M2, Fluid mechanics 2012/2013
Friday, December 7th, 2012

Multiscale Hydrodynamic Phenomena

Part I. : 30 minutes, NO documents

1. Quick Questions

In few words :

1.1 What is "dominant balance" ?
1.2 What is the dimension of the dynamic viscosity ?
1.3 What is the usual scale for pressure in incompressible NS equation ?
1.4 What is the usual scale for pressure in incompressible NS equation at small Reynolds ?
1.5 Which problem exhibits logarithms ?
1.6 What is "homogenisation"
1.7 What is the Friedrich equation ?
1.8 What is the Bürgers equation ?
1.9 What is the KDV equation ?
1.10 What is the natural selfsimilar variable for heat equation ?
1.11 In which one of the 3 decks of Triple Deck is flow separation ?

2. Exercice

Let us look at the following ordinary differential equation :

$$(E_\varepsilon) \qquad \varepsilon \frac{d^2y}{dx^2} + 1 - y = 0,$$

valid for $0 \leq x$, with boundary conditions $y(0) = 0$ and $y(\infty) = 1$. Of course ε is a given small parameter. We want to solve this problem with the Matched Asymptotic Expansion method (if you prefer use Multiple Scales or WKB).
2.1) Why is this problem singular ?
2.2) What is the outer problem obtained from (E_ε) and what is the possible general form of the outer solution ?
2.3) What is the inner problem of (E_ε) and what is the inner solution ?
2.4) Solve the problem at first order (up to power ε^0).
2.5) Suggest the plot of the inner and outer solution.
2.6) What is the exact solution for any ε.

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Part II. : 2h30min all documents.

Flow in elastic tubes blood flow in Arteries

The five sections are independent (at first order). They all correspond to the papers given at the end. Read the Kundu Cohen chapter (KC08) as introduction, the Ling and Atabek (LA72) paper, and the Womersley (W55) seminal paper.
Starting from Navier Stokes equations we want to obtain the LA72 equations (Question 1) and show that if integrated across the section (Question 2) and linearized we have the KC08 equations (Question 4), and that a viscous solution is W55 (Question 3). A long time an distance analysis is in Question 5.

Equations
1.1 What are the hypothesis to write equations (1) (2) and (3) in LA72 ?
1.2 There are two lengths of scale in the problem : the unperturbed radius say R_0, and a long scale, say λ corresponding to the blood pulse wave, we have $R_0 \ll \lambda$. Find in "2. Statement of the problem" of LA72 a clue of this and find in KC08 the relevant hypothesis. Note that in KC08, $A_0 = \pi R_0^2$ and $a_0 = R_0$.
1.3 We have another scale which is not always small : the variation of radius $R - R_0$ we define $R = R_0\bar{R}$. But we define as well $\bar{R} = 1 + \varepsilon\bar{R}_1$ This is a ε which is not always small. Find in "2. Statement of the problem" a clue of this. Find in KC08 the discussion of the small perturbation of radius.
1.4 Write (3) in LA72 with scale R_0 and λ for r and z. Introduce the blood flow velocity scale W_0.
What is the relevant scale for U_0 ? (note that KC08 uses u for w).
1.5 We use T the time of the pulse flow as the natural time scale (why not ?). Let us call P_0 the scale of pressure (around a given p_e pressure KC08).
Write (1) (2) and (3) from (LA72) with scales T, λ R_0 and W_0.
1.6 Present the equation (2) from (LA72) like this :

$$\frac{\partial \bar{w}}{\partial \bar{t}} + A\bar{w}\frac{\partial \bar{w}}{\partial \bar{z}} + B\bar{u}\frac{\partial \bar{w}}{\partial \bar{r}} = -E\frac{\partial \bar{p}}{\partial \bar{z}} + C(\frac{\partial^2 \bar{w}}{\partial \bar{r}^2} + \frac{\partial \bar{w}}{\bar{r}\partial \bar{r}} + D\frac{\partial^2 \bar{w}}{\partial \bar{z}^2})$$

