# Master SdI mention M2FA, Fluid Mechanics program

Hydrodynamics. P.-Y. Lagrée and S. Zaleski.

Test

December 4, 2009

All documentation is authorized except for reference [1]. Internet access is not allowed.

General remark: there is no need to repeat in your paper the proofs and derivations given in the course. Questions are of variable difficulty. Answers to this part of the exam should be given on a separate paper.

**1** In all the following questions we consider slow, viscous, Newtonian, incompressible flow. Consider a swimming microorganism that moves by distortion of its body shape. The equations are, with standard notations

$$-\nabla p + \mu \nabla^2 \mathbf{u} = 0 = \nabla \cdot \sigma \tag{1}$$

and

$$\nabla \cdot \mathbf{u} = 0 \tag{2}$$

where $\mathbf{u}$ and $p$ are the velocity and the pressure and $\sigma$ is the stress tensor. Let $(\mathbf{u}, \sigma)$ be the velocity and stress fields that are the solution to the above eqations such that there is no net force or torque on the swimming body, and let $(\hat{\mathbf{u}}, \hat{\sigma})$ be the solution of the equations for translation of the same body with the same shape at velocity $\hat{\mathbf{U}}$ when acted upon by an external force $\hat{\mathbf{F}}(t)$.

The reciprocal theorem (Batchelor) states that the solutions $(\mathbf{u}, \sigma)$ and $(\hat{\mathbf{u}}, \hat{\sigma})$ are related by

$$\int_{S(t)} \mathbf{n} \cdot \hat{\sigma} \cdot \mathbf{u} \, \mathrm{dS} = \int_{S(t)} \mathbf{n} \cdot \sigma \cdot \hat{\mathbf{u}} \, \mathrm{dS} \tag{3}$$

where $S(t)$ is the instantaneous boundary of the swimming object,and $\mathbf{n}$ is the unit outward normal to $S$. Show that the right hand side of (3) vanishes. (Hint: the body is self propelled).

**2** The surface velocity for the self propelled swimmer is then decomposed into a translational velocity $\mathbf{U}(t)$ and a disturbance motion $\mathbf{u}'$. Show that

$$\hat{\mathbf{F}}(t) \cdot \mathbf{U}(t) = -\int_{S(t)} \mathbf{n} \cdot \hat{\sigma} \cdot \mathbf{u}' \, \mathrm{dS}. \tag{4}$$

**3** Consider now a sphere of radius $a$. The surface of the spherical microorganism may in effect have complex motions because of the presence of motor proteins or cilia. Show that

$$\mathbf{U}(t) = -\frac{1}{4\pi a^2} \int_S \mathbf{u}' \, \mathrm{dS}. \tag{5}$$

**Reference** [1] Stone H. and Samuel, A. 1996, Propulsion of microorganisms by surface distortions, Phys. Rev. Lett. **77**, 4102.

**M2, Fluid mechanics** 2009/2010
Friday, December 4th, 2009

**Multiscale Hydrodynamic Phenomena**

### Exercice 1

Let us look at the following ordinary differential equation :

$$(E_\varepsilon) \qquad \varepsilon\frac{d^2y}{dx^2} + x^2\frac{dy}{dx} - y = 0,$$

valid for $0 \leq x \leq 1$, with boundary conditions $y(0) = 1$ and $y(1) = 1$. Of course $\varepsilon$ is a given small parameter.
We want to solve this problem with the Matched Asymptotic Expansion method.
1) Why is this problem singular ?
2) What is the outer problem obtained from $(E_\varepsilon)$ and what is the possible general form of the outer solution ?
3) Discuss the position of the boundary layer ($x = 0$ or $x = 1$) find the new local scale $\delta(\varepsilon)$ at the singular point.
4) What is the inner problem of $(E_\varepsilon)$ and what is the inner solution ?
5) Solve the problem at first order (up to power $\varepsilon^0$).
6) Suggest the plot of the inner and outer solution.
7) Construct the composite expansion and draw it for a small $\varepsilon$.

### Exercice 2

## Separation of Jets or thermal boundary layers from a Wall

*The following sentences are extracted from Smith & Duck (Q.Jl Mech appl. Math. Vol XXX Pt 2, 1977 pp 143 -156) "Separation of Jets or thermal boundary layer from a Wall" :*
"The need to investigate the structure of a jet flow near its point of departure from a wall arises in many diverse situations. Examples are found in rotating fluids in free convection boundary layers as well for wall jet *per se* near a concave corner or other discontinuity in wall conditions. In the problem we have in mind a steady planar jet-like boundary layer flow has been established along a fixed smooth surface by some imposed constraint. the jet must then leave the wall. The cause of this expulsion from the wall may be an impending collision with a jet moving in the opposite direction or a finite change in the wall slope (...)."
*See figure* **??** *for a sketch. Then the basic flow at high Reynolds number is presented :*
"For these problems there exists a basic scale $U_\infty$, say, a typical length scale $L$ and, if $\nu$ represents the kinematic viscosity of the fluid, the Reynolds number $Re = U_\infty L/\nu$ is large. It is supposed that, due to one the agencies mentioned above, the viscous jet flow has been set up in a boundary layer of typical thickness $O(LRe^{-1/2})$, outside of which the motion is relatively slow, and there is no slip at the wall."
"We then ask : how does the jet react ahead of its departure from the wall and what is the form of

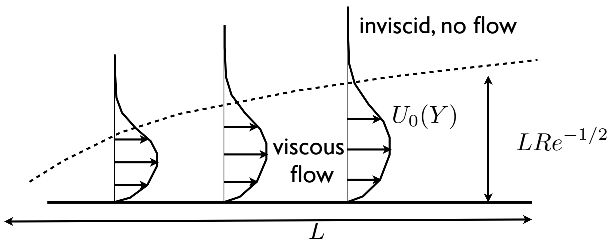


FIG. 1 – A jet developing along a wall

the separation ? (...) We propose below that the upstream response in fact takes places over a length scale $O(LRe^{-3/7})$ ." *Of course this scaling is one of the solution of the exam, we will have to prove it in the responses to the questions.*
"The flow is assumed to be steady, laminar and two-dimensional and incompressible."
"The flow develops an interaction over a streamwise length scale $O(LRe^{-3/7})$ centered about the point of separation. The interaction has a double deck structure in which the unknown induced pressure and displacement are linked by a novel relation peculiar to the jet flow situation. During the interaction the fluid near the wall forms a viscous sublayer, driven along by the induced local pressure gradient, whereas the majority of the boundary layer reacts in an invisicid displaced fashion."

*The interaction self induced by the flow :*
"Upstream of separation the sublayer pressure rises slightly, causing a decrease in the skin friction, and the sublayer expands. The associated movement of fluid in the inviscid region then induces a pressure fall across the jet, but, because the pressure at the edge of the jet does not alter, the transverse pressure gradient reinforces the pressure rise at the wall. So the process is mutually reinforcing. "
*Let us define the point $x_0 = O(1)$ where there is the very small accident in the wall jet (like a bump, a wedge...) ; we set up a multi Deck structure around this point. We change the scales to look closer at this point.*

"We set $x = x_0 + \varepsilon^3 X$ where the coordinate $X$ is $O(1)$ within the interaction zone" (*$L$ is the scale of $x$).*"Then the presence of the jet-like profile upstream implies the conditions, for $X \to -\infty$

$$u \to U_0(Y) + ... \qquad v \to O(Re^{-1/2}) \qquad p \to O(Re^{-1})$$

with $Y = Re^{1/2}y$ defining the boundary layer coordinate. Here $U_0(Y)$ has the properties :
$U_0(Y) = U_0'Y + O(Y^2)$ as $Y \to 0$ and $U_0(Y) \to 0$ as $Y \to \infty$. $U_0'$ is a given positive constant. Outside the boundary layer the motion is inviscid and relatively slow.

1°) *Write the non dimensional Navier Stokes equations and discuss the boundary conditions for the basic flow profile $U_0(Y)$*
2°) *Discuss the boundary conditions in $Y$ and $X$*
3°) First when $Y = O(1)$ (region I), we have the form", *with $\varepsilon$ and $\pi$ unknown up to now :*

$$u = U_0(Y) + \varepsilon u_1 + ... \qquad v = \varepsilon^{-2} v_1 Re^{-1/2} \qquad p = \pi p_1$$

"The Navier-Stokes equations show that the small disturbances here are effectively inviscid and the streamwise pressure gradient is negligible, but the transverse pressure gradient acts significantly."

