} \qquad \text{as } s \searrow -\infty$$

At $\mu^{-2/3}$: $\quad z_1'' + 2z_1' z_1 + 1 = 0$
This can be integrated once choosing the constant by matching backwards:

$$z_1' + z_1^2 + s = 0$$

This Ricatti equation can be turned into the Airy equation with the substitution $z_1 = \zeta'/\zeta$:

$$\zeta'' + s\zeta = 0$$

The general solution can be expressed in terms of $K_{1/3}$ and $I_{1/3}$ for $s < 0$. Matching backwards rules out $I_{1/3}$, so for $s < 0$

$$\zeta = (-s)^{1/2} K_{1/3}\left(\tfrac{2}{3}(-s)^{3/2}\right)$$

This solution continues into $s > 0$ in the form

$$\tfrac{1}{\sqrt{3}}(s)^{1/2}\left[J_{1/3}\left(\tfrac{2}{3}s^{3/2}\right) + J_{-1/3}\left(\tfrac{2}{3}s^{3/2}\right)\right]$$

Because $z = \zeta'/\zeta$, there is trouble with $z \searrow -\infty$ as $s \nearrow s_0 = 2.34$, the first root of $J_{1/3} + J_{-1/3} = 0$. From the equation for ζ we find

$$z_1 \sim -\frac{1}{s_0 - s} + \tfrac{1}{3}s_0(s_0 - s) \qquad \text{as } s \nearrow s_0$$

Matching forward, we reconstruct the above solution in the transition phase in terms of the original time variable as it approaches its breakdown:

$$x \sim 1 + \mu^{-2/3}\left[-\frac{\mu^{-1/3}}{\mu^{-1/3}s_0 - t} + \mu^{1/3}\tfrac{1}{3}s_0(\mu^{-1/3}s_0 - t)\right]$$

The asymptoticness is broken when $\mu^{-1/3}s_0 - t = \mathrm{ord}(\mu^{-1})$ where $x = \mathrm{ord}(1)$.

Fast phase. With the scaling suggested above, $t = \mu^{-1/3}s_0 + \mu^{-1}\tau$, the governing equation becomes

$$x_{\tau\tau} + x_\tau(x^2 - 1) + \mu^{-2}x = 0$$

Matching backwards into the end of the transition phase

$$x \sim 1 + \tau^{-1} - \mu^{-4/3}\tau\tfrac{1}{3}s_0 \qquad \text{as } \tau \searrow -\infty$$

This suggests an asymptotic expansion

$$x(t,\mu) \sim x_0(\tau) + \mu^{-4/3}x_1(\tau) + \mu^{-2}x_2(\tau)$$

At μ^0: $\quad x_0'' + x_0'(x_0^2 - 1) = 0$

Integrating the equation once and choosing the constant of integration by matching backwards to $x_0 \sim 1 + \tau^{-1}$ as $\tau \searrow -\infty$

$$x_0' + \tfrac{1}{3}x_0^3 - x_0 = -\tfrac{2}{3}$$

Integrating again and matching backwards yields an implicit solution

$$\tfrac{1}{3}\ln\frac{2 + x_0}{1 - x_0} + \frac{1}{1 - x_0} = -\tau$$

The fast phase ends when $\tau \nearrow \infty$ where

$$x_0 \sim -2 + 3e^{-3\tau - 1}$$

At $\mu^{-4/3}$: $\quad x_1'' + x_1'(x_0^2 - 1) + 2x_0'x_1x_0 = 0$

Integrating and matching backwards to $x_1 \sim -\tfrac{1}{3}s_0\tau$ as $\tau \searrow -\infty$

$$x_1' + x_1x_0^2 - x_1 = -s_0$$

This linear equation can be solved, but we only need the form as $\tau \nearrow \infty$. In this limit the equation becomes

$$x_1' + 3x_1 = -s_0$$

and so $x_1 \sim -\tfrac{1}{3}s_0$ as $\tau \nearrow \infty$.

At μ^{-2}: $\quad x_2'' + x_2'(x_0^2 - 1) + 2x_0'x_2x_0 = -x_0$

Again we need only the form of the solution as $\tau \nearrow \infty$. In this limit the equation becomes

$$x_2'' + 3x_2' = 2$$

and so $x_2 \sim \tfrac{2}{3}\tau$ as $\tau \nearrow \infty$.

Matching forwards we reconstruct our solution in the fast phase in terms of the original time variable as it approaches its breakdown as $\tau \nearrow \infty$:

$$\begin{aligned} x \sim{} & [-2 + \cdots] + \mu^{-4/3}\left[-\tfrac{1}{3}s_0 + \cdots\right] \\ & + \mu^{-2}\left[\tfrac{2}{3}\mu(t - \mu^{-1/3}s_0) + \cdots\right] + \cdots \end{aligned}$$

The asymptoticness is thus broken when $t - \tfrac{3}{2}\mu^{-1/3}s_0 = \mathrm{ord}(\mu)$; broken by the particular integral in the correction term (the effect of the restoring force, neglected in the lowest approximation of the fast phase).

Second slow phase. Following the fast phase there is a repetition of the slow phase with x reversed in sign and time shifted by half the period, $\tfrac{1}{2}\mathcal{T}$. At the lowest approximation we therefore have a solution

$$T - \mu^{-1}\tfrac{1}{2}\mathcal{T} = \ln(-X_0) - \tfrac{1}{2}(X_0^2 - 1)$$

Matching backwards into the fast phase which ends near $x = -2$, this second slow phase solution has

$$X_0 \sim -2 + \tfrac{2}{3}\left[T - \mu^{-1}\tfrac{1}{2}\mathcal{T} - \ln 2 + \tfrac{3}{2}\right]$$

So comparing with the end of the fast region, we see that the period of the relaxation oscillator is

$$\mathcal{T} \sim \mu(3 - 2\ln 2) + 3\mu^{-1/3}s_0 \qquad \text{as } \mu \to \infty$$

The first term in the period comes from the slow phase, and in some sense $2\mu^{-1/3}s_0$ comes from a delay in the transition phase, while the other $\mu^{-1/3}s_0$ comes from overshooting $x = -2$ slightly.

From our solution we can see that the maximum displacement of the relaxation oscillation is at the end of the fast phase near $x = -2$:

$$\max|x| \sim 2 + \mu^{-4/3}\tfrac{1}{3}s_0 \qquad \text{as } \mu \to \infty$$

And the maximum velocity occurs in the fast phase near $x = -1$:

$$\max|\dot{x}| \sim \tfrac{4}{3}\mu + \mu^{-1/3}s_0 \qquad \text{as } \mu \to \infty$$

Higher order terms in the expansion can be obtained with some effort. The next term is just larger than the indicated μ^{-2} – it is $\mu^{-2}\ln\mu$ followed by μ^{-2}.