SCIENCES SORBONNE UNIVERSITÉ

M2, Fluid mechanics 2018/2019
Friday, December 7th, 2018, Room 55-66-112

Multiscale Hydrodynamic Phenomena

Part I. : 90 minutes, NO documents

1. Quick Questions In few words :
1.1 What is the usual scale for friction for an incompressible flow at small Reynolds around a sphere ?
1.2 What is the usual scale for friction for an incompressible flow at large Reynolds around a sphere ?
1.3 What is the usual scale for friction for an incompressible flow at small Reynolds around a cylinder ?
1.4 What is the usual scale for friction for an incompressible flow at large Reynolds around a cylinder ?
1.5-6 ∂'Alembert, Heat : give the equation and a physical example of use of this equation.
1.7 Solution of Laplace equation in the upper half domain with Neumann BC in $y = 0$, Dirichlet 0 at infinity ?
1.8 What is the KDV equation ? Which balance is it ?
1.9 Find the self similar variable for linear KDV equation.

2. Exercice
Let us look at the following ordinary differential equation : (E_ε) $\quad \dfrac{d^2y}{dt^2} + 4\pi^2 y = -2\varepsilon \dfrac{dy}{dt},$ valid for any $t > 0$ with boundary conditions $y(0) = 0$ and $y'(0) = 2\pi$. Of course ε is a given small parameter.
We want to solve this problem with Multiple Scales.
2.1 Expand up to order ε : $y = y_0(t) + \varepsilon y_1(t)$, show that there is a problem for long times.
2.2 Introduce two time scales, $t_0 = t$ and $t_1 = \varepsilon t$
2.3 Compute $\partial/\partial t$ and $\partial^2/\partial t^2$
2.4 Solve the problem.
2.5 Suggest the plot of the solution.

3. Exercice
Consider the following equation (of course ε is a given small parameter)

$$(E_\varepsilon) \qquad \varepsilon^2 \frac{d^2u}{dx^2} + \frac{du}{dx} = \frac{1}{2}\cos(x) \text{ with } u(0) = 0 \quad u(\pi) = 1.$$

We want to solve this problem with the Matched Asymptotic Expansion method.
3.1 Why is this problem singular ?
3.2 What is the outer problem and what is the possible general form of the outer solution ?
3.3 What is the inner problem of (E_ε) and what is the inner solution ?
3.4 Suggest the plot of the inner, outer and composite solution.

4. Exercice
Solve with WKB approximation the problem

$$\varepsilon y''(x) = y(x) \text{ with } y(0) = 0, y(1) = 1$$

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Part II. : 1h 15 min all documents.

Multiscale Hydrodynamic Phenomena

Double Deck

This is a part of " FLOW THROUGH CONSTRICTED OR DILATED PIPES AND CHANNELS : PART 1 By F. T. SMITH " The Quarterly Journal of Mechanics and Applied Mathematics August 1976. We consider the flow in a pipe or between two parallel plates (in practice we will study the channel 2D case). The characteristic length is h_0 (either radius or half distance between plates) in an incompressible fluid with vanishing viscosity (Reynolds number $Re \to \infty$). First we consider and establish the classical Poiseuille profile (flow under a given pressure gradient, say $-\Pi$) say $\bar{U}_P(\bar{y})$. Then we will write the double deck equations (here, no triple deck).
1.1 Write 2D steady Navier Stokes equations with boundary conditions, with and without dimension (use h_0, use Q/h_0 as velocity scale, where Q is the flow rate).
1.2 Suppose a velocity profile invariant by translation in $\bar{x}$, find it by integration.
1.3 Write pressure and shear stress.
1.6 Write with dimensions the wall shear stress τ_w and pressure.
We consider the equations of double deck (as a variant of triple deck). We denote x_3 the relative small size of an indentation in $\bar{x} = 0$ on the wall which creates a perturbation so that $\bar{x} = x_3\hat{x}$ and in the lower deck $\hat{y} = \varepsilon\hat{y}$.
2.1 Main deck, $\bar{x} = x_3\hat{x}$ and $\bar{y}$ we do a perturbation of the Poiseuille flow substituted in NS equations, show