*Subsititute this Ansatz in the incompressibility equation and in the longitudinal Navier Stokes equation and show that $u_1$ and $v_1$ may be written with the help of an unknown function $-A$ as in the classical Triple Deck theory. Show then that the hypotheses of the sentence leads to a transverse pressure gradient $(\frac{\partial p_1}{\partial Y})$ proportional to $-A'$. Propose the relation between the scale of pressure $\pi$ and $\sqrt{Re}$ and $\varepsilon$.*
4°)*Lower Deck problem, show that with $u = \varepsilon\hat{u}$, $v = \varepsilon^{\beta} Re^{-1/2}\hat{v}$, $Y = \varepsilon\hat{y}$ and $p = \varepsilon^{\alpha}\hat{p}$, the Lower Deck equations are then :*

$$\frac{\partial \hat{u}}{\partial X} + \frac{\partial \hat{v}}{\partial \hat{y}} = 0, \quad \hat{u}\frac{\partial \hat{u}}{\partial X} + \hat{v}\frac{\partial \hat{u}}{\partial \hat{y}} = -\frac{d\hat{p}}{dX} + \frac{\partial^2 \hat{u}}{\partial \hat{y}^2}.$$

*What is the value of $\alpha$ and $\beta$ ?*
5°) *What are the boundary conditions for $\hat{u}$ and $\hat{v}$ in $\hat{y} = 0$ and $\hat{y} \to \infty$ and in $X \to -\infty$ ?*
6°) *In matching the pressure between the top of the Lower Deck and the bottom of the Main Deck deduce the scaling of the pressure $\pi$ as a function of the Reynolds number. Deduce the value of $\varepsilon$ as a function of the Reynolds number. Note here that there is no inviscid Upper Deck.*
7°) *By integration across the jet, show that the pressure through the Main Deck verifies :*

$$p_1(x,\infty) = \hat{p}(X) + [\int_0^{\infty} U_0(Y)^2 dY](A'')$$

*taking into account that there is no inviscid disturbance outside the jet deduce the relation between $\hat{p}(X)$ the pressure in the Lower Deck and $A$ the displacement function.*
8°) *Verify that all the scalings of figure* **??** *are correct. Verify that the longitudinal pressure gradient is negligible in the Main Deck as it was supposed.*
9°) *Extra question, linearise the Lower Deck equations around the basic state $\hat{u} = U_0'\hat{y}$, $\hat{v} = 0$ and $\hat{p} = 0$ and show that perturbation of the longitudinal velocity in $e^{Kx}\phi'(\hat{y})$ and of the transverse velocity in $-e^{Kx}\phi(\hat{y})$ may be written. The solution of this system gives the value of the exponential $K$. As says Stewartson :* "the possibility of a free interaction has been established, it is analogous to the study of the supersonic free interaction."

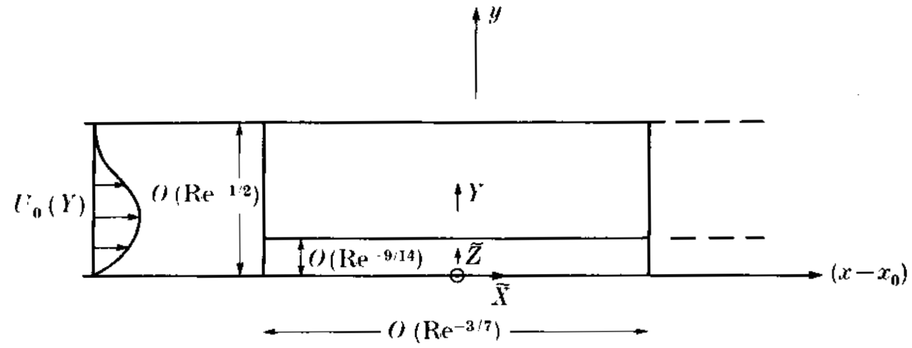


**FIG. 1. Schematic diagram (not to scale) of the two-tiered flow structure I, II set up during the free interaction.**

FIG. 2 – The interacting Double Deck structure, read : $Re^{-1/2}$ $Re^{-9/14}$ and $Re^{-3/7}$

Exercice 1

1) When $\varepsilon$ is 0, the outer problem is an ODE of degree 1, and we have two BC.
2) The outer problem $E_0$ is $x^2\frac{dy}{dx} - y = 0$, so that $y'/y = 1/x^2$,
after integration $ln(y) = C - 1/x$ so that $y(x) = Ae^{-1/x}$
3) So $y_{out}(x) = e^{1-1/x}$ so that $y_{out}(1) = 1$ but $y_{out}(0) = 0 \neq 1$.
The boundary layer is likely to be in $x = 0$. It is not possible to solve $y(x) = Ae^{-1/x}$ with $y(0) = 1$.
Let us introduce $x = \delta\tilde{x}$ and $y = \tilde{y}(\tilde{x})$. So $\varepsilon\frac{d^2y}{dx^2} + x^2\frac{dy}{dx} - y = 0$ is $\varepsilon\delta^{-2}\frac{d^2\tilde{y}}{d\tilde{x}^2} + \delta^2\delta^{-1}\tilde{x}^2\frac{d\tilde{y}}{d\tilde{x}} - \tilde{y} = 0$ The first term (the term with the second order derivative) is indeterminate up to now, but it was small, and we want it back. The second is small (with $y'$), the third is or order unity. The dominant balance gives then with the first and the third $\varepsilon\delta^{-2}$.
4) The thickness of the boundary layer is $\delta = \sqrt{\varepsilon}$. The inner problem is with : $x = \sqrt{\varepsilon}\tilde{x}$ $\frac{d^2\tilde{y}}{d\tilde{x}^2} - \tilde{y} = 0$ so $\tilde{y} = Be^{-\tilde{x}} + De^{\tilde{x}}$.
5) The boundary condition that we want to impose is $\tilde{y}(0) = 1$. The solution is $\tilde{y} = Be^{-\tilde{x}} + (1-B)e^{\tilde{x}}$.
The matching : $y_{out}(x \to 0) = \tilde{y}(\tilde{x} \to \infty)$ so $B = 1$.
6) and 7) The composite solution is $y_{comp} = e^{-x/\sqrt{\varepsilon}} + e^{1-1/x} - 0$.

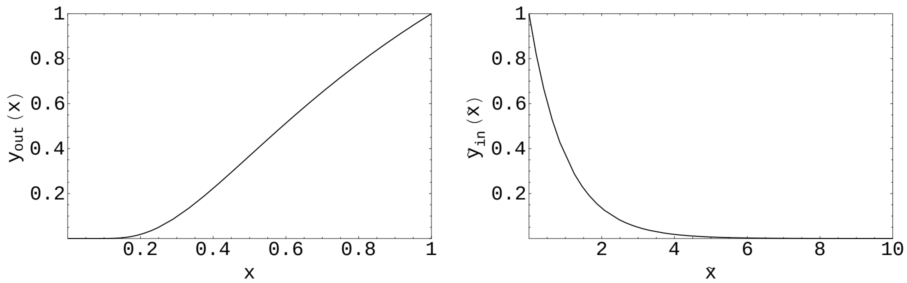


FIG. 3 – Left the outer solution $y_{out}(x) = e^{1-1/x}$, right the inner solution $\tilde{y} = e^{-\tilde{x}}$

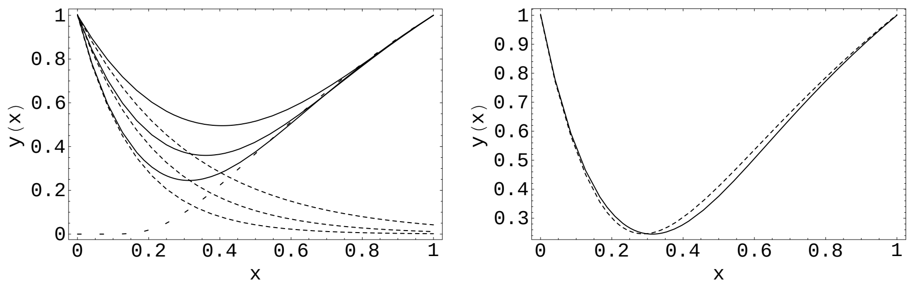


FIG. 4 – Left the full solution for $\varepsilon = 0.1,\ 0.05\ 0.025$ plain line, the outer solution large dashing line and the inner solution written in outer variable $e^{-x/\sqrt{\varepsilon}}$. Right the composite solution ($\varepsilon = 0.025$) and the full solution.

