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HOMOGENIZATION IN MECHANICS
A SURVEY OF SOLVED AND OPEN PROBLEMS

Abstract. *We give an account of recent results and open problems in mechanics of composite media from the view point of asymptotic methods.*

1. Introduction

Generally speaking, homogenization is the study of the relationship between the local structure of a non-homogeneous medium and its macroscopic behavior. More specifically, homogenization denotes the mathematical techniques for the asymptotic study of physical media with a periodic (or nearly periodic) micro-structure. It is to be noticed that homogenization was one of the first non trivial examples of the so-called *G*-convergence of solutions of Partial Differential Equations. (De Giorgi and Spagnolo [24]). *G*-convergence developped into new concepts (Γ-convergence) concerning solutions of variational inequalities. As for this theory, mainly developped by De Giorgi and his co-workers, the reader is refered to the recent book by Attouch [2] and perhaps to the papers contained in the E. De Giorgi colloquium (Kree [44]). But in any case, very many "homogenization problems" in mechanics are not associated with the minimization of a functional and consequently they are out of the scope of Γ-convergence. Such are, for instance problems giving "memory effects" (non local operators in time) by homogenization.

Another mathematical theory linked with homogenization is the so-called "compactness by compensation" for which the reader is refered to Tartar [93], Boccardo et Murat [7], Murat [72]. See also in this connection Zhikov, Kozlov and Oleinik [98, 99, 100].

In this paper we adopt mainly the point of view of asymptotic expansions (see Van Dyke [96], Cole [20], Cole and Kevorkian [21] for the corresponding theory). We start with a rapid account of symptotic methods (Sect. 2). The following sections are devoted to problems (or groups of problems) in mechaincs. For most of them we give an account of some results, references and open problems.

General references on homogenization are Bakhvalov and Panasenko [3], Bensoussan, Lions and Papanicolaou [6] and Sanchez [84]. This survey paper is widely inspired by the works [86, 87] by the author.

2. Two-scale asymptotic expansions and local periodicity.

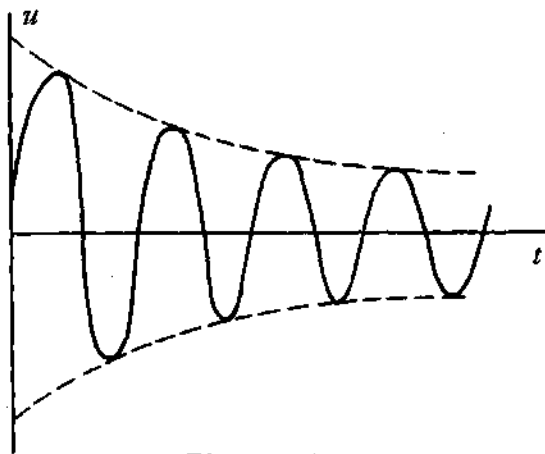This method is classical in mechanics of vibrations, when a small perturbation modify a motion which should be otherwise periodic in time, for instance, the motion of a pendulum submitted to a small damping is such that each *"period"* is almost analogous to the preceeding one, but the cumulative effect of the damping provokes important differences (of the amplitude, for instance) of the motion of two *"far located in time"* periods. To study this one introduces, aside the ordinary time t, two variables (the so-called fast and slow times)

Figure 2.1

$t^* = t$ and $\tau = \epsilon t$ (with ϵ small parameter) and we search for an asymptotic expansion of the solution $u^\epsilon(t)$ under the form

$$(2.1) \qquad u^\epsilon(t) = u^0(t^*(t), \tau(t)) + \epsilon u^1(t^*(t), \tau(t)) + \dots$$

and we try describing the local periodic phenomena by the dependence on t through t^*, and the slow modulation by the dependence on t through τ. We of course have

$$(2.2) \qquad \frac{d}{dt} = \frac{\partial}{\partial t^*} + \epsilon \frac{\partial}{\partial \tau} \ .$$

Moreover, as a *convention* for the sake of simplicity, we drope the star in t^*, and write

$$(2.3) \qquad u^\epsilon(t) = u^0(t, \tau) + \epsilon u^1(t, \tau) + \ldots \ ; \quad t = \epsilon t$$

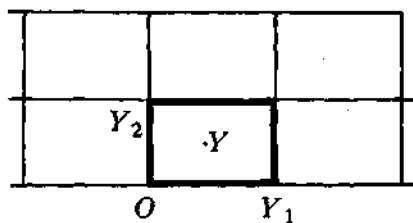$$(2.4) \qquad \frac{d}{dt} = \frac{\partial}{\partial t} + \epsilon \frac{\partial}{\partial \tau} \ .$$

Figure 2.2

According to analogous considerations, let Ω be a body made of a composite material in the R^3 space of the standard coordinates (x_1, x_2, x_3). Moreover, we assume that its mechanical properties are periodic with a small period, described with the aid of a small parameter ϵ as follows. In the auxiliar space of the variables (y_1, y_2, y_3) we consider a parallelepipedic period denoted by Y (with edges Y_1, Y_2, Y_3) as well as the parallelepipeds obtained by translations of an integer number of periods in the directions of the axes.