identify A B C E and D
1.7 In §2.3, LA72 argue that we can neglect "the term $\partial^2 w/\partial z^2$,which is negligible in comparison with the radial derivatives", why ?
1.8 A special regime corresponds to the Womersley problem, were the flow is linearized, but viscosity present and were the pressure gradient is a given harmonic function e^{int}, this is equation (1) from W55. Show that $n = 2\pi/T$.
1.9 W55 defines what is called now the Womersley's number : α. Write it with T, R_0 and ν.
1.10 What suggests this linearized study from W55 and linearized analysis from KC08 about the magnitude of A and B from Quest. 1.5 ? Define an ε_2 with A and B (value of ε_2 according to KC08 ?).
1.11 E should be equal to one. Why ?

1.12 One of the next sentences is "Because of the small radial velocity and acceleration, the radial variation of pressure within the artery can also be neglected", prove it from LA72 (1).
1.13 Write the final system from (1) (2) and (3) with $R_0/\lambda \ll 1$, t with $E = 1$, with ε_2 and with $1/\alpha^2$. This system should look like a boundary layer system.
1.14 Is it consistent with LA72 (5) ?
1.15 Discuss the boundary conditions (6) (7) and (8) from LA72.
1.16 From (6), write a relation between W_0 and R_0 and λ and ε.
1.17 Write A and B with ε
1.18 Write P_0 with ε
1.19 Up now, we do not have λ the longitudinal scale. We turn now the interaction with the wall. In KC08 (17.55), the wall is supposed to be elastic, in LA72 (4) the tissues are supposed to have some weight. Define a small parameter relative to the mass m in LA72 (4).
1.20 Write KC08 (17.55) or (17.58) as $p - P_e = k(R - R_0)$
1.21 From this, show that we have a relation between λ/T and k and ρ and R_0.
1.22 Write the final system with all the boundary conditions and all the scales.

Equations before Integral method

Preparing the integral method, we take the system from LA72, and show that it can be integrated across the section. This will give an integral system.
2.1 Expand $\frac{d(\phi r)}{dr}$ and simplify $\frac{d\phi}{dr} + \frac{\phi}{r}$
2.2 From equation (3) (7) and (8) of LA72, show that $Q = \int_0^R 2\pi r w dr$ the flux of mass is linked to $\partial R/\partial t$.
2.3 Show that LA72 (2) is

$$\frac{\partial}{\partial \bar{t}}(\bar{r}\bar{w}) + \varepsilon(\frac{\partial}{\partial \bar{z}}(\bar{r}\bar{w}^2) + \frac{\partial}{\partial \bar{r}}(\bar{r}\bar{u}\bar{w})) = -\bar{r}\frac{\partial}{\partial \bar{z}}\bar{p} + \frac{2\pi}{\alpha^2}(\frac{\partial}{\partial \bar{r}}(\bar{r}\frac{\partial}{\partial \bar{r}}\bar{w}))$$

2.4 We define $Q_2 = \int_0^R 2\pi r w^2 dr$ the flux of momentum. Write 2.3 with $\bar{Q}_2$ and $\bar{Q}$ and the value of $\bar{\tau}_w = \frac{\partial}{\partial \bar{r}}\bar{w}$ at the wall. Of course the final integral system is not closed, as we do not know the relation between Q and Q_2, and between τ_w and Q, this is done with Womersley profiles.

Womersley famous solution for pulsatile flow in tubes.