See Carl M. Bender,Steven A. Orszag, Advanced Mathematical Methods for Scientists and Engineers : Springer 1999

Exercice 2

1) No slip condition, matching with a fluid at rest at infinity. The $Re^{-1/2}$ allows to write a Prandt system of equations.
2) The condition in $X$ correspond to the beginning of the interaction. It will be useful for the Lower Deck equations.
3) incompressibility the order of magnitude $(\varepsilon)/((\varepsilon)^3) = (\varepsilon^{-2}Re^{-1/2})(Re^{-1/2})$ gives $\frac{\partial u_1}{\partial X} + \frac{\partial v_1}{\partial Y} = 0$. We neglect the transverse pressure $U_0\frac{\partial u_1}{\partial X} + v_1\frac{\partial U_0(Y)}{\partial Y} = 0$.
We deduce that $u_1 = AU_0'(Y)$ and $v_1 = -A'(X)U_0(Y)$ as in the classical triple Deck Theory.
The transverse equation is $U_0\frac{\partial v_1}{\partial X} = -\frac{\partial \hat{p}(Y)}{\partial Y}$ if the balance $\varepsilon^{-5}Re^{-1} = \pi$ holds.
4) $\beta = -1$ and $\alpha = 2$
5) no slip condition, $X \to -\infty$ : matching with the linear incoming profile $\hat{u} \to U_0'\hat{y}$
6) the pressure is of order $\varepsilon^2$ in the Lower Deck and of order $\varepsilon^{-5}Re^{-1}$ in the Main Deck. We then deduce $\varepsilon = Re^{-1/7}$
7) $\frac{\partial \hat{p}(Y)}{\partial Y} = -U_0\frac{\partial v_1}{\partial X} = U_0^2 A''$ so by integration we have the proposed equation ($\hat{p}$ is the value of the pressure in $Y = 0$ by matching). At infinity there is no perturbation of pressure, so the interacting relation is : $\hat{p} = -[\int_0^\infty U_0(Y)^2 dY](A'')$.

See the original article :
Smith F.T. & Duck P.W. (1977) "Separation of Jets or thermal boundary layer from a Wall" : Quartely Journal of Mechanics and Applied. Mathematics. Vol XXX Pt 2, 1977 pp 143 -156

# SEPARATION OF JETS OR THERMAL BOUNDARY LAYERS FROM A WALL

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SUMMARY

Consideration is given to the nature of the separation and subsequent reversed flow occurring when a jet-like boundary layer on a wall encounters a concave corner of finite angle $\alpha$ or collides with an opposing jet. Separation is caused by a nonlinear upstream response, within the jet, wherein the motion acquires a double-deck structure of streamwise extent $O(L^*\mathrm{Re}^{-\frac{3}{7}})$, $L^*$ and $\mathrm{Re} \gg 1$ being a characteristic length scale and Reynolds number respectively. The upstream pressure rise at the wall is sustained by the inviscid displacement of most of the jet because the displacement generates an adverse pressure gradient across the jet. Throughout most of the flow the induced pressure is proportional to the curvature of the jet displacement, the external motion exerting little influence. Numerical solutions of the key problem of the upstream interaction lead to an apparently self-consistent description of the reversed flow far downstream in the double-deck. This description in turn leads to a tentative account of the complete departure and reattachment of the jet, and the separation point is predicted to occur at a distance $O(L^*\mathrm{Re}^{-\frac{3}{14}})$ from the corner (or line of symmetry in the jet collision problem). The results apply to some well-known jet situations in rotating fluids, oscillatory motions and free convection boundary layers.

## 1. Introduction

THE need to investigate the structure of a jet flow near its point of departure from a wall arises in many diverse situations. Examples are found in rotating fluids (**1, 2**), oscillatory motions (**3** to **5**) and free convection boundary layers (**6** to **8**), as well as for wall jets *per se* (**9**) near a concave corner or other discontinuity in wall conditions. In the problems we have in mind, a steady planar (or effectively planar) jet-like boundary-layer flow has been established along a fixed smooth surface by some imposed constraint. The jet must then leave the wall. The cause of this expulsion from the wall may be an impending collision with a jet moving in the opposite direction, as in (**1** to **7**), or a finite change in the wall slope, as in (**8**) or as would arise for any jet-style motion along a cornered surface. Concrete examples of these jet-flow situations occur in the fluid motion induced on a rotating sphere (**1**) or rotating disc (**2**), the steady streaming associated with an oscillating cylinder (**3, 4**) or with an oscillatory pressure gradient acting in a curved pipe (**5**), the thermal boundary layer on a heated inclined plate (**6, 7**) or in a cavity (**8**), and the plane or axisymmetric wall jet (**9**) encountering a cornered surface. In the first three cases above, the jet is promoted by an effective slip velocity along the wall and, although the slip itself reduces to zero on approaching the point of departure, a nonzero jet-like profile remains. In the thermal boundary layer the jet profile is provoked by buoyancy forces. For the wall jet, there is assumed a source of momentum far upstream.

For all these problems there exists a basic velocity scale $U^*$, say, a typical length scale $L^*$ and, if $\nu$ represents the kinematic viscosity of the fluid, the Reynolds number $\mathrm{Re}=U^*L^*/\nu$ is large (for the thermal boundary layer, $\mathrm{Re}=\mathrm{Gr}^{\frac{1}{2}}$, where $\mathrm{Gr}\gg 1$ is the Grashof number). It is supposed that, due to one of the agencies mentioned above, the viscous jet flow has been set up in a boundary layer of typical thickness $O(L^*\mathrm{Re}^{-\frac{1}{2}})$, outside of which the motion is relatively slow, and there is no slip at the wall. (In (**6, 7**), incidentally, the boundary-layer thickness is $O(L^*\mathrm{Re}^{-\frac{2}{5}})$. However, the basic structure described in this paper still holds when the scalings are altered appropriately.) We then ask: how does the jet react ahead of its departure from the wall, and what is the form of the separation? Stewartson (**1**), in studying the motion outside a rotating sphere, suggested the necessary upstream response in the jet might take place through a small inviscid zone of streamwise extent $O(L^*\mathrm{Re}^{-\frac{1}{2}})$ near the equator, with a viscous slip layer of standard boundary-layer character adjoining the wall. However he also pointed out the likelihood of the slip layer separating rather than simply circulating at the equator. His suggestion has since been applied implicitly in many other jet-style problems (**2** to **7**). We propose below, however, that the upstream response in fact takes place over a greater length scale, namely $O(L^*\mathrm{Re}^{-\frac{3}{7}})$, and that the jet flow separates at a distance upstream of the equator much greater than the boundary-layer thickness $O(L^*\mathrm{Re}^{-\frac{1}{2}})$.

We let $L^*(x, y)$ denote distances along and perpendicular to the wall and let $U^*(u, v)$ be the corresponding velocities. The pressure is written as $\rho U^{*2}p$, where $\rho$ is the fluid density. The flow is assumed to be steady, laminar and two-dimensional (the flow near the departure points in (**1, 2**) is effectively planar). Also, we suppose, purely for convenience, that the fluid is incompressible; in the free convection boundary layer the compressibility can be accommodated satisfactorily in the Boussinesq approximation, in which case the extra buoyancy forces are found to have virtually no influence on the local results derived below. In section 2 it is shown that the jet flow can develop a free interaction over a streamwise length scale $O(L^*\mathrm{Re}^{-\frac{3}{7}})$ centred about the point of separation $x=x_0$. The interaction has a double-deck structure (**10, 11**) in which the unknown induced pressure and displacement are linked by a novel relation ((2.5c) below) peculiar to the jet flow situation. During the interaction the fluid near the wall forms a viscous sublayer, driven along by the induced local pressure gradient, whereas the majority of the boundary layer reacts in an inviscid displaced



fashion. Upstream of separation the sublayer pressure rises slightly, causing a decrease in the skin friction, and the sublayer expands. The associated movement of fluid in the inviscid region then induces a pressure fall across the jet, but, because the pressure at the edge of the jet does not alter, the transverse pressure gradient reinforces the pressure rise at the wall. So the process is mutually reinforcing. The scalings involved here may be derived from an order-of-magnitude argument analogous to that used by Smith (**12**) in a channel flow problem. The present work in fact has some connections with the last named; however, it differs crucially in that the oncoming boundary layer does not have a contained profile but instead extends, on the boundary-layer scale, from 0 to $\infty$. Further, there is here the possibility that the boundary layer may move far from the wall. This suggestion is confirmed by numerical results (see section 3), for the fundamental nonlinear problem determining the interaction, which show that the jet flow separates in a regular fashion. An asymptotic description in section 4 of the motion far beyond separation, on the interaction scale, then leads to a discussion (section 5) of the complete departure of the jet from the wall.