Let ϵY be the homothetic of Y with ratio ϵ. We consider the body Ω with the ϵY-periodic structure. Thus, some property $u^\epsilon(x)$ (here u may denote displacement, stress or some other property of the mechanical process under consideration) is searched under the form of an asymptotic expansion
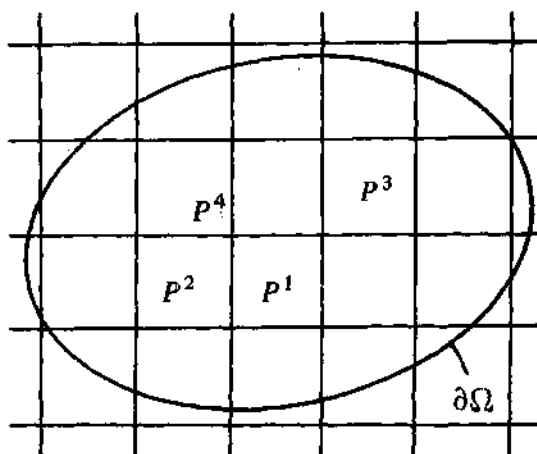
Figure 2.3

$$(2.5) \qquad \begin{cases} u^\epsilon(x) = u^0(z(x), y(x)) + u^1(z(x), y(x)) + \ldots \\ z(x) = x \ ; \quad y(x) = x/\epsilon \end{cases}$$

or merely (with the preceeding convention)

$$(2.6) \qquad \begin{cases} u^\epsilon(x) = u^0(x, y) + \epsilon u^1(x, y) + \ldots \\ y = \dfrac{x}{\epsilon} \ ; \quad \dfrac{d}{dx_i} = \dfrac{\partial}{\partial x_i} + \dfrac{1}{\epsilon} \dfrac{\partial}{\partial y_i} \end{cases}$$

and moreover, we intend to describe the influence of the periodic structure

(resp. of the other non periodic causes of the phenomenon, as the boundary $\partial\Omega$ and so on) by the microscopic variable (resp. the macroscopic variable z or merely x) in (2.5), (2.6). To this end, we search for an expansion (2.5) or (2.6) with functions u^i Y-periodic with respect to the variable y and smooth with respect to x. Indeed, *each $u^i(x,y)$ is defined on $\Omega \times Y$ (or on $\Omega \times R^3$, which amounts to the same, as it is Y-periodic).*

It is worhtwhile to see that *each term $u^i(x,y)$ is a locally periodic* function in the following sense. Let us compare the values of $u^i(x,y)$ at two points P^1, P^2 (Fig. 2.3) homologous by periodicity corresponding to two contiguous periods. By periodicity, the dependence on y is the same and the dependence on x is *"almost the same"* because the distance P^1P^2 is small and u^i is a smooth function of x. On the other hand, let P^3 be a point homologous to P^1 by periodicity, but located far from P^1. The dependence of u^i on y is the same, but the dependence on x is very different because P^1, P^3 are not near to each other. Finally we compare the values of u^i at two different points P^1, P^4 of the same period. The dependence on x is almost the same, but dependence on y is very different because P^1 and P^4 are not homologous by periodicity (in fact, the distance P^1, P^4 is *"large"* when measured with the variable y!).

It is evident that this locally periodic expansion is fit to describe the solution in regions of Ω far from its boundary, or from regions where the local effects are not ϵY-periodic, such as discontinuities of the microscopic structure. *I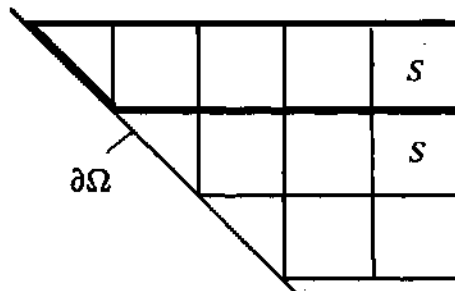n such regions,* the appropriate asymptotic expansions are *almost periodic* in the microscopic variable y *only with respect to displacements which are tangential to the boundary,* because the medium in fact is not periodic as for displacements normal to the boundary, and there is no reason for the solution to be *"almost invariant"* with respect to such displacements. As a consequence, near the boundary of the body (Fig. 2.3) we must

Figure 2.4

consider *boundary layers* where the solution is searched under the form (2.5) or (2.6), but now x runs in $\partial\Omega$ and y in the strip S (Fig. 2.4), and u^i is searched to be S-periodic. (Note, in Fig. 2.4 for instance, that S is a semi-infinite strip formed by Y-periods (plus perhaps *"parts"* of periods at the intersection with $\partial\Omega$). This situation is *easily described for boundaries parallel to a coordinate plane,* for instance $x_3 = \text{cost}$ (Fig. 2.5).