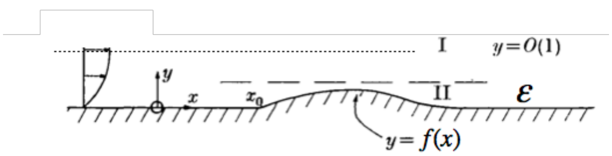


Fig. 1. The regions of flow through an indented channel (or symmetric pipe) when the Reynolds number ≫ 1.

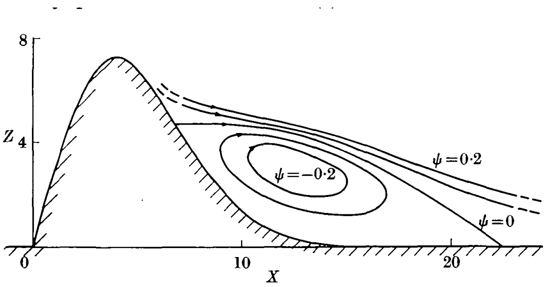


Fig. 6. Flow streamlines near and within the reversed-flow eddy for the localized constriction (4.1) when h = 3.

that :

$$\bar{u} = \bar{U}_P(\bar{y}) + \varepsilon_u \bar{A}(\hat{x})\frac{d\bar{U}_P(\bar{y})}{d\bar{y}} + ..., \quad \bar{v} = -\varepsilon_v \frac{d\bar{A}}{d\hat{x}}\bar{U}_P(\bar{y}) + ...,$$

Value of $\varepsilon_u, \varepsilon_v$? (suppose that pressure gradient is negligible)
2.2 Write the behavior/ boundary condition for the velocities of (2.1) at the bottom.
2.2 Write the behavior/ boundary condition for the velocities of (2.1) in $\bar{y} = 1$ the top. This is a symmetry line, deduce that $\bar{v} = 0$ in $\bar{y} = 1$. Deduce that $\bar{A}(\hat{x}) = 0$. Hence we notice that there is no upper deck (Poiseuille flow is the equivalent of Blasius, but there is no ideal fluid).

3.1 Lower deck in 2D : show that the Poiseuille profile is linear near the wall : consider a layer of thickness ε (for example $\bar{y} = \varepsilon\hat{y}$), and expand Poiseuille flow.
3.2 Deduce that the perturbation of longitudinal velocity are of order ε in the lower deck.
3.3 Show by dominant balance that $x_3 = Re^{-1}\varepsilon^3$. Then if the bump if of length $O(1)$ (i.e. if $\bar{x} = \hat{x}$), then $\varepsilon = Re^{-1/3}$.
3.4 Show that the pressure is $\varepsilon^2 = Re^{-2/3}$
3.5 Show that the velocity is linear (downstream) and that the velocity is linear by matching with the main deck.
3.6 The wall is defined by $\hat{y} = \hat{f}(\hat{x})$; boundary condition at the wall for velocity ?

4.1 Gather previous results and conclude that the final system is :

$$\frac{\partial \hat{u}}{\partial \hat{x}} + \frac{\partial \hat{v}}{\partial \hat{y}} = 0 \text{ and } \hat{u}\frac{\partial \hat{u}}{\partial \hat{x}} + \hat{v}\frac{\partial \hat{u}}{\partial \hat{y}} = -\frac{d\hat{p}}{d\hat{x}} + \frac{\partial^2 \hat{u}}{\partial \hat{y}^2}$$

