M2, Fluid mechanics 2016/2017

Friday, December 2nd, 2016

Multiscale Hydrodynamic Phenomena

Part I. : 90 minutes, NO documents

1. Quick Questions In few words :

1.1 Write incompressible NS equations in 2D, in developed formulation.

1.2 What is "dominant balance" ?

1.3 What is the usual scale for pressure in incompressible NS equation at small Reynolds ?

1.4 What is the usual scale for pressure in incompressible NS equation at large Reynolds ?

1.5 Write Prandtl equations with no pressure gradient (Blasius problem)

1.6 Show that the self similar solution is $\eta = y/\sqrt{x}$ (do not prove $f''' + ff'' = 0$)

1.7-8-9 ∂'Alembert, Laplace, Heat : give the equation and any simple solution of it.

1.10 What is the Bürgers equation ? Which balance is it ?

2. Exercice

Let us look at the following ordinary differential equation : (E_ε) $\quad \dfrac{d^2y}{dt^2} + \pi\varepsilon\dfrac{dy}{dt} + y = 0,$ valid for any $t > 0$ with boundary conditions $y(0) = 0$ and $y'(0) = 1$. Of course ε is a given small parameter.

We want to solve this problem with Multiple Scales.

2.1 Expand up to order ε : $y = y_0(t) + \varepsilon y_1(t)$, show that there is a problem for long times.

2.2 Introduce two time scales, $t_0 = t$ and $t_1 = \varepsilon t$

2.3 Compute $\partial/\partial t$ and $\partial^2/\partial t^2$

2.4 Solve the problem.

2.5 Suggest the plot of the solution.

2.6 What is the exact solution for any ε, compare.

3. Exercice

Consider the following equation (of course ε is a given small parameter)

$$(E_\varepsilon) \qquad \varepsilon^2\frac{d^2u}{dx^2} + \frac{du}{dx} = 1. \text{ with } u(0) = 0 \quad u(1) = \pi.$$

We want to solve this problem with the Matched Asymptotic Expansion method.

3.1 Why is this problem singular ?

3.2 What is the outer problem and what is the possible general form of the outer solution ?

3.3 What is the inner problem of (E_ε) and what is the inner solution ?

3.4 Suggest the plot of the inner and outer solution.

3.5 Next order solution.

3.6 What is the exact solution for any ε.

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Multiscale Hydrodynamic Phenomena

M2, Fluid mechanics 2014/2015
Friday, December 5th, 2014

Part II. : 1h 15 min all documents.

Drop Impact

This is a part of "Drop dynamics after impact on a solid wall : Theory and simulations" Jens Eggers, Marco A. Fontelos, Christophe Josserand, and Stéphane Zaleski, PoF 22 2010

Here we do not write Navier Stokes equations, we just estimate rough orders of magnitude.
1.1 What is the name of We Fr and Re defined by (1) ? What do they scale or balance ?
1.2 What is their dimension ?
1.3 With the values given in the text give the range of numerical values for them.
1.4 Conclude about the regimes (influence of viscosity large or small, influence of surface tension large or small, etc) ?
1.5 Write the equation of a sphere moving along z at velocity U downwards (see J. Philippi sketch, page 4)
1.6 Deduce by a first order expansion of the intersection of this sphere with the plane at very small time that the intersection locus is $r_j(t) \simeq \sqrt{t}$ (small time compared to a time τ defined with $R = D/2$ and U).
1.7 From a simple balance of Newton's law (time variation of momentum is the force, force is the mean pressure times the surface), deduce the estimate $P(t)/(\rho U^2) \sim \sqrt{\tau/t}$.

Liquid sheet extension.
2.1 Comment the choice of velocity field (2). Is in incompressible ? Rotational ?
2.2 Write Euler equation and deduce the pressure field.
2.3 Prove the formula (3).
2.4 Give a proof to the self similar equation (4).
2.5 Comment the scales for (4) and discuss figure 5.

Boundary Layer. The paper is with dimension, simplifications are more clear without dimension.
3.1 Discuss the equation (6) (7). Which equation is not written ? What will be the result for $\partial p/\partial z$?.
3.2 Give a proof to (8) (write equations without dimension).
3.3 Check the validity of $\partial_r p = 0$.
3.4 Check (9).
3.5 Check (10).
3.6 Check (11) and (12).
3.7 Check (13) and (14).
3.8 Figure 7 is not reproduced, can you imagine it ?
3.9 Comment the last paragraph about stability.
3.10 Final conclusion ?