3.1 Show from question 1.X to 2.X that equation (1) of W55 is relevant under some hypothesis, note that the factor $2\pi/\alpha^2$ that you have maybe, comes from the choice of time scale. Use now Womersley notations.
3.2 Verify that (3) is a solution of (1).
3.3 Suppose that α is small. What does it mean in terms of frequency and viscosity ?
3.4 Suppose $\alpha = 0$, show that W55 (2) gives Poiseuille flow in this case, is it a regular or singular problem ?
3.5 Suppose that α is large. What does it mean in terms of frequency and viscosity ?
3.6 Suppose $1/\alpha = 0$, show that W55 (2) is a singular problem ?
3.7 Introduce a boundary layer near the wall $y = 1 - \varepsilon\tilde{y}$, why this form ?
3.8 Show that the inner problem is exponential.
3.9 Plot the solution.
3.10 Expand (3) and show that $\alpha \to 0$ gives Poiseuille.
3.11 Compute Q_2 and τ_w as a function of Q in Poiseuille case.
3.11 Expand (3) and show that $\alpha^{-1} \to 0$ gives the previous exponential solution (difficult).

The linear wave solution

Along questions 1.X we established the long wave approximation, in 2.X we established the integral equations. At this point, we needed some information destroyed by the integration, this information is the shape of the velocity. A good idea is to say that the velocity profile looks like a Womersley profile that we established in 3.X. In fact we supposed a Poiseuille profile, with this closure one closes the system, so we have (17.53) of KC08 in full form :

$$\frac{\partial \pi R^2}{\partial t} + \frac{\partial}{\partial z} Q = 0, \text{ and } \frac{\partial}{\partial t} Q + \frac{\partial}{\partial z}\left(\frac{4}{3}\frac{Q^2}{\pi R^2}\right) = \pi R^2 \frac{\partial}{\partial z} p - 8\nu \frac{Q}{R^2} \text{ and } p - p_e = k(R - R_0).$$

4.1 Show that the previous system is the one we obtained. Deduce that KC08 (17.54) is wrong.
4.2 What hypothesis allow us to write (17.56) and (17.57) ?
4.3 Compute the Moens-Korteweg velocity with k.
4.4 Write a ∂'Alembert equation for the pressure.
4.5 General solution of 4.4 ?
4.6 The artery is supposed to be infinite, what does it mean in term of time for a pulse given at the entrance ?
4.7 A pulse is given in $z = 0$, $p = p_0 \sin(2\pi t/T)$ for $0 < t < 1/2$, what is the solution in z t ?
4.8 Of course arteries are not infinite, estimate λ from KC08, conclusions ?

Long distance behaviour

In fact the pressure may be expressed as $\bar{p} = \bar{R} + \varepsilon_v \frac{\partial \bar{R}}{\partial t}$ if we suppose a Kelvin Voigt model for the relation between the pressure and the change of radius.
5.1 Write the constant of the dimensional Kelvin-Voigt law with the previous scales and ε_v.
5.2 With suitable scales, small perturbations of the flow (neglect non linear terms in the advection) in a viscoelatic artery are :

$$\begin{cases} \dfrac{\partial \bar{R}}{\partial \bar{t}} &= -\dfrac{\partial \bar{Q}}{\partial \bar{x}} \\ \dfrac{\partial \bar{Q}}{\partial \bar{t}} &= -\dfrac{\partial \bar{R}}{\partial \bar{x}} + \varepsilon_v \dfrac{\partial^2 \bar{Q}}{\partial \bar{x}^2} \end{cases}$$

is it correct ?
5.3 Show that a multiple scale analysis may be done to obtain the behaviour of a pulse wave going to the right in the tube.
5.4 Deduce that in the rigth moving frame, with suitable variables $\bar{\tau}$ and $\bar{\xi}$:

$$\frac{\partial}{\partial \bar{\tau}} \bar{R}_1 = \frac{1}{2} \frac{\partial^2}{\partial \bar{\xi}^2} \bar{R}_1$$

5.5 Show that we can define a selfsimilar solution of 5.4 of constant integral on the domain in ξ (i.e. $\int_0^\infty \bar{R}_1 d\bar{\xi} = 1$).
5.6 Plot the propagation of a pulse along an infinite artery.