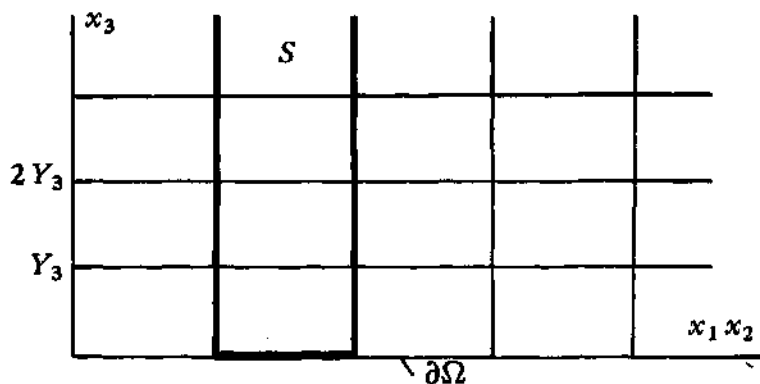In this case, the solution in the boundary layer region takes the form

Figure 2.5

(the superscript BL is for "Boundary Layer"):

$$(2.7) \qquad u^\epsilon(x) = u^{0BL}(x,y) + \epsilon u^{1BL}(x,y) + \dots$$

with

$$(2.8) \qquad u^{iBL}(x,y) = u^{iBL}(x_1, x_2, y_1, y_2, y_3),$$

y_1 and y_2 periodic with periods Y_1, Y_2, but not necessarily x_1, x_2, y_3 periodic.

REMARK 2.1. As for the expansion (2.6) far from the boundaries, the *"boundary conditions"* for the y variable amounts to the Y-periodicity. But in the boundary layer (2.7), the *"boundary conditions"* for y amounts to periodicity in y_1, y_2, genuine boundary conditions for $y_3 = 0$ and *"matching"* between (2.7) and (2.6) as $y_3 \to +\infty$ and $x_3 \to 0$. This amounts to saying that there is a *"transition region"* of the layer towards the bulk solution (2.6) far from the boundary. ∎

3. Matched asymptotic expansions.

We saw in the preceeding section that a function $u(x)$ may have asymptotic expansions of different nature in different regions, for instance, in the boundary layer near $\partial\Omega$ and the bulk region at the interior of Ω. It is clear that two such expansions *"must agree"*, i.e. the boundary layer contains a transition region between the genuine boundary layer and the *"outer"* region (outer to the boundary layer). As for this relation between the boundary

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layer and the bulk region, the tangential variables $(x_1, x_2, y_1, y_2$ in the case of (2.6), (2.7)) play the role of parameters, and x_3, y_3 are the relevant variables. We write

(3.1) $\qquad u^\epsilon(x) = u^0(x_3) + \epsilon u^1(x_3) + ...$ (outer or bulk expansion)

(3.2) $\qquad u^\epsilon(x) = u^{0BL}(y_3) + \epsilon u^{1BL}(y_3) + ...$ (inner or boundary layer expansion)

and we emphasize that the outer (resp. inner) expansion only depends on the outer or bulk variable x_3 (resp. on the inner or boundary layer variable $y_3 = x_3/\epsilon$).

We now give some *definitions. The outer (resp. inner) limit* of a function $u(x)$ is the limit as $\epsilon \to 0$ *for fixed outer variable x_3 (resp. inner variable y_3).* In the same way, the *m-term outer (resp. inner) expansion* is the asymptotic expansion of m terms of u^ϵ for $\epsilon \to 0$ *with fixed outer variable x_3 (resp. inner variable y_3).* For instance, $u^0(x_3)$ is the outer limit, and $u^{0BL}(y_3) + \epsilon u^{1BL}(y_3)$ is the 2-term inner expansion. As sometimes we deal with expansions the first term of which are not of order 0(1), we also define the *outer (resp. inner) representation* as the first non-zero term of the outer (resp. inner expansion).

We now give the *"matching rules"* expressing that the outer and inner expansions (3.1), (3.2) agree in some intermediate transition region. Justification of these rules may be seen in the general references given in sect. 1.

The matching at order 0(1) is:

(3.3) $\qquad$ Inner limit of (the outer limit) =
$\qquad\qquad$ = Outer limit of (the inner limit) .