The tentative conclusion from section 5 is that separation (followed by a sizeable eddy of reversed flow) is most likely to take place, within the double-deck, at a distance $O(L^*\mathrm{Re}^{-\frac{3}{14}})$ before the actual change in boundary conditions is reached. The conclusion holds, we believe, for all the problems (**1** to **9**) detailed above. Three-dimensionality and slipping effects exert practically no influence on this result.

## 2. Flow structure during the interaction

We set $x=x_0+\mathrm{Re}^{-\frac{3}{7}}\tilde{X}$, where the coordinate $\tilde{X}$ is $O(1)$ within the interaction zone. Then the presence of the jet-like profile upstream implies the conditions, for $\tilde{X}\to-\infty$,

$$u\to U_0(Y)+O(\mathrm{Re}^{-\frac{1}{7}}),\qquad v\to O(\mathrm{Re}^{-\frac{1}{2}}),\qquad p\to O(\mathrm{Re}^{-1}),\qquad (2.1)$$

with $Y=\mathrm{Re}^{\frac{1}{2}}y$ defining the boundary layer coordinate. Here $U_0(Y)$ has the properties $U_0(Y)\sim\frac{1}{2}\kappa Y+O(Y^2)$ as $Y\to 0$ and $U_0(Y)\to 0$ as $Y\to\infty$. $\kappa$ is a given positive constant. Outside the boundary layer upstream, the velocities $u$, $v$ are $O(\mathrm{Re}^{-\frac{1}{2}})$ and the pressure, $p$, is $O(\mathrm{Re}^{-1})$, so that the motion is inviscid and relatively slow.

When $\tilde{X}$ is finite the solution divides essentially into two parts. First, when $Y$ is $O(1)$ (region I), we have the form

$$u=U_0(Y)+\mathrm{Re}^{-\frac{1}{7}}u_1(\tilde{X}, Y),\qquad v=\mathrm{Re}^{-\frac{3}{14}}v_1(\tilde{X}, Y),\qquad p=\mathrm{Re}^{-\frac{2}{7}}p_1(\tilde{X}, Y),\qquad (2.2)$$

with relative errors $O(\mathrm{Re}^{-\frac{2}{7}})$. The Navier-Stokes equations show that the small disturbances here are effectively inviscid and the streamwise pressure gradient is negligible, but the transverse pressure gradient acts significantly.

The solutions are therefore

$$u_1 = \tilde{A}(\tilde{X})U_0'(Y), \qquad v_1 = -\tilde{A}'(\tilde{X})U_0(Y), \qquad \partial p_1/\partial Y = -U_0(Y)\partial v_1/\partial \tilde{X}, \tag{2.3}$$

where $\tilde{A}$ is an unknown function of $\tilde{X}$, with $\tilde{A}(-\infty)=0$. The second solution zone (II) occurs near the wall when $Y$ is small and $O(\mathrm{Re}^{-\frac{1}{7}})$. There $\tilde{Z} = \mathrm{Re}^{\frac{9}{14}}y$ is $O(1)$ and to first order

$$u = \mathrm{Re}^{-\frac{1}{7}}\tilde{U}(\tilde{X}, \tilde{Z}), \qquad v = \mathrm{Re}^{-\frac{5}{14}}\tilde{V}(\tilde{X}, \tilde{Z}), \qquad p = \mathrm{Re}^{-\frac{2}{7}}\tilde{P}(\tilde{X}). \tag{2.4}$$

The pressure $\tilde{P}$ is independent of $\tilde{Z}$ from the transverse momentum equation. The continuity and streamwise momentum equations become

$$\frac{\partial \tilde{U}}{\partial \tilde{X}} + \frac{\partial \tilde{V}}{\partial \tilde{Z}} = 0, \qquad \tilde{U}\frac{\partial \tilde{U}}{\partial \tilde{X}} + \tilde{V}\frac{\partial \tilde{U}}{\partial \tilde{Z}} = -\frac{d\tilde{P}}{d\tilde{X}} + \frac{\partial^2 \tilde{U}}{\partial \tilde{Z}^2}, \tag{2.5a}$$

with the boundary conditions

$$\left.\begin{array}{l} \tilde{U} = \tilde{V} = 0 \quad \text{at} \quad \tilde{Z} = 0, \qquad \tilde{U} \sim \frac{1}{2}\kappa\{\tilde{Z} + \tilde{A}(\tilde{X})\} \quad \text{as} \quad \tilde{Z} \to \infty, \\ \tilde{U} \to \frac{1}{2}\kappa\tilde{Z}, \qquad \tilde{V} \to 0, \qquad \tilde{P} \to 0 \quad \text{as} \quad \tilde{X} \to -\infty. \end{array}\right\} \tag{2.5b}$$

These constraints reflect the no-slip condition at the wall, the matching with the inviscid zone I and the upstream conditions (2.1). In addition, however, a relation is needed between the pressure $\tilde{P}(\tilde{X})$ and the displacement $\tilde{A}(\tilde{X})$. It is found as follows. Integrating (2.3) for $p_1$ we have

$$p_1 = \tilde{P}(\tilde{X}) + \tilde{A}''(\tilde{X})\int_0^Y U_0^2(Y_1)\, dY_1, \tag{2.6}$$

the value of $p_1(\tilde{X}, 0)$ ensuring the continuity of pressure between I and II. But since $u_1$ and $v_1 \to 0$ as $Y \to \infty$, the velocities induced just outside the boundary layer are expected to be $o(\mathrm{Re}^{-\frac{1}{7}})$. Hence from Bernoulli's equation in the inviscid flow there, the pressure must then be $o(\mathrm{Re}^{-\frac{2}{7}})$. Therefore we require $p_1(\tilde{X}, \infty)$ to be zero. This gives, from (2.6), the relation

$$d^2\tilde{A}/d\tilde{X}^2 = -\gamma\tilde{P}(\tilde{X}), \tag{2.5c}$$

where $\gamma^{-1} = \int_0^\infty U_0^2(Y_1)\, dY_1$ is a known positive constant. With (2.5a) and (2.5b), (2.5c) fixes the nature of the free interaction upstream of the particular disturbance.

The pressure-displacement law (2.5c), though remarkably similar to Ackeret's law (where $\tilde{A}' \propto -\tilde{P}$) for supersonic flows with a uniform mainstream (**10**), appears to be a quite novel relation for a thermal boundary layer or jet flow adjoining a plane wall. It is remarkable too in having a *parabolic* character despite the incompressibility of the fluid. That it yields, like Stewartson's (**10**) work on supersonic boundary layers, a mode of upstream response within the boundary layer is readily seen by considering the possibility of a small disturbance from the original motion $U = \frac{1}{2}\kappa\tilde{Z}$



$\tilde{V} = \tilde{P} = \tilde{A} = 0$. This takes place when $\tilde{X}$ is large and negative. There

$$\tilde{U} \sim \tfrac{1}{2}\kappa\tilde{Z} + e^{\lambda\tilde{X}}\tilde{f}'(\tilde{Z}), \qquad \tilde{V} \sim -\lambda e^{\lambda\tilde{X}}\tilde{f}(\tilde{Z}), \qquad \tilde{P} \sim be^{\lambda\tilde{X}}, \tag{2.7}$$

if terms $O(e^{2\lambda\tilde{X}})$ are neglected. The constants $\lambda$ (assumed positive) and $b$ are unknowns. We require $\tilde{f}(0) = \tilde{f}'(0) = 0$ and $\tilde{f}'(\infty) = -\frac{1}{2}\kappa\gamma b\lambda^{-2}$. The latter constraint follows from the implied displacement behaviour $\tilde{A} \sim -\gamma b\lambda^{-2}\exp(\lambda\tilde{X})$, from (2.5c). Substitution into (2.5a) and neglect of terms $O(e^{2\lambda\tilde{X}})$ yields the solution

$$\tilde{f}'(\tilde{Z}) = \frac{b\lambda^{\frac{1}{3}}2^{\frac{2}{3}}}{\mathrm{Ai}'(0)\kappa^{\frac{2}{3}}}\int_0^{\bar{Z}} \mathrm{Ai}(q)\, dq,$$