Bibliography

Womersley 1955 Philosophical Magazine 46 : 199–221.
Ling & Atabek 1972, J.. Fluid Mech. (1972), vol. 55, part 3, pp493-511
Kundu & Cohen 2008 Fluid Mechanics

W55

XXIV. Oscillatory Motion of a Viscous Liquid in a Thin-W a Elastic Tube—I : The Linear Approximation for Long Waves

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[Revised MS. received October 19, 1954]

§ 1. Introduction and Summary

The problem of determining the motion of a liquid in an elastic tube when subjected to a pressure-gradient which is a periodic function of the time arises in connection with the flow of blood in the larger arteries (Helps and McDonald 1954, Womersley 1954). Attempts have been made in the past to measure the rate of flow in the aorta and femoral artery of the dog and rabbit (Shipley, Gregg and Schroeder 1943) and to relate these observations to the varying pressure. In the absence of an adequate mathematical theory, these were not very successful. More recent determinations by direct observation of the motion through the translucent arterial wall, using high-speed cinematography (Helps and McDonald 1954, Womersley 1954) have been accompanied by measurements, not only of pulse-pressure, but of pressure-gradient. Fair agreement has been shown between the observed rates of flow and a simple solution for oscillatory motion of a viscous liquid in a tube with rigid walls (Womersley, in press).

In this paper the corresponding solution for a thin-walled elastic tube is given, it being assumed that the effect of the inertia terms in the equations of viscous fluid motion can be neglected. An approximate correction for the effect of the inertia terms will be presented in Part II of this communication.

It is shown that, when the liquid contained in the tube is viscous, the pressure-wave cannot be propagated without distortion. Not only is the motion damped, but the wave-velocity rises as the frequency increases. For constant frequency, the wave-velocity rises as the viscosity of the liquid decreases, tending to an asymptotic value equal to that for a perfect fluid given by Lamb (1898). It is also shown that the longitudinal oscillation of the walls of the tube (caused by the viscous drag on its inner surface) is important in determining the rate of flow, which may be 10% greater than that in a rigid tube under the same pressure-gradient.

* Communicated by Sir Geoffrey Taylor.

§ . The Solution for the Rigid Tube

The si ple solution for the oscillatory motion of a vis s liquid in a rigid tu , under a simple-harmonic pressure gradient, was given by Lambossy (1952) who gave the formulae for velocity and viscous drag. He did not apply his result to the motion of the blood in arteries, being solely concerned with the effect of the viscous drag on the frequency response of pressure recording instruments. The author obtained the same result independently, in a different form, and derived the expression for the rate of flow. This result has been used to predict rates of flow from observed pressure-gradient (Womersley, in press). This solution is repeated here for completeness. Let R be the radius of the tube, w the velocity along the tube, Ae^{int} the pressure-gradient, μ the viscosity of the liquid, ρ the density of the liquid, $\nu=\mu/\rho$ the kinematic viscosity.

The equation of motion is

$$\frac{\partial^2 w}{\partial r^2}+\frac{1}{r}\frac{\partial w}{\partial r}-\frac{1}{\nu}\frac{\partial w}{\partial t}=\frac{A}{\rho}e^{int}. \qquad (1)$$

Let $y=r/R$ and $w=ue^{int}$ and let the non-dimensional quantity $R\sqrt{(n/\nu)}$ be denoted by α. The equation for u is

$$\frac{d^2u}{dy^2}+\frac{1}{y}\frac{du}{dy}-i\alpha^2 u=\frac{AR^2}{\mu} \qquad (2)$$

and therefore

$$w=\frac{AR^2}{\mu}\cdot\frac{1}{i^3\alpha^2}\left\{1-\frac{J_0(\alpha i^{3/2}y)}{J_0(\alpha i^{3/2})}\right\}e^{int}. \qquad (3)$$

