M2, Fluid mechanics 2019/2020

Multiscale Hydrodynamic Phenomena

Wed., December 4th, 2019, 8 :30am - 12 :30pm, Room 55-66-112

Part I. : 90 minutes, NO documents

1. Quick Questions In few words :

1.1 Usual scales for pressure and friction for an incompressible flow at small Reynolds around a sphere ?

1.2 Usual scales for pressure and friction for an incompressible flow at large Reynolds around a sphere ?

1.3 Usual scales for pressure and friction for an incompressible flow at small Reynolds around a cylinder ?

1.4 Usual scales for pressure and friction for an incompressible flow at large Reynolds around a cylinder ?

1.5 ∂'Alembert equation : write the equation and the generic solution of it

1.6 Heat equation in a 1D domain, temperature imposed in 0 and at infinity : write the equation and show that there is a self similar solution

1.7 Remind without demonstration the solution of Laplace equation in the upper half domain ($\forall x$ and $y \geq 0$) with a Neumann BC in $y = 0$, and a Dirichlet Boundary Condition equal to 0 at infinity ?

1.8 What is the Burgers equation ? Which balance is it ?

2. Exercice

Let us look at the following ordinary differential equation : $(E_\varepsilon)\quad \dfrac{d^2y}{dt^2} + y = -2\varepsilon\dfrac{dy}{dt},\quad$ valid for any $t > 0$ with boundary conditions $y(0) = 1$ and $y'(0) = 0$. Of course ε is a given small parameter.

We want to solve this problem with Multiple Scales.

2.1 Expand up to order ε : $y = y_0(t) + \varepsilon y_1(t)$, show that there is a problem for long times.

2.2 Introduce two time scales, $t_0 = t$ and $t_1 = \varepsilon t$

2.3 Compute $\partial/\partial t$ and $\partial^2/\partial t^2$

2.4 Solve the problem.

2.5 Suggest the plot of the solution.

3. Exercice

Consider the following equation (of course ε is a given small parameter)

$$(E_\varepsilon)\qquad \varepsilon\frac{d^2u}{dx^2} + \frac{du}{dx} + \varepsilon u = \frac{1}{2}(1 + x) \text{ with } u(0) = 0 \quad u(1) = 1.$$

We want to solve this problem with the Matched Asymptotic Expansion method.

3.1 Why is this problem singular ?

3.2 What is the outer problem and what is the possible general form of the outer solution ?

3.3 What is the inner problem of (E_ε) and what is the inner solution ?

3.4 Suggest the plot of the inner, outer and composite solution.

4. Exercice

Solve with WKB approximation the problem

$$\varepsilon y''(x) = y(x) \text{ with } y(0) = 0, y(1) = 1$$

M2, Fluid mechanics 2019/2020
Friday, December 4th, 2018
Part II. : 1h 15 min all documents.

Multiscale Hydrodynamic Phenomena

Flow in elastic tubes (arteries)

This is a part of "On some model equations for pulsatile flow in viscoelastic vessels" by Mitsotakis et al. Wave Motion 90 (2019) 139-151. We consider the flow in a viscoelastic pipe. The behaviour of the flow is very similar to the free surface water flow.

1.0 Write incompressible Navier Stokes equations.
1.1 The viscous longitudinal term in axi symetrical incompressible NS is :

$$V_x = \nu \left(\frac{\partial^2 u}{\partial r^2} + \frac{\partial u}{r \partial r} + \frac{\partial^2 u}{\partial x^2} \right)$$

justify that for large Reynolds number (to be defined) this term is negligible.
1.2 Using scaling (2.5) of the paper, show that one term is smaller than the others.
1.3 Take the mean value of V_x (i.e. evaluate $\int_0^{r_w}(V_x) r dr$), show that we obtain the wall shear stress.
1.4 As the scale of laminar wall shear stress is proportional to the scale of the velocity, the authors introduce an empirical "damping coefficient" κ (formula 1.1). From previous questions what should be the scale of κ with ν and R ?
1.5 As they do after a inviscid analysis, they claim that the total friction is proportional to $-\kappa u^w$. This point of view is to my opinion strange (I was not referee !), and even false. Explain why ?
1.6 Write a sentence to justify this very crude approximation, that we take for granted from now.
1.7 The small perturbation of the radius r_0 of the artery is η. As we suppose an ideal fluid, the normal velocity of the wall is the normal velocity of the fluid. Justify (1.5).
1.8 The acceleration of the wall is due to the normal pressure on it plus elastic and visco elastic forces. Identify each term in (1.6). What term is here neglected as viscosity of the fluid is neglected ?