(where $\bar{Z} = (\frac{1}{2}\kappa\lambda)^{\frac{1}{3}}\tilde{Z}$ and Ai is the Airy function), which satisfies the governing equation and wall conditions. The outer constraint then determines the exponent $\lambda$ as

$$\lambda = \{-3\gamma\,\mathrm{Ai}'(0)\}^{\frac{3}{7}}(\tfrac{1}{2}\kappa)^{\frac{5}{7}} = 0{\cdot}5652\gamma^{\frac{3}{7}}\kappa^{\frac{5}{7}}. \tag{2.8}$$

The pressure constant $b$ remains unknown and is fixed by the conditions prevailing downstream in much the same way as occurs in triple-deck interactions. In fact, the entire situation confronting us, now that the possibility of a free interaction has been established, is analogous to the study of the supersonic free interaction (**11**). If the undisturbed state $\tilde{U} = \frac{1}{2}\kappa\tilde{Z}$, $\tilde{V} = \tilde{P} = \tilde{A} = 0$ is slightly perturbed, then we can anticipate that the ensuring solution will develop into a nonlinear free interaction. Further, since the effective skin friction $\tilde{\tau}_0\,(= \partial\tilde{U}/\partial\tilde{Z}$ at $\tilde{Z} = 0)$ is given by

$$\tilde{\tau}_0 = \tfrac{1}{2}\kappa\{1 - 3\gamma b\lambda^{-\frac{5}{3}}\mathrm{Ai}(0)e^{\lambda X}(\tfrac{1}{2}\kappa)^{\frac{1}{3}}\} \tag{2.9}$$

as $\tilde{X} \to -\infty$, a small pressure rise $(b > 0)$ induces a small drop in skin friction. So setting $b$ to be a positive constant suggests a trend towards separation of the boundary layer. The numerical investigation of this suggestion is described in section 3 below.

The physical basis of the upstream interaction is evident now. If the pressure suffers a small increase, the adverse pressure gradient causes the skin friction to drop slightly and so the viscous layer II expands. The expansion leads to fluid in the main layer I being displaced away from the wall and this transverse convection sets up a transverse pressure gradient. Since the pressure at the outer edge of the boundary layer must remain unchanged, however (as the displacement of the jet profile has virtually no effect there), the pressure rise at the inner edge is accentuated by the transverse pressure variation. Hence the interaction process is self-sustaining. The "double-deck" structure is drawn in Fig. 1.

For a thermal boundary layer, the imposed wall temperature plays little part as far as the local interaction process is concerned. It merely serves to

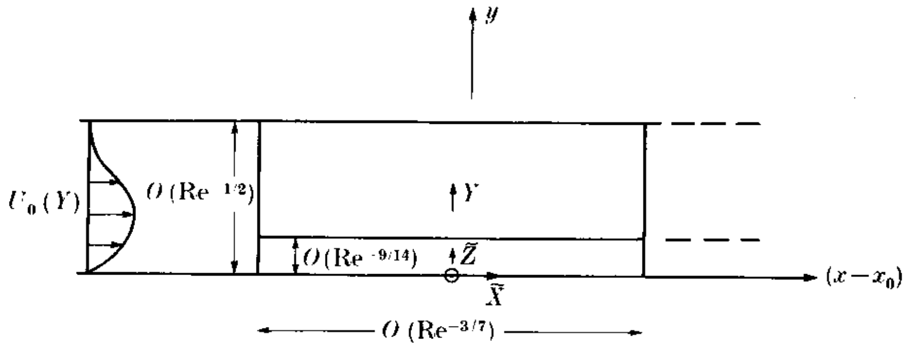



FIG. 1. Schematic diagram (not to scale) of the two-tiered flow structure I, II set up during the free interaction.

Similarly, the three-dimensional rotation or oscillatory effects in the problems of (**1** to **5**) exert little influence during the interaction once the jet-style flow has been established. In all these situations the free interaction properties appear to be dominated by the local planar flow, largely independent of the original cause of the jet-like motion.

## 3. Numerical Results

First, the factors $\gamma$, $\kappa$ are removed from the fundamental problem (2.5a, b, c) by the normalization

$$(\tilde{U}, \tilde{V}, \tilde{P}, \tilde{A}, \tilde{X}, \tilde{Z}) = (\gamma^{-\frac{1}{7}}\kappa^{\frac{3}{7}}U, \gamma^{\frac{1}{7}}\kappa^{\frac{4}{7}}V, \gamma^{-\frac{2}{7}}\kappa^{\frac{6}{7}}P, \gamma^{-\frac{1}{7}}\kappa^{-\frac{4}{7}}A, \gamma^{-\frac{3}{7}}\kappa^{-\frac{5}{7}}X, \gamma^{-\frac{1}{7}}\kappa^{-\frac{4}{7}}Z).$$

Then we have the equations of motion

$$\frac{\partial U}{\partial X}+\frac{\partial V}{\partial Z}=0, \qquad U\frac{\partial U}{\partial X}+V\frac{\partial U}{\partial Z}=-\frac{dP}{dX}+\frac{\partial^2 U}{\partial Z^2}, \tag{3.1}$$

with

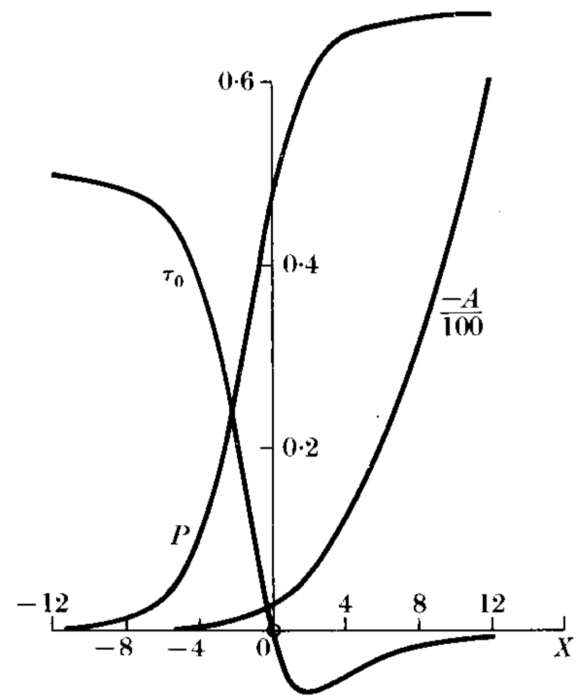
$$\left.\begin{aligned} &U=V=0 \quad \text{at} \quad Z=0, \qquad U\sim\tfrac{1}{2}\{Z+A(X)\} \quad \text{as} \quad Z\to\infty,\\ &U\sim\tfrac{1}{2}Z \quad \text{as} \quad X\to-\infty \quad \text{and} \quad A''(X)=-P(X). \end{aligned}\right\} \tag{3.2}$$

The problem (3.1), (3.2) was treated numerically by writing the governing equations in terms of $\psi$, $U$ and $\tau$, where $\psi$ is the scaled stream function ($U=\partial\psi/\partial Z$, $V=-\partial\psi/\partial X$) and $\tau=\partial U/\partial Z$ is the stress. Thus (3.1) may be replaced by three first-order differential equations. These were then represented by a centred-difference scheme with boundary conditions prescribed at $Z=0$ and at $Z=Z_\infty$, where $Z_\infty$ is a suitable large value for the edge of the layer. The solution was determined by forward marching in the $X$-direction, solving iteratively the nonlinear equations for $\psi$, $U$, $\tau$, $A$ and $P$ at each $X$-station until successive iterates were sufficiently close in value. The integrations were started by prescribing a small negative jump $A_{-\infty}$ in the displacement function $A$ at some initial station $X_{-\infty}$. Typical values chosen were $A_{-\infty}=-0{\cdot}001$, $Z_\infty=40$ and mesh widths $\Delta X=0{\cdot}1$, $\Delta Z=0{\cdot}1$.