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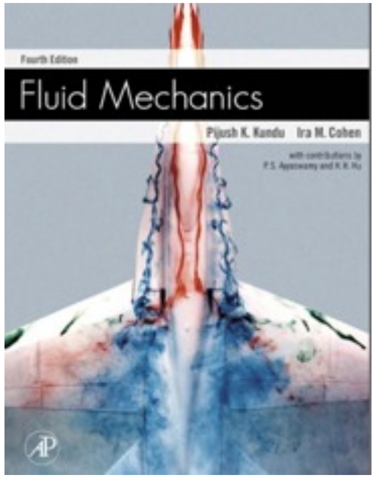

KC08

Pulse Wave Propagation in an Elastic Tube: Inviscid Theory

Consider a homogeneous, incompressible, and inviscid fluid in an infinitely long, horizontal, cylindrical, thin walled, elastic tube. Let the fluid be initially at rest. The propagation of a disturbance wave of small amplitude and long wave length compared to the tube radius is of interest to us. In particular, we wish to calculate the wave speed. Since the disturbance wave length is much greater than the tube diameter, the time dependent internal pressure can be taken to be a function only of (x, t).

Before we embark on developing the solution, we need to understand the inviscid approximation. For flow in large arteries, the Reynolds and Womersley numbers are large, the wall boundary layers are very thin compared to the radius of the vessel. The inviscid approximation may be useful in giving us insights in understanding such flows. Clearly, this will not be the case with arterioles, venules and capillaries. However, the inviscid analysis is strictly of limited use since it is the viscous stress that is dominant in determining flow stability in large arteries.

Under the various conditions prescribed, the resulting flow may be treated as one dimensional.

Let $A(x, t)$ and $u(x, t)$ denote the the cross sectional area of the tube and the longitudinal velocity component, respectively. The continuity equation is:

$$\frac{\partial A}{\partial t} + \frac{\partial (Au)}{\partial x} = 0, \tag{17.53}$$

and, the equation for the conservation of momentum is:

$$\rho A\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x}\right) = -\frac{\partial\left((p - p_e)\,A\right)}{\partial x}, \tag{17.54}$$

where $(p - p_e)$ is the transmural pressure difference. Since the tube wall is assumed to be elastic (not viscoelastic), under the further assumption that A depends on the transmural pressure difference $(p - p_e)$ alone, and the material obeys Hooke's law, we have from equation (17.41), the pressure—radius relationship (referred to as "tube law"),

$$p - p_e = \frac{Eh}{a_0}\left(1 - \frac{a_0}{a}\right) = \frac{Eh}{a_0}\left[1 - \left(\frac{A_0}{A}\right)^{\frac{1}{2}}\right], \tag{17.55}$$

where $A = \pi a^2$, and $A_0 = \pi a_0^2$. The equations (17.53), (17.54), and (17.55) govern the wave propagation. We may simplify this equation system further by linearizing it. This is possible if the pressure amplitude $(p - p_e)$ compared to p_0, the induced fluid speed u, and $(A - A_0)$ compared to A_0, and their derivatives are all small. If the pulse is moving slowly relative to the speed of sound in the fluid, the wave amplitude is much smaller than the wave length, and the distension at one cross section has no effect on the distension elsewhere, the assumptions are reasonable. As discussed by Pedley (2000), in normal human beings, the mean blood pressure, relative to atmospheric, at the level of the heart is about 100 mmHg, and there is a cyclical variation between 80 and 120 mmHg, so the amplitude-to-mean ratio is 0.2,

which is reasonably small. Also, in the ascending aorta, the pulse wave speed, c, is about 5 m/s, and the maximum value of u is about 1 m/s, and (u/c) is also around 0.2. In that case, the system of equations reduce to

$$\frac{\partial A}{\partial t} + A_0\frac{\partial u}{\partial x} = 0, \tag{17.56}$$

and,

$$\rho\frac{\partial u}{\partial t} = -\frac{\partial p}{\partial x}, \tag{17.57}$$

and,

$$p - p_e = \frac{Eh}{2a_0A_0}\left(A - A_0\right), \quad \text{and} \quad \frac{\partial p}{\partial A} = \frac{Eh}{2a_0A_0} \tag{17.58}$$