2.1 Using small disturbance theory with a small ε, scaling (2.5-2.6), and supposing a plug velocity profile $u(x,t)$ (no r) show that we can obtain a wave equation for $\bar{\eta}$. What is the scale of the wave speed (its name is Moens -Korteweg celerity) ?

3.1 Demonstrate (2.1) from Euler.
3.2 Show that (2.7) is the good scaling of (2.1).
3.3 For each following equation : (2.8), (2.9) (2.10), give its name and check the scaling
3.4 Justify (2.18)
3.5 Justify (2.22) and (2.23). What are the differences with the case of water in channel ?

4.1 After some algebra, a kind of KdV equation (or BBM Benjamin Bona Mahony) is obtained, check (4.10) and (4.11).
4.2 Comment (4.12-4.14).

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On some model equations for pulsatile flow in viscoelastic vessels

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HIGHLIGHTS

- A derivation of Boussinesq systems for pulsatile flow in viscoelastic vessels is presented.
- The derivation is based on formal asymptotic expansions of the velocity potential.
- Simpler unidirectional model equations are also derived.
- Dissipative effects due to viscous stresses in bio-fluids are also taken into account.
- The dissipation effects are explored numerically.

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ABSTRACT

Considered here is the derivation of partial differential equations arising in pulsatile flow in pipes with viscoelastic walls. The equations are asymptotic models describing the propagation of long-crested pulses in pipes with cylindrical symmetry. Additional effects due to viscous stresses in bio-fluids are also taken into account. The effects of viscoelasticity of the vessels on the propagation of solitary and periodic waves in a vessel of constant radius are being explored numerically.

1. Introduction

The description of fluid flows in pipes with viscoelastic wall material is motivated mainly by the studies of hemodynamics [1]. A cardiac cycle consists of the systolic phase where the heart ventricles contract and pump blood to the arteries and the diastolic phase where the heart ventricles are relaxed and the heart fills with blood again. During the systolic phase the large arteries are deformed and store elastic energy that is released during the diastolic phase. This property of the vessels is usually referred to as the compliance of the vessels. Modelling the viscoelastic properties of the vessels appears to have significant difficulties of mathematical and numerical nature [1–6]. The mathematical modelling of such flows suggests the use of the equations of continuum mechanics for incompressible fluid flow known as the Navier–Stokes equations.

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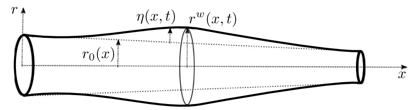


Fig. 1. Sketch of the physical domain for a single vessel segment with elastic and impenetrable wall.

The Navier–Stokes equations in three dimensions are too complicated to be used in practical situations and for this reason several simplified mathematical models have been derived [7–12]. The main simplifications that have been made are based on the assumption of the axial (or cylindrical) symmetry of the vessels. This assumption and using approximations of the averaged velocity of the fluid led to the derivation of simple one-dimensional (1 + 1D) models [13–20]. The models include unidirectional [12,21–24], and bidirectional models [25–27]. Bidirectional models can approximate accurately reflections of pulses occurred in the presence of non-uniformities in the vessels wall as opposed to unidirectional models such as the KdV equation [21,28] where it is assumed that the pulses propagate mainly in one direction. In this paper we focus on the derivation of bidirectional equations. Further simplifications were made by assuming that the velocity of the fluid is very large. This assumption led to lumped-parameter (or zero-dimensional) models [29,30] and the references therein. The later models are used in practice to predict the flow and the pressure of the blood in operational situations [4].