for $\hat{x} \to -\infty$ we have $\hat{u} \to \bar{U}_p'(0)\hat{y}$ and for $\hat{y} \to \infty$ we have $\hat{u} \to \bar{U}_p'(0)\hat{y}$ and $\hat{u}(\hat{x}, \hat{f}(\hat{x},)) = \hat{v}(\hat{x}, \hat{f}(\hat{x},)) = 0$.
4.2 Show that $(\hat{u}, \hat{v}, \hat{p}) = (\hat{y}, 0, 0)$ is a base flow. interpretation ?
4.3 Show that for a small perturbation of the wall $\hat{f}(\hat{x}) = a\hat{f}_1(\hat{x})$, say $a \ll 1$ we linearise :

$$\hat{u} = U_P'(0)\hat{y} + au_1 \text{ and } \hat{v} = 0 + av_1 + ... \text{ and } \hat{p} = 0 + ap_1 + ...$$

Find the differential equation with u_1 (by elimination of v_1 and p_1)
4.4 By linearisation of the boundary condition find $u_1(\hat{x}, 0)$ as a function of f_1.
4.5 write the differential equation with u_1 in fourier space, so deal with modes $u_{1k}(\hat{y})e^{ik\hat{x}}$. show that $\partial_{\hat{y}} u_{1k}(\hat{y})$ solves an Airy equation $Ai''(\eta) = \eta Ai(\eta)$.
Find the relation between the perturbed value of the shear at the wall $\partial\hat{u}/\partial\hat{y}|_0$ and f_{1k}. Discussion of figure extracted from the original paper

FLOW THROUGH CONSTRICTED OR DILATED PIPES AND CHANNELS: PART 1

By F. T. SMITH

(*Mathematics Department, Imperial College, London*)

[Received 6 May 1975. Revise 26 September 1975]

SUMMARY

The motion of fluid through an indented non-symmetric channel or symmetric pipe is considered when the flow ahead of the indentation is fully-developed and the typical Reynolds number, K, is large. The theoretical description, for steady flows and slowly-varying indentations, is founded on a three-region structure, according to which the main core of fluid suffers a small inviscid displacement of its streamlines while the viscous motion close to the walls is nonlinear and forced along by the induced pressure-gradient. The displacement can be shown to be the average of the wall displacements, but the pressure must be calculated together with the viscous problem. Numerical solutions are presented both for linear constrictions or dilatations and for more confined ones, and flow separation, if it occurs, appears to be regular, with, for the more local indentation, a physically sensible eddy and reattachment ensuing downstream. The theory, which is believed to set out a rational approach to the solution, is valid provided the small inclination α of the indentation lies between $O(K^{-1})$ and $O(K^{-2/7})$ for a non-symmetric distortion of the wall, or between $O(K^{-1})$ and $O(K^{-1/7})$ for a symmetric distortion, in which ranges there is no substantial upstream influence. A companion paper (1) considers these limitations further and extends the theory to unsteady flows.

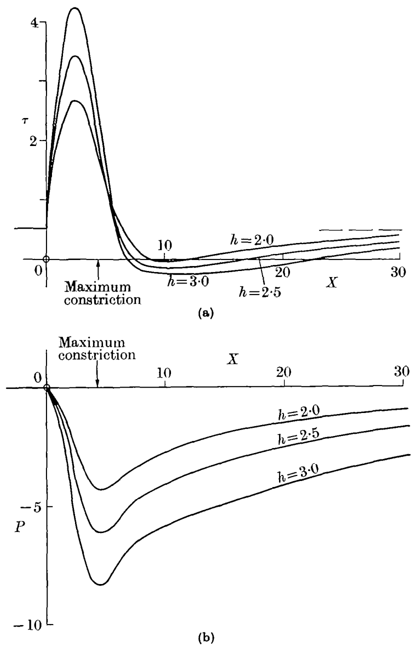


FIG. 5. Variation of (a) skin friction $\tau(X)$, (b) pressure $P(X)$ for the localized indentations (4.1) when $h = 2, 2{\cdot}5, 3$ (constricted motions).