Differentiating equation (17.56) with respect to t and equation (17.57) with respect to x, and subtracting the resulting equations, we get,

$$\frac{\partial^2 A}{\partial t^2} = \frac{A_0}{\rho}\frac{\partial^2 p}{\partial x^2}, \tag{17.59}$$

and with equation (17.58), we obtain,

$$\frac{\partial^2 p}{\partial t^2} = \frac{Eh}{2a_0A_0}\frac{\partial^2 A}{\partial t^2} = \frac{\partial p}{\partial A}\frac{A_0}{\rho}\frac{\partial^2 p}{\partial x^2}. \tag{17.60}$$

Combining equations (17.59) and (17.60), we produce,

$$\frac{\partial^2 p}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 p}{\partial t^2}, \quad \text{or,} \quad \frac{\partial^2 p}{\partial t^2} = c^2(A_0)\frac{\partial^2 p}{\partial x^2}, \tag{17.61}$$

where, $c^2 = \frac{Eh}{2\rho a_0} = \frac{A}{\rho}\frac{dp}{dA}$. Equation (17.61) is the wave equation, and the quantity,

$$c = \sqrt{\frac{Eh}{2\rho a_0}} = \sqrt{\frac{A}{\rho}\frac{dp}{dA}}, \tag{17.62}$$

is the speed of propagation of the pressure pulse. This is known as the Moens-Korteweg wave speed. If the thin wall assumption is not made, following Fung (1997), by evaluating the strain on the midwall of the tube,

$$c = \sqrt{\frac{Eh}{2\rho\left(a_0 + h/2\right)}}, \tag{17.63}$$

Next, similar to equation (17.61), we can develop,

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}, \tag{17.64}$$

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A nonlinear analysis of pulsatile flow in arteries

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LA72

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An approximate numerical method for calculating flow profiles in arteries is developed. The theory takes into account the nonlinear terms of the Navier–Stokes equations as well as the nonlinear behaviour and large deformations of the arterial wall. Through the locally measured values of the pressure, pressure gradient and pressure–radius function the velocity distribution and wall shear at a given location along the artery can be determined. The computed results agree well with the corresponding experimental data.

1. Introduction

The study of blood flow in arteries has occupied the attention of the researchers for over 150 years. Like most of the problems of life sciences, it is a complex one and has defied all attempts at a completely satisfactory solution. Mathematical treatment of the problem has been subjected to constant changes and modifications to account for new evidence uncovered through improved experimental measurements. One can trace the history and development of the problem from numerous review articles. The most consistent treatment of the problem was given by Womersley (1957). Later, his analysis was extended by others to include the effect of initial stresses, perivascular tethering and orthotropic and viscoelastic behaviour of the arterial wall. A detailed comparison of this group of articles is given by Cox (1969).

Womersley's theory and its extensions are based on the linearized Navier–Stokes equations and small elastic deformations. Although they are shown to be satisfactory in describing certain aspects of the flow in small arteries, they fail to give an adequate representation of the flow field, especially in large arteries, see Fry, Griggs & Greenfield (1964) and Ling, Atabek & Carmody (1969). Because of the large dynamic storage effect of these arteries, the nonlinear convective acceleration terms of the Navier–Stokes equations are no longer negligible. Moreover, the walls of arteries undergo large deformations. As a result of this, both the geometric and elastic nonlinear effects come into play, see Ling (1970).

To take these factors into account an approximate numerical method is developed. The method, assuming axially symmetric flow, predicts the velocity distribution and wall shear at a given location in terms of locally measured values of the pressure, pressure gradient and pressure–radius relation. The results of computations show good agreement with the corresponding experimental data. The simplicity of the method may make it useful in circulatory research, where detailed flow characteristics are required under a wide range of arterial pressures and heart rates.