In Fig. 2 we show the solutions for $P(X)$, $A(X)$, and the skin friction $\tau_0=\tau(X,0)$. Initially they develop as in (2.7) to (2.9), and then the trends set there are continued as the motion becomes nonlinear. In particular, separation ($\tau_0=0$) is eventually reached (and we take the origin for $X$ to be at the separation point), with $P=P_s=0{\cdot}4720$ and $A=A_s=-2{\cdot}854$. The solution at separation is regular, avoiding the possibility of the Goldstein (**13**) singularity for reasons similar to those given in (**11**). Thereafter the forward marching scheme is strictly invalid, due to the presence of reversed flow in $X>0$, but the reversed flow velocities were sufficiently small here to allow progress into the separated part, c.f. (**11**). This progress, like that in (**14**), was improved by adopting the Flügge-Lotz and Reyhner (**15**) technique of neglecting $U\partial U/\partial X$ in (3.1) if $U<0$. Beyond separation the skin friction reaches a negative minimum (approximately $-0{\cdot}069$) and then gradually rises, seemingly towards zero, as $X$ increases further. Meanwhile the pressure apparently tends to a constant value downstream.

Some of the velocity profiles obtained during the calculations are presented in Fig. 3. Because of the ever-increasing displacement, especially beyond separation, the outer boundary $Z_\infty$ had to be enlarged to the value 80 to ensure a satisfactory approach to the uniform shear in (3.2) up to



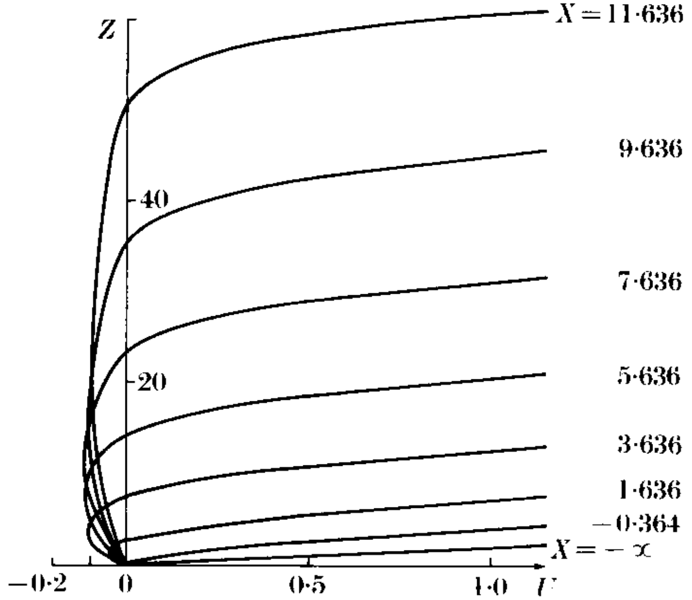



FIG. 3. Velocity profiles during the free interaction.

about $X=14$. The results were checked by altering $\Delta X$, $\Delta Z$ and $A_{-\infty}$ independently and their accuracy is believed to be to within 0·0002 in $P(X)$.

Table 1 gives values of $\tau_0$, $P$ and $U_{min}$, the minimum value of $U$ at various stations. It appears that an asymptotic form is being approached as $X$ increases and that it is being attained quite rapidly (at least, compared with (**14**)). This is a fortunate feature, since the integrations did not need to be taken excessively far beyond separation before a fairly clear suggestion for the asymptotic form, emerging as $X \to \infty$, became evident. We consider the asymptotic behaviour below.

## 4. The separated structure for $X \to \infty$.

The numerical evidence indicates that, when $X \to \infty$, the pressure $P$ acquires a constant value $P_0 > 0$ (a representative value is $P_0 = 0{\cdot}6750$). Based on that assumption and the assumption that the solution must be self-preserving far downstream, the separated flow structure for $X \gg 1$ is akin to that of the supersonic boundary layer. We omit, therefore, the derivation of the powers of $X$ involved below (the derivation follows the lines set out in (**14**)) and present the asymptotic form directly.

For $X \gg 1$ we propose that

$$P = P_0 - \tfrac{1}{2}P_1 X^{-\frac{8}{3}} + O(X^{-\frac{10}{3}}), \qquad A = -\tfrac{1}{2}P_0 X^2 + O(1), \tag{4.1}$$

where $P_1$ is an unknown constant. The motion in II then subdivides into



TABLE 1. *Values of $\tau_0$, $P$, $U_{min}$ at various X-stations (Here $X_{-\infty} = -12{\cdot}364$).*

| $X$ | −12·364 | 0 | 1·636 | 3·636 | 5·636 | 7·636 | 9·636 | 11·636 |
|---|---|---|---|---|---|---|---|---|
| $\tau_0$ | 0·5 | 0 | −0·0671 | −0·0534 | −0·0324 | −0·0197 | −0·0129 | −0·0089 |
| $P$ | 0 | 0·4720 | 0·5922 | 0·6482 | 0·6644 | 0·6696 | 0·6717 | 0·6726 |
| $U_{min}$ | — | 0 | −0·0464 | −0·1021 | −0·1162 | −0·1118 | −0·1023 | −0·0913 |

three basic zones (i) to (iii). In (i), the detached shear layer of thickness $O(X^{\frac{1}{3}})$ centred about $Z = -A(X)$,

$$\psi = X^{\frac{2}{3}} G_0(\xi) + O(1). \tag{4.2}$$

Here $\xi = \hat{Z} X^{-\frac{1}{3}}$ is $O(1)$ in (i) and $\hat{Z} = Z + A(X)$. From (3.1), the function $G_0(\xi)$ satisfies

$$G_0''' + \tfrac{2}{3}G_0 G_0'' - \tfrac{1}{3}G_0'^2 = 0, \qquad G_0 - \tfrac{1}{4}\xi^2 \to 0 \quad \text{as} \quad \xi \to \infty, \qquad G_0'(-\infty) = 0. \tag{4.3}$$

The constraint as $\xi \to \infty$ matches the solution in (i) to the shear $\psi = \frac{1}{4}\hat{Z}^2 + 2P(X)$ holding identically above (i), while for $\xi \to -\infty$ we require the

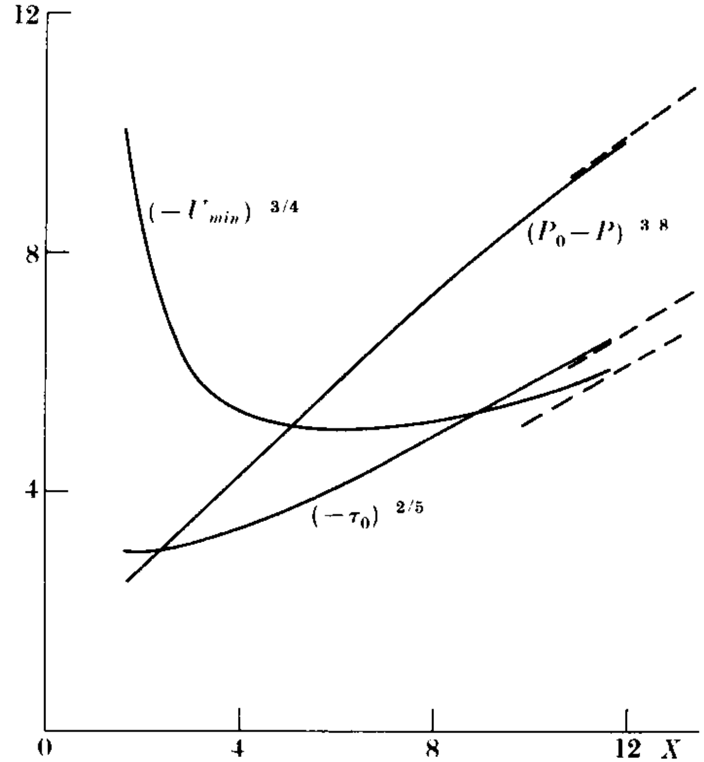



FIG. 4. Graphs of values of $(P_0 - P)^{-\frac{3}{8}}$, $(-\tau_0)^{-\frac{2}{5}}$ and $(-U_{min})^{-\frac{3}{4}}$ computed in $X > 0$. The dashed straight lines give the slopes predicted asymptotically for $X \gg 1$ in section 4.