For practical reasons, the inclusion of the dissipative effects in the flow can be done by assuming a laminar flow and small viscosity. For example assuming a parabolic profile for the horizontal velocity of the fluid it has been shown that the Navier–Stokes equations can be reduced to a modified system which is very similar to the Euler equations [6], (we also refer to the Poiseuille solution for the justification of this parabolic profile of the horizontal velocity). Specifically, denoting $u = u(x, r, t)$, $v = v(x, r, t)$ the horizontal and radial velocity respectively, and $u^w(x, t) = u(x, r^w, t)$ the horizontal velocity of the fluid on the vessel wall (at radius $r = r^w(x, t)$), then assuming that $u(x, r, t)$ is proportional to $((r^w)^2 - r^2)u^w(x, t)$ these equations can be written in cylindrical coordinates in the form:

$$u_t + uu_x + vu_r + \frac{1}{\rho} p_x + \kappa u^w = 0, \tag{1.1}$$

$$v_t + uv_x + vv_r + \frac{1}{\rho} p_r = 0, \tag{1.2}$$

$$u_x + v_r + \frac{1}{r} v = 0, \tag{1.3}$$

where $p = p(x, r, t)$ is the pressure of the fluid, ρ is the constant density of the fluid and κ is the viscous frequency parameter (also known as the Rayleigh damping coefficient) with dimensions $[s^{-1}]$.

A sketch of the physical domain for this problem is presented in Fig. 1, where the distance of vessel's wall from the centre of the vessel in a cross section is denoted by $r^w(x, t)$ and depends on x and t while the radius of the vessel at rest is given by the function $r_0(x)$. In general the deformation of the wall will be a function of x and t. If we denote the radial displacement of the wall by $\eta(x, t)$ then the vessel wall radius can be written as $r^w(x, t) = r_0(x) + \eta(x, t)$.

The governing equations (1.1)–(1.3) combined with initial and boundary conditions form a closed system. A compatibility condition is also applied at the centre of the vessel (due to cylindrical symmetry). Specifically, we assume that

$$v(x, r, t) = 0, \quad \text{for} \quad r = 0. \tag{1.4}$$

In general we assume for consistency purposes that $v(x, r, t) = O(r^{1+\tau})$, with $\tau \geq 0$ as $r \to 0$. The impermeability of the vessel wall can be described by the equation:

$$v(x, r, t) = \eta_t(x, t) + (r_0(x) + \eta(x, t))_x u(x, r, t), \text{ for } r = r^w(x, t), \tag{1.5}$$

and expresses that the fluid velocity equals the wall speed $v = r^w_t$. The system is accompanied also by a second boundary condition, which is the Newton second law applied on the vessel wall:

$$\rho^w h \eta_{tt}(x, t) = p^w(x, t) - \frac{E_\sigma h}{r_0^2(x)} (\eta(x, t) + \gamma \eta_t(x, t)), \tag{1.6}$$

where ρ^w is the wall density, p^w is the transmural pressure, h is the thickness of the vessel wall, $E_\sigma = E/(1-\sigma^2)$ where E is the Young modulus of elasticity with σ denoting the Poisson ratio of the viscoelastic wall. In this study we assume that E is a constant and in general we simplify the notation by denoting E_σ with E. The last term in (1.6) models the viscous nature of the vessel wall and can be derived by using a simple Kelvin–Voigt model (spring-dashpot model). In this setting $\gamma = n_w/E_v$ where n_w is the dashpot coefficient of viscosity and E_v is the Young modulus of the viscous part of Kelvin body. In practical situations the parameter γ is very small and usually can be taken to be of order $O(10^{-4})$. It is noted that because the flow is pressure-driven the effect of gravity is neglected. For more information about the derivation of the Euler equations and the boundary conditions we refer to [2,31].

Although the dispersion of the flow can be ignored from the majority of mathematical models derived from the Navier–Stokes equations or from the Euler equations resulting into very simple systems of conservation laws, the need for more accurate description of the waves and their reflections suggests the inclusion of this fundamental property. A first attempt towards the derivation of bi-directional weakly-nonlinear and weakly-dispersive system of equations was presented in [27]. Using asymptotic techniques more general asymptotic models were derived in [32]. The systems derived in [32] appeared to justify the non-dispersive models of [33,34] with asymptotic reasoning. It was also shown that the inclusion of dispersive terms can describe more accurately the effects of the vessel wall variations within the flow. One basic ingredient that was ignored in both works [27,32] is the viscosity effects of the vessels by assuming simple elastic vessels.