2. Statement of the problem

Pulse propagation phenomena in arteries are caused by the interaction of blood with the elastic arterial wall. Therefore, the mathematical statement of the problem should include equations which govern the motion of blood and the motion of the arterial wall, and also the relations (boundary conditions) which connect these two motions with each other. This set of equations and conditions make a formidable boundary-value problem. However, the problem can be greatly simplified through the following three experimental observations.

(i) The radial motion of the arterial wall is primarily dictated by the pressure wave.

(ii) The perivascular tethering has a strong dampening effect on the longitudinal motion of the arterial wall, hence this motion may be neglected, see Patel, Greenfield & Fry (1964).

(iii) To a large extent velocity profiles are developed locally as the pressure wave propagates along the artery, hence they do not carry a significant amount of momentum history from far upstream. This somewhat unusual behaviour of the flow can be explained in terms of the combined effects of fast propagation of the pressure wave and large distensibility and taper of the arterial wall. For example, during systole, the heart of a medium-sized dog ejects approximately 25 ml of blood into the ascending aorta. Assuming that the cross-sectional area of the root of the aorta during systole to be 4·5 cm^2, the corresponding displacement of the blood along the aorta will be only 5·5 cm. During this time a fast-rising pressure-gradient wave front, approximately 12 cm in width, accelerates blood locally as it sweeps along the aorta with a speed of $\sim$ 400 cm/s. As a result, in most parts of the aorta, the momentum boundary layer is developed locally with a minor contribution from the preceding cardiac cycles. This momentum layer is significantly reduced by the local convective accelerations which are generated through both the natural taper of the vessel and taper due to the wave front. In addition, the radial velocity of the flow near the expanding wall will generate a similar effect. These two latter effects will be discussed in detail in §4.3. After closure of the aortic valve, blood in the root of aorta is essentially at rest. At distal locations, the overall passive contraction of the arterial wall will create a basic flow which will be increasing with distance owing to the integration of wall flux. The magnitude of this diastolic flow is small and, as before, the momentum boundary layer is developed locally and is reduced by the local convective acceleration due to arterial taper. Thus, within a cardiac cycle, the mean momentum defect produced by the mean wall shear is effectively absorbed by the mean positive convective accelerations. For this reason, little information about the flow is convected far downstream, and the entrance effect is essentially confined to a displacement distance corresponding to one heart beat. The asymmetrical velocity profiles created by an arterial branch are found to be confined

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to a distance of 10 diameters, which is again approximately equal to the displacement length of blood for one heart beat, see Ling, Atabek & Carmody (1969). Similarly, asymmetrical velocity profiles and secondary flows developed by the aortic arch and arterial branches are found to be localized and are not convected into the descending aorta.

The first two of the above observations will permit one to decouple the motion of the arterial wall from the motion of the blood, while the third observation will allow one to simplify the equations governing the motion of blood.

2.1. *Equations governing the motion of blood*

For this problem blood can be taken as an incompressible Newtonian fluid. We shall use the cylindrical co-ordinates r, θ and z, with z along the axis of the vessel. Since our aim is to use locally measured quantities to predict the local flow characteristics, the choice of the origin of z is immaterial.

The motion of blood is governed by the Navier–Stokes equations and the equation of continuity. We shall assume that the flow is axially symmetric and body forces are absent. Under these assumptions the governing equations have the following form:

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial r}+w\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial r}+\nu\left(\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{\partial^2 u}{\partial z^2}-\frac{u}{r^2}\right), \quad (1)$$

$$\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial r}+w\frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z}+\nu\left(\frac{\partial^2 w}{\partial r^2}+\frac{1}{r}\frac{\partial w}{\partial r}+\frac{\partial^2 w}{\partial z^2}\right), \quad (2)$$

$$\frac{\partial u}{\partial r}+\frac{u}{r}+\frac{\partial w}{\partial z} = 0. \quad (3)$$

Here t denotes time, u and w denote the components of the fluid velocity in the r and z directions, respectively, p is the pressure, ρ is the density and ν is the kinematic viscosity of blood.