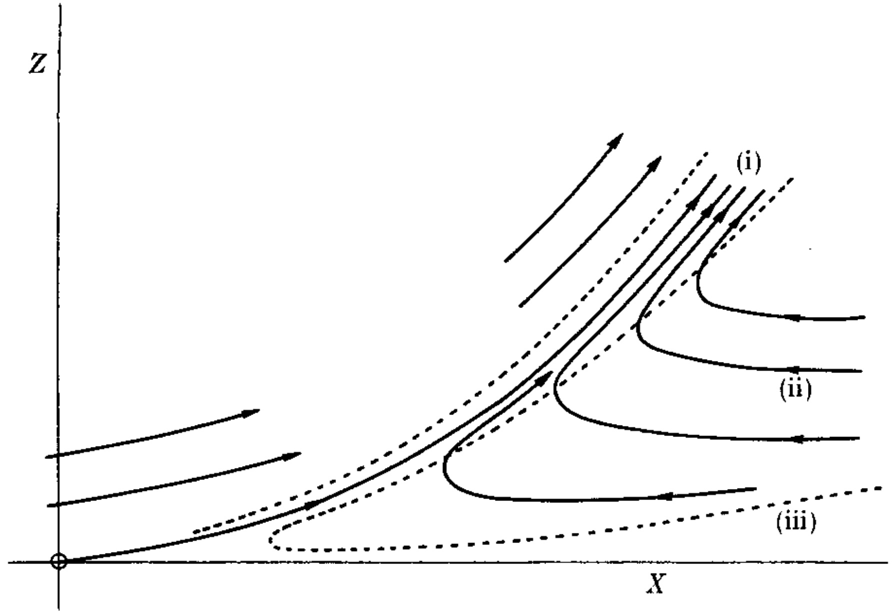



FIG. 5. Sketch of the flow structure and streamlines when $X$ is large. Zones (i) to (iii), of thicknesses $\propto X^{\frac{1}{3}}, X^2, X^{\frac{7}{6}}$ respectively, are marked by dots.

velocity $O(X^{\frac{1}{3}})$ in (i) to be reduced to zero to merge with the slower reversed flow below the shear layer. The solution of (4.3) gives $G_0(-\infty) = -2^{-\frac{1}{3}}C_0$ ($C_0 = 1{\cdot}2521$, from (**14**)). Beneath the shear layer, the motion in the inviscid zone (ii), of thickness $O(X^2)$, is determined by the solution in a thinner viscous sublayer (iii), of thickness $O(X^{\frac{7}{6}})$, astride the wall. Within (iii),

$$\psi = X^{-\frac{1}{6}}F(\eta) + O(X^{-\frac{5}{6}}), \tag{4.4}$$

where $\eta = ZX^{-\frac{7}{6}}$ is $O(1)$. The function $F(\eta)$ satisfies, from (3.1), the Falkner-Skan equation

$$F''' - \tfrac{1}{6}FF'' + \tfrac{4}{3}(F'^2 - P_1) = 0, \tag{4.5}$$

and the boundary conditions $F(0) = F'(0) = 0$ (for no-slip) and $F'(\infty) = -P_1^{\frac{1}{2}}$. The condition for $\eta \to \infty$ joins the solution in (iii) to the zone (ii) where the velocity $U$ is practically uniform, negative and $O(X^{-\frac{1}{3}})$. The solution of (4.5), obtained numerically, has the properties $F''(0) = -1{\cdot}349P_1^{\frac{3}{4}}$. $F(\eta) + P_1^{\frac{1}{2}}\eta \to 0{\cdot}664P_1^{\frac{1}{4}}$ as $\eta \to \infty$. Equating the value of $\psi$ at the lower edge of the shear layer ($\psi \approx -C_0X^{\frac{2}{3}}2^{-\frac{1}{3}}$) with its value at the upper edge of the inviscid zone (ii) (where $Z \to \frac{1}{2}P_0X^2$, so that $\psi \to -\frac{1}{2}P_0P_1^{\frac{1}{2}}X^{\frac{2}{3}}$), we obtain $P_1$ in terms of $P_0$ as

$$P_1 = 2^{\frac{4}{3}}C_0^2/P_0^2. \tag{4.6}$$

In consequence the skin friction $\tau_0$ for $X \gg 1$ is predicted as

$$\tau_0 = -2{\cdot}698(C_0/P_0)^{\frac{3}{2}}X^{-\frac{5}{2}} + O(X^{-\frac{19}{6}}). \tag{4.7}$$



Further terms in the expansions above are obtainable in principle, with eigenfunctions appearing at lower order as in (**14**). Comparisons of the predictions (4.1) and (4.7) with the numerical results are made in Fig. 4. The agreement is sufficiently good overall to encourage the belief in the asymptotic structure set out in (4.1) to (4.7). A sketch of the flow pattern is presented in Fig. 5.

Because the shear layer is continually moving away from the wall as $X$ increases, the underlying assumptions of the double-deck must break down on a length scale $x - x_0 \gg O(\mathrm{Re}^{-\frac{3}{7}})$. We discuss that, and the complete mode of departure of the jet flow from the wall, next.

## 5. Implications of the separated flow structure

Downstream of the double-deck, a number of possible reasons for breakdown of the above structure are apparent, but the chief one seems to be associated with the position of the detached shear layer. To first order, the distance of the shear layer from the wall is $\frac{1}{2}P_0(x - x_0)^2\,\mathrm{Re}^{\frac{3}{14}}\gamma^{\frac{5}{7}}\kappa^{\frac{9}{7}}$ (from (4.1)) as the double-deck is left. Hence some new features must certainly evolve in the inviscid flow underneath when $x - x_0$ is positive and $O(\mathrm{Re}^{-\frac{3}{14}})$, because then the transverse length scale becomes comparable with the streamwise scale $x - x_0$, and so the inviscid motion develops an elliptic character. That this should also define the stage at which the entire flow evolves a new character is perhaps intuitively obvious. It is not definite *a priori*, however. The whole flow within and above the shear layer could in principle continue on (see paragraph following (5.6) below), effectively unchanged from the form generated asymptotically by the double-deck solution in section 4. We suggest *tentatively* however that the entire motion does indeed change significantly when $x - x_0$ is $O(\mathrm{Re}^{-\frac{3}{14}})$. The clearest picture of the downstream flow seems to emerge for the problem of the jet encountering a concave corner of angle $\alpha$, where $0 < \alpha < \frac{1}{2}\pi$.

The nature of the flow field then appears to be established as follows. Let the distance of the shear layer from the wall be $\mathrm{Re}^{-\frac{3}{14}}S(\bar{X})$, where $\bar{X} = \mathrm{Re}^{\frac{3}{14}}(x - x_0)$ is $O(1)$ and let $p = \mathrm{Re}^{-\frac{2}{7}}\bar{P}(\bar{X}, \bar{Y})$ in the region (a) between the shear layer and the wall where $y = \mathrm{Re}^{-\frac{3}{14}}\bar{Y}$ and $0 \leqslant \bar{Y} \leqslant S(\bar{X})$.

Above the shear layer, in the detached jet flow (b), the solution consists of the original jet-style profile, but displaced. We omit the details of the motion in (b), which yield the relation

$$\frac{d^2S}{d\bar{X}^2} = \gamma\bar{P}(\bar{X}, S(\bar{X})) \tag{5.1}$$

between pressure and the jet-flow displacement. Throughout, the slow inviscid motion outside the jet has no active role in the basic structure.

Now we argue that, in (a) $\bar{P}$ must remain uniform and equal to its original value, $P_0\gamma^{-\frac{2}{7}}\kappa^{\frac{6}{7}}$, at $\bar{X} = 0+$. For if $\bar{P}$ is nonuniform then the motion induced in (a) leads to the often-conjectured result that between separation and reattachment an inviscid eddy of recirculating fluid is formed. Batchelor's **(16)** work then implies that the vorticity in (a) is uniform, and so the flow comprises a slow vortex motion. From physical considerations, this vortex rotates so that the slip velocity $u$ is negative at the wall. Between (a) and the wall a viscous slip layer is therefore promoted. However, since the slip velocity has to reduce to zero as $\bar{X} \to 0+$ (as the velocities $O(\mathrm{Re}^{-\frac{1}{7}})$ in (a) are much greater than those present in the reversed flow of the double-deck in section 4), the slip layer would almost certainly separate. This does not seem a reasonable occurrence to expect in the eddy downstream of separation. Nor is it clear just how the reversed flow in the double-deck (in (ii)) can be sustained in the process. Moreover, the momentum flux of the vortex flow is $O(\mathrm{Re}^{-\frac{1}{2}})$, comparable with the flux of the detached jet itself, which again seems unreasonable.