In this paper we extend the work [32] and derive some new asymptotic one-dimensional equations of Boussinesq type (weakly non-linear and weakly dispersive) that approximate the system (1.1)–(1.3) with boundary conditions (1.4)–(1.6). The derivation is based on formal expansions of the velocity potential as in [35]. The new systems generalise the previously derived Boussinesq systems of [32] as they coincide with them when the viscoelastic property of the vessel wall is ignored. The new mathematical models are of significant importance since they include all the necessary ingredients for the accurate description of regular fluid flows in pipes with viscoelastic walls.

Since the dissipation caused by the fluid viscosity and the dissipation caused by the viscoelasticity of the vessel wall are different in their nature there is a question on whether the different dissipative terms have also different effects on the propagation of pulses. The answer to this question is explored computationally by studying the effects caused by the two dissipative terms on the propagation of a solitary wave.

The paper is organised as follows: In Section 2 we present the derivation of a new system of Boussinesq type for the description of the velocity and the deviation of the vessel wall for fluid flow in a viscoelastic vessel. This system is further improved in Section 3 by computing the fluid velocity at different levels of radius. In Section 4 further simplifications lead to unidirectional equations that depend only on the deviation of the vessel wall, while the velocity of the fluid can be computed explicitly using a simple asymptotic formula. Section 5 demonstrates the dissipation effects on the propagation of solitary and periodic waves. We close this paper with some conclusions and perspectives.

2. Derivation of the new mathematical models

Here we proceed with the derivation of the new equations. The derivation is based on the assumption that the flow is irrotational and therefore we assume the existence of a smooth velocity potential $\phi(x,r,t)$ such that $(u,v)^T = \nabla\phi$, i.e. we assume that $u = \phi_x$ and $v = \phi_r$. Then, as in [36] the velocity potential can be chosen appropriately such that the Eqs. (1.1)–(1.2) can be integrated into the generalised Cauchy–Lagrange integral:

$$\phi_t + \frac{1}{2}\phi_x^2 + \frac{1}{2}\phi_r^2 + \frac{1}{\rho}p + \kappa\phi = 0, \quad \text{for } r = r^w. \tag{2.1}$$

The mass conservation (continuity) equation is then reduced to the elliptic equation

$$r\phi_{xx} + (r\phi_r)_r = 0, \quad 0 < r < r^w, \tag{2.2}$$

and boundary conditions for the velocity are written as

$$\phi_r = 0, \ \text{for } r = 0, \tag{2.3}$$

and

$$\phi_r = \eta_t + (r_0(x) + \eta)_x\phi_x, \ \text{for } r = r^w(x,t). \tag{2.4}$$

In order to make simplifications to the previous equations we consider the following non-dimensional (scaled) variables:

$$\eta^* = \frac{\eta}{a},\ x^* = \frac{x}{\lambda},\ r^* = \frac{r}{R},\ t^* = \frac{t}{T},\ \phi^* = \frac{1}{\lambda\varepsilon\tilde{c}}\phi,\ p^* = \frac{1}{\varepsilon\rho\tilde{c}^2}p, \tag{2.5}$$

where a is a typical deviation of the vessel wall from its rest position, λ a typical wavelength of a pulse, R is a vessel's typical radius, $T = \lambda/\tilde{c}$ the characteristic time scale, while $\tilde{c} = \sqrt{Eh/2\rho R}$ is the Moens–Korteweg characteristic speed, [1]. It is noted that the external pressure is considered zero and is neglected. The parameters ε and δ characterise the nonlinearity and the dispersion of the system:

$$\varepsilon = \frac{a}{R}, \quad \delta = \frac{R}{\lambda}. \tag{2.6}$$

Usually, ε and δ are very small. Specifically, we assume that $\varepsilon \ll 1$, $\delta^2 \ll 1$, while the Stokes–Ursell number is of order 1: $\varepsilon/\delta^2 = O(1)$. The system of Eqs. (2.1)–(2.5) along with the boundary condition (1.6) is then written in dimensionless variables in the form:

$$\phi^*_{t^*} + \frac{\varepsilon}{2}\phi^{*2}_{x^*} + \frac{\varepsilon}{\delta^2}\frac{1}{2}\phi^{*2}_{r^*} + p^* + \varepsilon\kappa^*\phi^* = 0, \quad \text{for } r^* = r^{*w}, \tag{2.7}$$

$$\delta^2 r^*\phi^*_{x^*x^*} + (r^*\phi^*_{r^*})_{r^*} = 0, \quad 0 < r^* < r^{*w}, \tag{2.8}$$

$$\phi^*_{r^*} = 0, \ \text{for } r^* = 0, \tag{2.9}$$

$$\phi^*_{r^*} = \delta^2\eta^*_{t^*} + \delta^2(r_0^*(x^*) + \varepsilon\eta^*)_{x^*}\phi^*_{x^*}, \ \text{for } r^* = r^{*w}, \tag{2.10}$$

$$p^* = \alpha^*\delta^2\eta^*_{t^*t^*} + \beta^*(\eta^* + \delta^2\gamma^*\eta^*_{t^*}), \ \text{for } r^* = r^{*w}, \tag{2.11}$$

where $\kappa^* = \kappa\lambda/\tilde{c}\varepsilon$, $\alpha^* = \rho^w h/\rho R$, $\beta^* = \beta^*(x) = 2R^2/r_0^2(x)$ and $\gamma^* = \gamma/\delta^2 T$. For the sake of simplicity in the notation, we drop the asterisk from the new variables in the following derivations for the non-dimensional variables except if it is stated otherwise.

Following standard asymptotic techniques, cf. [37], we consider a formal expansion of the velocity potential [38]:

$$\phi(x,r,t) = \sum_{m=0}^{\infty} r^m\phi_m(x,t). \tag{2.12}$$

Demanding ϕ to satisfy Eq. (2.8) leads to the following recurrence relation

$$\delta^2\partial_x^2\phi_{2m} + (2m+2)^2\phi_{2m+2} = 0, \quad \phi_{2m+1} = 0, \tag{2.13}$$

for $m = 0, 1, 2, \ldots$, where ∂_x^j denotes the jth order derivative with respect to x. A direct application of the last relation is

$$\phi_2 = -\frac{\delta^2}{4}\partial_x^2\phi_0, \tag{2.14}$$

and

$$\phi_{2m+2} = \frac{\delta^4}{(2m+2)^2(2m)^2}\partial_x^4\phi_{2m-2} = O(\delta^4), \tag{2.15}$$

for $m = 1, 2, \ldots$. The last relation ensures that the terms ϕ_m of the velocity potential expansion for $m \geq 4$ are negligible. More general, we observe that

$$\phi_{2m} = (-1)^m\frac{\delta^{2m}}{2^{2m}}\partial_x^{2m}\phi_0, \tag{2.16}$$

for $m = 1, 2, \ldots$, and therefore

$$\phi(x,r,t) = \sum_{m=0}^{\infty} r^{2m}(-1)^m\frac{\delta^{2m}}{2^{2m}}\partial_x^{2m}\phi_0. \tag{2.17}$$

A 2nd order asymptotic approximation of the velocity potential is

$$\phi(x,r,t) = \phi_0(x,t) - \delta^2\frac{r^2}{4}\partial_{xx}\phi_0(x,t) + O(\delta^4). \tag{2.18}$$

Using the previous observations on the expansion of the velocity potential we observe that (2.10) can be approximated by the relation

$$\eta_t + r_x^w\phi_{0x} + \frac{r^w}{2}\phi_{0xx} - \delta^2\frac{r_0^2 r_{0x}}{4}\phi_{0xxx} - \delta^2\frac{r_0^3}{16}\phi_{0xxxx} = O(\delta^4, \varepsilon\delta^2). \tag{2.19}$$