PHYSICS OF FLUIDS 22, 062101 (2010)

Drop dynamics after impact on a solid wall: Theory and simulations

Jens Eggers,[1] Marco A. Fontelos,[2] Christophe Josserand,[3,4] and Stéphane Zaleski[3,4]
[1]School of Mathematics, University of Bristol, University Walk, Bristol BS8 1TW, United Kingdom
[2]CSIC-UAM-UC3M-UCM, Instituto de Ciencias Matemáticas (ICMAT), C/Serrano 123, 28006 Madrid, Spain
[3]CNRS, UMR 7190, Institut Jean Le Rond d'Alembert, F-75005 Paris, France
[4]UMR 7190, Institut Jean Le Rond d'Alembert, Université Paris 06, F-75005 Paris, France

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We study the impact of a fluid drop onto a planar solid surface at high speed so that at impact, kinetic energy dominates over surface energy and inertia dominates over viscous effects. As the drop spreads, it deforms into a thin film, whose thickness is limited by the growth of a viscous boundary layer near the solid wall. Owing to surface tension, the edge of the film retracts relative to the flow in the film and fluid collects into a toroidal rim bounding the film. Using mass and momentum conservation, we construct a model for the radius of the deposit as a function of time. At each stage, we perform detailed comparisons between theory and numerical simulations of the Navier–Stokes equation. © 2010 American Institute of Physics. [doi:10.1063/1.3432498]

I. INTRODUCTION

Understanding the impact of fluid drops on a solid wall is relevant to a large number of industrial and environmental processes. Examples include printing, cooling of surfaces by sprays, deposition of pesticides or nutrients on plant leaves, or natural rain. Of particular interest is the question of

(...)

Thus assuming a spherical drop upon impact, there remain three dimensionless parameters which determine the dynamics

$$\mathrm{We} = \frac{2\rho R U^2}{\gamma}, \quad \mathrm{Re} = \frac{2RU}{\nu}, \quad \mathrm{Fr} = \frac{U^2}{2gR}, \tag{1}$$

(...)

Our focus in this paper is on the regime of large We, Re, and Fr numbers. For example, for rain, the size and speed varies between R=0.5 mm and U=4.5 m/s for small drops and R=2 mm and U=9 m/s for large drops.[16] Thus

(...)

In this paper, numerical simulations will be used both as a guide to the proper modeling of impact and to compare to theoretical predictions quantitatively. We simulate the Navier–Stokes equation for the liquid with free surface boundary conditions at the interface (so that no outer fluid is

(...)

The analytical description of the first stage of impact is particularly difficult, as the drop undergoes a strong deformation and the flow is redirected from a vertical to a horizontal flow direction. The redirection of the flow is driven by very strong pressure gradients, as illustrated in Fig. 3. In agreement with classical impact theory,[20] the high pressure region occupies a volume with the same radius as the contact area of the drop with the solid. Using the horizontal momentum balance to such open domain and applying the pressure impact approach,[21] we obtain that the amplitude of pressure field $P(t)$ in this self-similar region behaves like

$$\frac{P(t)}{\rho U^2} \sim \sqrt{\frac{\tau}{t}},$$

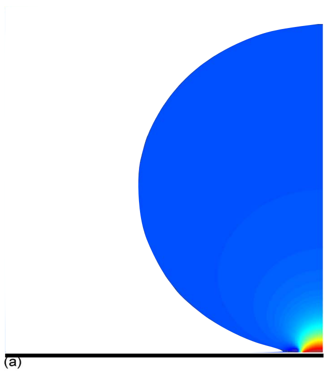


FIG. 3. (Color online) The pressure field corresponding to the same conditions in Fig. 1 for (a) t/τ=0.08, (b)

Calling the resulting maximum dimensionless height ℓ,

we divide the height by ℓ and the radius by $\sqrt{\ell}$. The result is equivalent to dividing the physical height by its maximum h_{max} and the physical radius by $(V/h_{max})^{1/2}$. As seen in Fig. 5 (right), the collapse to a self-similar profile is quite satisfactory over a period of very significant drop deformation. In both examples, the rescaled profiles for larger times come close to a profile of universal shape, which is well fitted by

$$H_s(x) = 1/(1 + Cx^2)^6, \tag{5}$$

II. LIQUID SHEET EXPANSION

We intend to describe the intermediate and long time dynamics of drop impacts. There, the pressure becomes insignificant as a driving force for the flow. This suggests the following hyperbolic flow pattern as the inviscid base flow, following the first interaction period:

$$v_r = \frac{r}{t}, \quad v_z = -\frac{2z}{t}. \tag{2}$$

We note the obvious fact that time can be replaced by $t+t_0$ here and in all of the following expressions. The physical significance of t_0 is the time it takes for the pressure to decay and the hyperbolic flow to establish itself. According to our previous arguments, t_0 is in the order of τ.