correction Ex 1

want to see the $Log(Re)$

Π theorem does not work on

$$\partial_t \eta + c_0 \partial_x \eta + (c_0 h_0^{-1}) \eta \partial_x \eta + (c_0 h_0^2) \partial_x^3 \eta = 0$$

but on

$$\partial_t \eta + c_0 \partial_x \eta + (c_0 h_0^{-1}) \eta \partial_x \eta + K \partial_x^3 \eta = 0$$

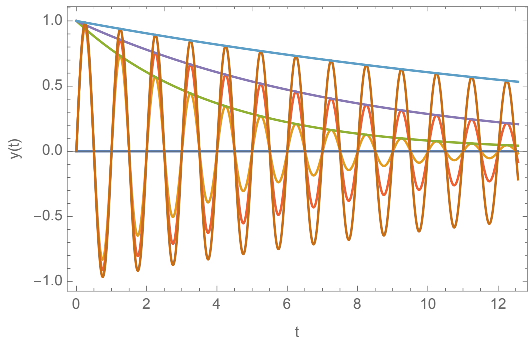
correction Ex 2

Exactly the curse with coefficient 2, $y_0 = \sin(2\pi t)$ and $y_1 = -\varepsilon t \sin(2\pi t)$.

so that the solution is $y = e^{-t_1} sin(t_0)$

```
 se = DSolve[{ y''[t] + 4 Pi^2 y[t] == - 2 e y'[t], y[0] == 0,
    y'[0] == 2 Pi}, y[t], {t, 0, 1}];
Plot[{0, y[t] /. se /. e -> .25,
  Exp[-t .25 ], y[t] /. se /. e -> .125,
  Exp[-t .125 ], y[t] /. se /. e -> .05,

  Exp[-t .05 ]}, {t, 0, 4 Pi}, Frame -> True,
 FrameLabel -> {"t", "y(t)"}]
```



correction Ex 3

If wi put $\varepsilon = 0$, we have an order one problem with 2 BC, so singular.

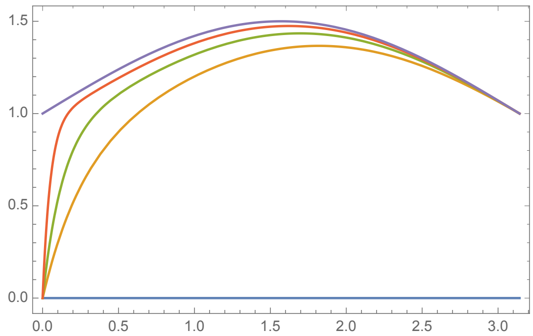
We find $u_{out} = 1 + \sin(x)/2$. We note that $u_{out}(0) = 1$, so we have to introduce an inner layer to full fit the 0 BC.

Change of scale $x = \delta \tilde{x}$, by dominant balance $\varepsilon^2 = \delta$, the problem is

$$\frac{d^2 \bar{u}}{d\bar{x}^2} + \frac{d\bar{u}}{d\bar{x}} = \varepsilon \cos(\varepsilon^2 \bar{x})$$

as $\varepsilon \to 0$ then $u''_{in} + u'_{in} = 0$ zolution is $u_{in} = A(1 - exp(-\tilde{x}))$. Matching gives $A = 1$. Hence composite expansion...

```
se = DSolve[{e u''[y] + u'[y] == Cos[y]/2, u[Pi] == 1, u[0] == 0},
   u[y], {y, 0, 1}];
s = DSolve[{u'[y] == Cos[y]/2, u[Pi] == 1}, u[y], {y, 0, 1}];
Plot[{0, u[y] /. se /. e -> .25, u[y] /. se /. e -> .125,
  u[y] /. se /. e -> .05, u[y] /. s}, {y, 0, Pi}, Frame -> True,
 FrameLabel -> {"x", "u(x)"}]
```



correction Ex 4

• with $\delta = \sqrt{\varepsilon}$, the eikonal $(S_0')^2 = 1$ then $S_0 = \pm x$ and $S_1 = cst$ hence the solution is the sum of $e^{\pm x/\sqrt{\varepsilon}}$