Since the momentum balance laws were reduced to the Cauchy–Lagrange integral equation (2.7) for $r = r^w$ we can eliminate the pressure using (2.11) and obtain

$$\phi_t + \frac{\varepsilon}{2}\phi_x^2 + \frac{\varepsilon}{\delta^2}\frac{1}{2}\phi_r^2 + \varepsilon\kappa\phi + \alpha\delta^2\eta_{tt} + \beta(\eta + \gamma\eta_t) = 0. \tag{2.20}$$

Substituting (2.18) into (2.20) we obtain the approximate momentum equation

$$\phi_{0t} - \delta^2\frac{r_0^2}{4}\phi_{0xxt} + \frac{\varepsilon}{2}\phi_{0x}^2 + \varepsilon\kappa\phi_0 + \alpha\delta^2\eta_{tt} + \beta(\eta + \delta^2\gamma\eta_t) = O(\delta^4, \varepsilon\delta^2). \tag{2.21}$$

Denoting the horizontal velocity at the centre of the vessel $u(x,0,t) = \phi_{0x}(x,t)$ by $w(x,t)$ we rewrite the Eqs. (2.19)–(2.21) in the following form:

$$\eta_t + r_x^w w + \frac{r^w}{2}w_x - \delta^2\frac{r_0^2 r_{0x}}{4}w_{xx} - \delta^2\frac{r_0^3}{16}w_{xxx} = O(\delta^4, \varepsilon\delta^2), \tag{2.22}$$

$$w_t + (\beta\eta)_x + \varepsilon w w_x + \varepsilon\kappa w - \delta^2\frac{r_0 r_{0x}}{2}w_{xt} - \delta^2\frac{r_0^2}{4}w_{xxt} + \alpha\delta^2\eta_{xtt} + \delta^2\gamma(\beta\eta_t)_x = O(\delta^4, \varepsilon\delta^2). \tag{2.23}$$

Although the system (2.22)–(2.23) is a valid approximation of the Euler equations for the prescribed asymptotic regime, it is not of much practical use due to the temporal derivatives of the wall deviations in the momentum equation (2.23). For this reason we proceed with further simplifications using low-order approximation for the velocity w and the of the vessel wall.

From the Eqs. (2.22)–(2.23) we observe that

$$w_t = -[\beta\eta]_x + O(\varepsilon, \delta^2), \qquad \eta_t = -r_x^w w - \frac{r^w}{2}\, w_x + O(\varepsilon, \delta^2).$$

Substituting these low-order approximations into (2.23) we obtain the simplified momentum equation

$$[1 - \delta^2 \alpha r_{0xx}]w_t + [\beta\eta]_x + \delta^2 \frac{(3\alpha + r_0)r_{0x}}{2}\,[\beta\eta]_{xx} + \varepsilon w w_x - \delta^2 \frac{(2\alpha + r_0)r_0}{4}\, w_{xxt} + \\ + \varepsilon\kappa w - \delta^2\gamma[\beta(r_{0x}w + \frac{r_0}{2}\, w_x)]_x = O(\delta^4, \varepsilon\delta^2).$$

The system (2.22)–(2.25) can be the base to other more amenable Boussinesq systems along the lines of [3 next section we derive a simplified Boussinesq systems with favourable properties in analogy to the classical Bo system for fluid flow in purely elastic vessels derived in [32].

■ ■ ■

Eqs. (4.5) and (4.6) coincide up to the order $O(\varepsilon, \delta^2)$ to a single equation for η^*, namely, the dimensionless BBM equation:

$$\eta^*_{t^*} + \eta^*_{x^*} + \varepsilon\frac{5}{2}\,\eta^*\eta^*_{x^*} - \delta^2\frac{4\alpha^* + 1}{16}\,\eta^*_{x^*x^*t^*} + \varepsilon\frac{\kappa^*}{2}\,\eta - \delta^2\frac{\gamma^*}{2}\,\eta^*_{x^*x^*} = 0. \tag{4.8}$$