The difficulties are resolved, however, if $\bar{P}$ is uniform, since the velocities in (a) are then much less than $O(\mathrm{Re}^{-\frac{1}{7}})$. We propose that in (a)

$$\left.\begin{aligned} p &= \mathrm{Re}^{-\frac{2}{7}} P_0 \gamma^{-\frac{2}{7}} \kappa^{\frac{6}{7}} + \mathrm{Re}^{-\frac{6}{7}} \bar{p}(\bar{X}, \bar{Y}), \\ (u, v) &= \mathrm{Re}^{-\frac{3}{7}} (\partial\bar{\psi}/\partial\bar{Y}, -\partial\bar{\psi}/\partial\bar{X}), \end{aligned}\right\} \tag{5.2}$$

to leading order. Then the governing equations for $\bar{p}$, $\bar{\psi}$ are inviscid, nonlinear and elliptic. The boundary conditions include $\bar{\psi} = 0$ at $\bar{Y} = 0$, for tangential flow at the wall, but at $\bar{Y} = S(\bar{X})$ $(= \frac{1}{2} P_0 \gamma^{\frac{5}{7}} \kappa^{\frac{6}{7}} \bar{X}^2$ from (5.1), (5.2)) fluid is entrained into the forward-moving viscous shear layer (c). Thus the reversed motion in (a) is similar to that occurring downstream of triple-deck separation, for injection into a boundary layer under a uniform mainstream **(17, 18)**. The solution in and above (c) carries on practically undisturbed from that of section 4. Batchelor's theorem does not apply now in (a)

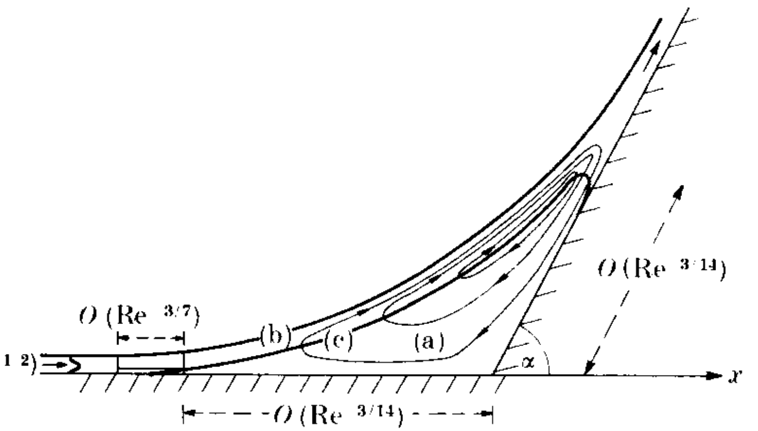



FIG. 6(a). Diagram of the conjectured flow pattern, from separation (within the double deck) to reattachment, for a jet encountering a concave corner of angle $\alpha$ $(0 < \alpha < \frac{1}{2}\pi)$. Here (a) is the inviscid region, from which fluid is entrained into the shear layer (c) and detached jet (b).



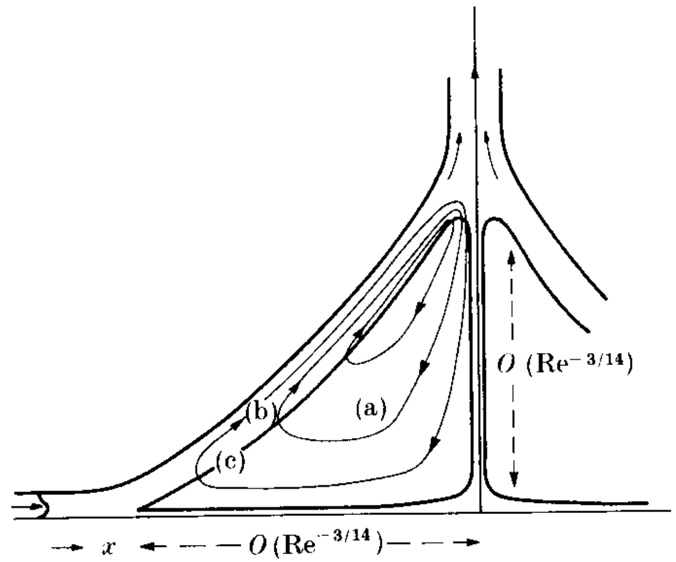



FIG. 6(b). The suggestion for the flow pattern induced during the collision of two opposing jets. (a), (b), (c) are as in Fig. 6(a).

because of the entrainment. Moreover, as $\bar{X} \to 0+$ the flow in (a) joins directly with that in (ii) of section 4, while the viscous slip layer, of thickness $O(\mathrm{Re}^{-\frac{11}{14}})$, between (a) and the wall must acquire the form implied by zone (iii), as $\bar{X} \to 0+$. Since in (a) the negative slip velocity at the wall increases as $\bar{X} \to 0+$, the slip layer is unlikely to separate.

The remaining boundary condition on $\bar{\psi}$ comes from downstream, where the mass flow necessary to preserve the entrainment into (c) must be supplied. This supply of fluid is associated with the unknown reattachment phenomenon. The reattachment itself poses a considerable problem, but we suggest that it must take place while $x - x_0$ is $O(\mathrm{Re}^{-\frac{3}{14}})$. It is certainly possible, on face value, for the flow in (b) and (c) to continue undisturbed throughout the regime in which $\bar{X}$ is finite, so that the shear-layer slope approaches $\frac{1}{2}\pi$. But, for the concave corner of angle $\alpha$ with $\alpha < \frac{1}{2}\pi$, the prospect of the shear layer being deflected through a right angle, effectively, is inconceivable. When $\alpha$ is $\frac{1}{2}\pi$, however, or in the jet collision problems (**1** to **7**), the right-angle deflection cannot be discounted immediately. On the other hand the region where $x - x_0$ is $O(\mathrm{Re}^{-\frac{3}{14}})$ may well decide the features of the eddy and reattachment phenomena (reattachment meaning, for the colliding jets, the merging of the two jets) for all the jet-flow separation problems considered above, and we conjecture that that is so. Figure 6 pictures the structure of the entire flow field between separation and reattachment, based on that conjecture. The determination of the complete reversed flow requires a further and comprehensive study, but we believe that the argument in favour of the relatively long scale response upstream (rather than the local response referred to in section 1) is fairly convincing.

**Acknowledgements**

The authors wish to thank Dr. D. S. Riley, with whom useful discussions on an application of this work have been held, and the referees for their suggestions. Professor N. Riley kindly pointed out the work in (**2**), and has informed us that Professor A. F. Messiter has deduced the same double-deck structure in a thermal boundary layer, as has Dr. S. H. Smith for (**2**).

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# ON MASS TRANSPORT VELOCITY DUE TO PROGRESSIVE WAVES

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SUMMARY

A double boundary-layer model is employed to investigate the mass transport velocity due to two-dimensional progressive gravity waves of large amplitude. For deep-water waves, the value of this velocity in the free-surface boundary layers is found to increase with distance from the generating region. In the far field, the mass transport velocity is found to dominate the Stokes drift velocity. Some qualitative discussion of the mass transport velocity field is also given for the case of shallow-water waves. Mathematical models are discussed both for fully-maintained and spatially-damped progressive waves.

## 1. Introduction

There exist many experimental and theoretical studies of mass transport velocity, the secondary velocity in gravity waves which causes fluid elements to migrate from some initial or specified location. However, amongst these studies, there are a number of errors and misleading statements, whilst some confusion still exists, especially with reference to the waves of larger amplitude.

Stuart (**1, 2**) and Riley (**3, 4**) introduced a double boundary-layer theory relating to solid bodies oscillating translationally in a viscous fluid which is otherwise at rest. The theory, which is valid for suitably large amplitudes, has been adapted by Dore (**5, 6**) to investigate the mass transport velocity due to *standing* surface and interfacial waves. In the present work, we postulate a simplified double boundary-layer model for *progressive* surface waves of sufficiently large amplitude. Mostly, attention is focussed on deep-water waves, but the case of progressive waves in finite depth is given some mathematical and qualitative discussion. We consider two-dimensional progressive wave motion in a homogeneous, incompressible fluid of infinite depth. The motion is first referred to a stationary system of Cartesian coordinates $(x', y', z')$ having origin in the mean free-surface level and $z'$-axis directed vertically upwards. Let $k$ denote the wave number, $\sigma$ the wave frequency, $\mathbf{q}'$ and $\mathbf{r}'$ the velocity and position vectors, $t'$ the time, and $\nu$ the kinematic viscosity. Henceforth, variables will be non-dimensionalized according to the scheme

$$\mathbf{r} = k\mathbf{r}', \qquad t = \sigma t', \qquad \mathbf{q} = (k/\sigma)\mathbf{q}', \qquad \epsilon = (\nu k^2/\sigma)^{\frac{1}{2}},$$