2.2. *Motion of the arterial wall*

As is indicated above, the longitudinal motion of the arterial wall is significantly arrested by the perivascular tethering. Here we shall neglect this component of the arterial motion and seek a simple relation connecting local values of the radial pressure force, mass and elastic response of the arterial wall. Let $R = R(z,t)$ denote the inner radius of the artery. We assume that the variation of R with pressure is known (determined experimentally). Let us denote this functional relation by $p = P(R)$. Although the effect of arterial taper (both the natural taper and the generated taper due to the wave front) on the motion of blood is important because of convective acceleration, its effect on the radial motion of artery is negligible. Therefore the equation of motion for the arterial wall can be written as

$$\frac{m}{2\pi R}\frac{\partial^2 R}{\partial t^2} = p(z,t)-P(R). \quad (4)$$

Here m denotes the effective mass of the artery per unit length in its natural state. Equation (4) is valid only locally (for a fixed z) and to emphasize this point we

use the partial derivative with respect to time. With p known as a function of time and the local elastic response of an artery, starting with homogeneous initial conditions, one can integrate this equation numerically to determine R as a function of time.

2.3. *Simplification of the equation of motion*

Equation (2) may be simplified by dropping the term $\partial^2 w/\partial z^2$, which is negligible in comparison with the radial derivatives. Because of the small radial velocity and acceleration, the radial variation of pressure within the artery can also be neglected. Therefore the longitudinal pressure gradient $\partial p/\partial z$ may be considered as a function of z and t only. Let us take $-\rho^{-1}(\partial p/\partial z) = F(z,t)$. Hereafter, we shall assume that $F(z,t)$ is an experimentally determined, known function. Then (2) may be written as

$$\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial r}+w\frac{\partial w}{\partial z} = F(z,t)+\nu\left(\frac{\partial^2 w}{\partial r^2}+\frac{1}{r}\frac{\partial w}{\partial r}\right). \quad (5)$$

As a result of the replacement of $\partial p/\partial z$ with a known function, (5) now contains only two unknown dependent variables, u and w. Equation (3) also contains only these dependent variables. Therefore, these two equations together are sufficient to determine both u and w. Of course we have to supplement them with proper boundary and initial conditions. In the radial direction the boundary conditions are

$$u(r,z,t)|_{r=R(z,t)} = \partial R/\partial t, \quad (6)$$

$$w(r,z,t)|_{r=R(z,t)} = 0, \quad (7)$$

$$[\partial w(r,z,t)/\partial r]_{r=0} = 0. \quad (8)$$

Boundary conditions in the z direction reflect the effect of upstream and downstream flows on the local flow. Since the aim is to determine the local flow from the locally measured flow properties, it is necessary to find a way to eliminate the need for boundary conditions on z. This will be accomplished, later, by eliminating all explicit z dependence from the equations.

1.4 continuity $W_0/\lambda = U_0/R_0$ so $U_0 = R_0 W_0/\lambda$

1.6 It is straightforward that $A = B = TW_0/\lambda$ and $E = P_0 T/(\rho\lambda W_0)$ $C = \nu T/R_0^2$ and $D = R_0^2/\lambda^2$.

1.10 $A = B = \varepsilon_2$.

1.16 LA72 (6) boundary condition for the transverse velocity $U_0 = \varepsilon R_0/T$ continuity $W_0/\lambda = U_0/R_0$ so $W_0 = \varepsilon\lambda/T$

1.17 $A = B = W_0 T/\lambda$ so $A = B = \varepsilon$ hence $\varepsilon_2 = \varepsilon$.

1.18 $P_0 = \rho\lambda W_0/T$ so that $P_0 = \varepsilon\rho\lambda^2/T^2$