In dimensional variables (4.8) takes the form

$$\eta_t + \tilde{c}\eta_x + \frac{5}{2}\,\frac{1}{r_0}\,\tilde{c}\eta\eta_x - \frac{(4\bar{\alpha} + r_0)r_0}{16}\,\eta_{xxt} + \frac{\kappa}{2}\,\eta - \frac{\tilde{c}}{2}\,\gamma\eta_{xx} = 0, \tag{4.9}$$

where here $\tilde{c} = \sqrt{\frac{Eh}{2\rho r_0}}$ is the standard Moens–Korteweg characteristic speed. The dispersion relation $\omega = \omega(k)$ can be easily computed:

$$\omega = \frac{\tilde{c}\,k}{1 + \dfrac{r_0(4\bar{\alpha} + r_0)}{16}\,k^2} - \frac{\mathrm{i}}{2}\,\frac{\kappa + \gamma\,\tilde{c}\,k^2}{1 + \dfrac{r_0(4\bar{\alpha} + r_0)}{16}\,k^2}. \tag{4.10}$$

In the absence of any form of dissipation, it is known that the BBM equation possesses classical solitary waves propagating with speed c_s given by the formula, [32],

$$\eta(x, t) = 3\frac{c_s - a}{b}\,\mathrm{sech}^2\left(\sqrt{\frac{c_s - a}{4c_s c}}\,(x - c_s t)\right), \tag{4.11}$$

with $a = \tilde{c}$, $b = 5\tilde{c}/2r_0$, and $c = \tilde{c}(4\bar{\alpha} + r_0)r_0/16$.

Observing that $\eta^*_{x^*} = -\eta^*_{t^*} + O(\varepsilon, \delta^2)$ from (4.8) and modifying accordingly the dispersive term of the BBM equation we obtain the analogous KdV equation:

$$\eta^*_{t^*} + \eta^*_{x^*} + \varepsilon\frac{5}{2}\,\eta^*\eta^*_{x^*} + \delta^2\frac{4\alpha^* + 1}{16}\,\eta^*_{x^*x^*x^*} + \varepsilon\frac{\kappa^*}{2}\,\eta - \delta^2\frac{\gamma^*}{2}\,\eta^*_{x^*x^*} = 0, \tag{4.12}$$

which in dimensional form becomes

$$\eta_t + \tilde{c}\eta_x + \frac{5}{2}\,\frac{1}{r_0}\,\tilde{c}\eta\eta_x + \tilde{c}\frac{(4\bar{\alpha} + r_0)r_0}{16}\,\eta_{xxx} + \frac{\kappa}{2}\,\eta - \frac{\tilde{c}}{2}\,\gamma\eta_{xx} = 0. \tag{4.13}$$

The dispersion relation $\omega = \omega(k)$ of the derived KdV equation (4.13) can be easily computed:

$$\omega = \tilde{c}\,k - \frac{r_0\,\tilde{c}\,(4\bar{\alpha} + r_0)}{16}\,k^3 - \frac{\mathrm{i}}{2}\,(\kappa + \gamma\,\tilde{c}\,k^2). \tag{4.14}$$

The imaginary part $\Im\omega(k)$ comes with the negative sign, which indicates that we have effectively introduced dissipation into the model.

6. Conclusions

In this paper we derived new weakly nonlinear and weakly dispersive asymptotic equations that describe the irrotational and dissipative flow of a fluid in pipes with viscoelastic walls. We also derived unidirectional equations of BBM and KdV type when the undisturbed radius is constant along the pipe. In order to study the dissipative effects due to fluid viscosity and the viscoelastic walls, we considered solitary and periodic waves propagating in a vessel of constant undisturbed radius and with parameters that resemble a large blood vessel. We observed that the dissipation caused by the viscoelastic wall is equally important compared to the dissipation caused by the viscosity of the fluid or more important, and therefore should not be neglected. It is also observed that the dissipative effects can be described very accurately by linear approximations. The new asymptotic models have the potential to contribute in the derivation of new lumped parameter models that can be used in operational situations where measurements of the pressure and flow of the fluid are required.