The flow (2) is an exact solution of the Euler equations with the pressure distribution $p(z,r,t)/\rho = -3z^2/t^2$. The pressure is thus decaying quickly in time, in agreement with the observation that the flow at intermediate times is no longer pressure-driven. The equation of motion for the convection of the free surface $h(r,t)$ by Eq. (2) is

$$\partial_t h + v_r \frac{\partial h}{\partial r} = v_z. \tag{3}$$

This equation has the similarity solution

$$h(r,t) = \frac{1}{t^2} H\left(\frac{r}{t}\right), \tag{4}$$

valid for any function H. Note that Eq. (4) permits to implement any initial condition for the shape of the drop at time t=0. It is an exact solution to the inviscid flow problem apart from the pressure boundary condition, which requires $p = -3h^2(z,t)/t^2$. As h goes down, this pressure quickly becomes insignificant and the boundary condition may be taken as one of vanishing pressure. This is consistent with the fact that at intermediate times, inertia dominates over surface tension, so the physical boundary condition is once more one of constant pressure.

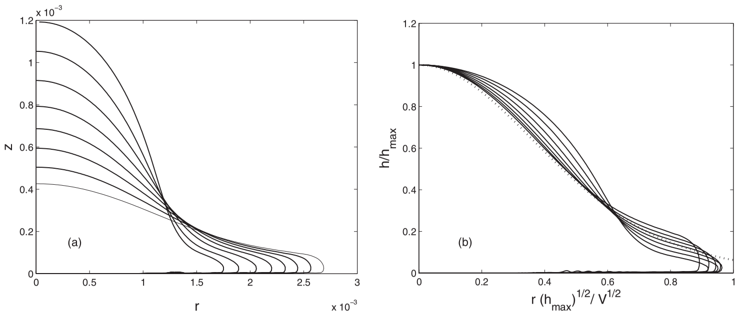


FIG. 5. drop profiles for different times. Right figures show the same profiles rescaled

III. BOUNDARY LAYER

In the Sec. II, we described an inviscid outer solution (2), which for large Reynolds numbers will develop a thin boundary layer near the solid surface. If the boundary layer is thinner than the drop thickness, we can neglect the effect of the free surface. Remarkably, solutions of this *time-dependent* boundary layer equations have been studied more than 50 years ago and can be found in Ref. 24; we repeat the axisymmetric version of the analysis here. Moreover, similar calculations have been proposed recently in the same context of drop impacts.[17]

The r component of the axisymmetric Navier–Stokes equation reads[25]

$$\partial_t v_r + v_r \partial_r v_r + v_z \partial_z v_r = -\partial_r p/\rho + \nu(\partial_r^2 v_r + \partial_z^2 v_r + \partial_r v_r/r - v_r/r^2) \quad (6)$$

$$\partial_r v_r + \partial_z v_z + v_r/r = 0 \quad (7)$$

is the incompressibility condition. According to the boundary layer theory of Prandtl,[26] a typical length scale in the z-direction (normal to the solid surface) is smaller by a factor of $1/\sqrt{\mathrm{Re}}$ than a corresponding scale in the r-direction. According to Eq. (7), on the other hand, $v_r/v_z = O(\sqrt{\mathrm{Re}})$. As a result, all terms on the left hand side of Eq. (6) are of the same order, but of the viscous terms, only the one with the highest number of z-derivatives survives.

Thus the boundary layer equation becomes

$$\partial_t v_r + v_r \partial_r v_r + v_z \partial_z v_r = -\partial_r p/\rho + \nu \partial_z^2 v_r. \quad (8)$$

As usual, the pressure distribution is that of the inviscid problem, which does not have any radial gradients, so it drops out from the equation. To satisfy incompressibility, it is most convenient to look for the stream function ψ

$$v_r = -\frac{\partial_z \psi}{r}, \quad v_z = \frac{\partial_r \psi}{r}. \quad (9)$$

In the inviscid case, $\psi = -r^2 z/t$. Moreover, the typical length scale for diffusion of vorticity is $\delta = \sqrt{\nu t}$, which suggests the ansatz

$$\psi = \sqrt{\nu}\frac{r^2}{\sqrt{t}} f\left(\frac{z}{\sqrt{\nu t}}\right). \quad (10)$$

For $f(\xi) = -\xi$, the inviscid result is recovered. Inserting Eq. (10) into the boundary layer Eq. (8), we find

$$f' + \eta f''/2 + f'^2 - 2ff'' = -f'''. \quad (11)$$

The boundary conditions are

$$f'(\infty) = -1, \quad f(0) = 0, \quad f'(0) = 0. \quad (12)$$

The numerical solution of Eq. (11), subject to Eq. (12), is shown in Fig. 6. We are not able to solve the equation exactly but report an empirical function which matches the true solution closely.

We now compare this boundary layer solution with our numerical simulations of the impacting drop. The z component of the velocity field is given in the boundary layer theory by

$$v_z = 2\sqrt{\frac{\nu}{t+t_0}} f\left[\frac{z}{\sqrt{\nu(t+t_0)}}\right], \quad (13)$$

so that its derivative becomes

$$\partial_z v_z = \frac{2}{t+t_0} f'\left[\frac{z}{\sqrt{\nu(t+t_0)}}\right].$$

Since the minimum of f' is -1 from the asymptotic matching, we can write for any time

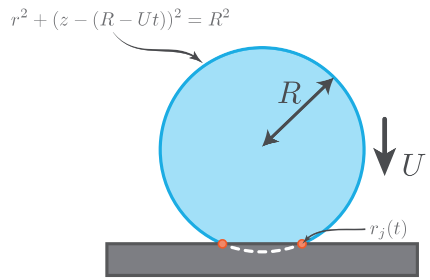
$$\partial_z v_z = (-M) f'\left[\frac{z}{\sqrt{2\nu(-M)}}\right], \quad (14)$$

where M is the minimum of $\partial_z v_z$. Therefore, rescaling the numerical velocity derivative profiles by $-M$ and the vertical coordinate z by $\sqrt{2\nu(-M)}$, all profiles should collapse onto the master curve f'. This is true for any time and or any value of the parameters Re, We.

Figure 7 (left) shows the numerical profiles $\partial_z v_z$ for two different sets of parameter values at different times. On the right, the profiles which have been rescaled according to Eq. (14) are compared to the theoretical boundary layer profile. We find good collapse as well as good agreement with the predicted similarity profile $f'(\xi)$. We have also checked the collapse for Reynolds numbers between 200 and 8000 and Weber numbers between 400 and 16 000, and found the results comparable to those shown in the two representative examples of Fig. 7.

Note that we have assumed the boundary layer to remain laminar, which we believe to be realistic. Namely, according to Ref. 27, p. 95, the critical Reynolds number Re_δ based on the *boundary layer thickness* is typically 400. On account of the smallness of δ, Re_δ is much smaller than Re, and well below the critical value.

J. Philippi PhD 2015



Impact of a drop on a plane. Sketch of the intersection of the spherical drop with the plane. At first order the drop remains spherical, and the corolla is so small that it is negligible. The wet radius $r_j(t)$ is then in square root of time.

FIGURE 1.14 – Schéma illustrant le raisonnement géométrique utilisé pour déterminer la position de l'intersection entre la goutte et la couche liquide $r_j(t)$. La goutte de rayon R tombe à vitesse constante U donc l'équation de la sphère dans le plan (r, z) est donnée par $r^2+(z-(R-Ut))^2 = R^2$ puisqu'aux temps courts la goutte a la forme d'une sphère tronquée. En prenant $z = 0$ et en négligeant le terme d'ordre 2 en temps, on obtient $r_j(t) = \sqrt{DUt}$.

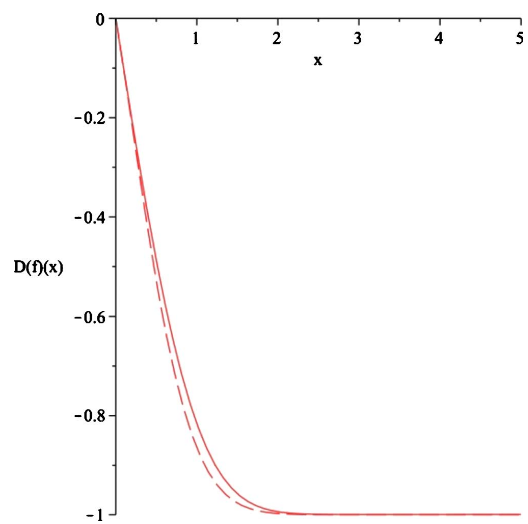


FIG. 6. (Color online) The similarity profile. The dashed line is $f'(\xi) = -1 + \exp\{-\xi - \xi^2\}$.

correction