correction Ex 1

want to see the $Log(Re)$

correction Ex 2

Exactly the curse with cos,

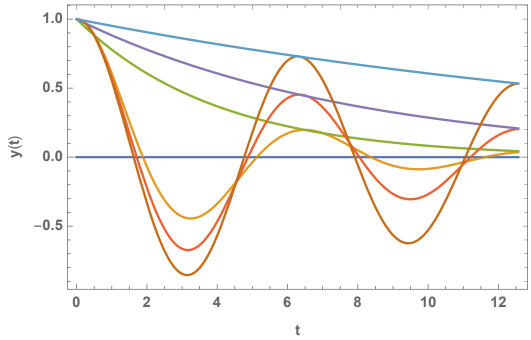
```
In[18]:= Simplify[DSolve[y''[t] + y[t] == 2 Sin[t] , y[t], t ]]

Out[18]= {{y[t] -> (-t + C[1]) Cos[t] + 1/2 (1 + 2 C[2]) Sin[t]}}
```

$y_0 = \cos(t)$ and $y_1 = -t\cos(t)$.

so that the solution is $y = e^{-t_1}\cos(t_0)$

```
se = DSolve[{y''[t] + y[t] == -2 e y'[t], y[0] == 1, y'[0] == 0},
   y[t], {t, 0, 1}];
Plot[{0, y[t] /. se /. e -> .25, Exp[-t .25], y[t] /. se /. e -> .125,
   Exp[-t .125], y[t] /. se /. e -> .05, Exp[-t .05]}, {t, 0, 4 Pi},
 Frame -> True, FrameLabel -> {"t", "y(t)"}]
```



correction Ex 3

If wi put $\varepsilon = 0$, we have an order one problem with 2 BC, so singular. We find $u_{out} = x/2 + x^2/4 + 1/4$. We note that $u_{out}(0) = 1/4$, so we have to introduce an inner layer to full fit the 0 BC.

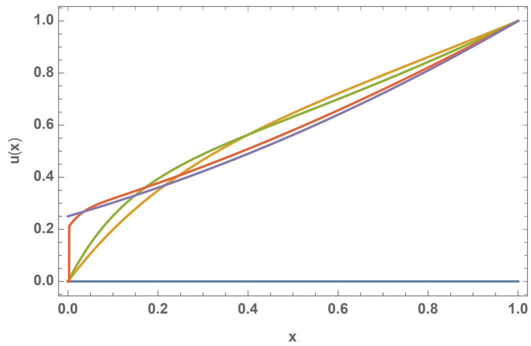
Change of scale $x = \delta\tilde{x}$, by dominant balance $\varepsilon^2 = \delta$, the problem is

$$\frac{d^2\bar{u}}{d\bar{x}^2} + \frac{d\bar{u}}{d\bar{x}} = 0$$

as $\varepsilon \to 0$ then $u''_{in} + u'_{in} = 0$ zolution is $u_{in} = A(1 - exp(-\tilde{x}))$. Matching gives $A = 1/4$. Hence composite expansion...

```
se = DSolve[{e u''[y] + u'[y] + e u[y] == (1 + y)/2, u[1] == 1,
    u[0] == 0}, u[y], {y, 0, 1}];
s = DSolve[{u'[y] == (1 + y)/2, u[1] == 1}, u[y], {y, 0, 1}];
```

```
Plot[{0, u[y] /. se /. e -> .25, u[y] /. se /. e -> .125,
  u[y] /. se /. e -> .05 u[y] /. se /. e -> .025, u[y] /. s}, {y, 0,
  1}, Frame -> True, FrameLabel -> {"x", "u(x)"}]
```



correction Ex 4

• with $\delta = \sqrt{\varepsilon}$, the eikonal $(S_0')^2 = 1$ then $S_0 = \pm x$
and $S_1 = cst$ hence the solution is the sum of $e^{\pm x/\sqrt{\varepsilon}}$:

$$y(x) = \frac{e^{x/\sqrt{\varepsilon}} - e^{-x/\sqrt{\varepsilon}}}{e^{1/\sqrt{\varepsilon}} - e^{-1/\sqrt{\varepsilon}}}$$

c'est exactement la solution exacte !

$$y(x) = \frac{\sinh(x/\sqrt{\varepsilon})}{\sinh(1/\sqrt{\varepsilon})}$$