Of course, the outer limit of u^ϵ in (3.1), (3.2) is $u^0(x_3)$; in order to compute its inner limit, we write it in the inner variable $y_3 = x_3/\epsilon$, and we compute the limit as $\epsilon \to 0$ for fixed y_3; this gives $\lim u^0(\epsilon y_3) = u^0(0)$ which is the left side of (3.3). Analogously, the right hand side is $u^{0BL}(+\infty)$. Thus, (3.3) amounts to

(3.4) $\qquad\qquad\qquad u^0(0) = u^{0BL}(\infty)$

or which amounts to the same, u^0 at the boundary $\partial\Omega$ equals the boundary layer first term far from the wall (far in the small variable y_3). It is easily seen that (3.3) or (3.4) amounts to the existence of an *"intermediate variable"* z small (resp. large) with respect to x_3 (resp. y_3) such that (3.1) and (3.2) give at the first order, the same information for $z = 0(1)$. We may take, for instance, $z = x/\epsilon^{1/2}$.

The *general matching rule* for m terms inner and outer expansions amounts to express that they agree, at the considered order, in the region where the intermediate variable z is $O(1)$.

Of course, in specific problems it may be useful writting the boundary layer terms as sum of their corresponding limits plus a complementary term, for instance in (3.4) $u^{OBL}(y) = u^0(0) + u^{0c}(y)$; matching $u^{0c}(y) \to 0$.
$$y \to \infty$$

4. Model equation - Elliptic equation of steady diffusion in divergence form.

We consider the boundary value problem

$$-\frac{\partial}{\partial x_i} \left[a_{ij}^\epsilon(x) \frac{\partial u}{\partial x_j} \right] = f \qquad \text{in } Q$$

$$u \big|_{\partial Q} = 0 \qquad \text{on } \partial Q = \text{boundary of } Q$$

where u is the unknown (the temperature, for instance) f is the given source term and a_{ij} are the (in general x-dependent) coefficients and ϵ is a parameter which is irrelevant for the time being. It is useful to write the equation:

$$(4.1) \qquad -\frac{\partial p_i}{\partial x_i} = f \quad ; \quad p_i = a_{ij} \frac{\partial u}{\partial x_j} .$$

Using the notation

$$(4.2) \qquad a_{ij}^\epsilon(x) = a_{ij}(x/\epsilon)$$

the ϵ-problem is:

Find u^ϵ satisfying

$$(4.3) \qquad -\frac{\partial}{\partial x_i} \left[a_{ij}^\epsilon(x) \frac{\partial u^\epsilon}{\partial x_j} \right] = f$$

$$(4.4) \qquad u^\epsilon \big|_{\partial Q} = 0 .$$

We look for the solution in the form (2.6), but in this problem we have in fact

$$(4.5) \qquad u^\epsilon(x) = u^0(x) + \epsilon u^1(x,y) + \dots$$

The expansion process induces:

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$$\begin{cases} \dfrac{d}{dx_i} = \dfrac{\partial}{\partial x_i} + \dfrac{1}{\epsilon}\dfrac{\partial}{\partial y_i} \Rightarrow \\[2mm] \dfrac{du^\epsilon}{dx_i} = \left(\dfrac{\partial u^0}{\partial x_i} + \dfrac{\partial u^1}{\partial y_i}\right) + \epsilon\left(\dfrac{\partial u^1}{\partial x_i} + \dfrac{\partial u^2}{\partial y_i}\right) + \dots \\[2mm] \underline{p}^\epsilon = \underline{p}^0(x,y) + \epsilon\,\underline{p}^1(x,y) + \dots \quad ; \quad p_i^0 = a_{ij}\left(\dfrac{\partial u^0}{\partial x_j} + \dfrac{\partial u^1}{\partial y_j}\right) . \end{cases} \tag{4.6}$$

At the present state, it should be noticed that when considering an expansion of the form (2.6), (under appropriate hypotheses) we have

Figure 4.1
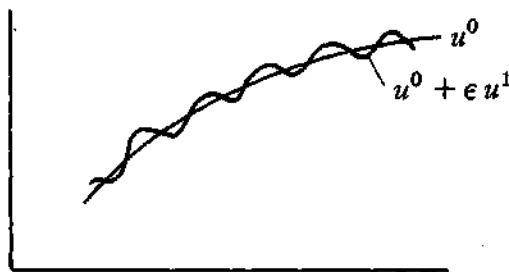
$$u^\epsilon(x) \longrightarrow u^0(x) \tag{4.7}$$

in L^2 strongly and H^1 weakly but not in H^1 strongly, and of course *grad* u^ϵ does not converge uniformly to *grad* u^0 (it does only in L^2 weakly). This is the reason why *the local gradient is very different of grad* u^0; as in mechanical problems the gradient is usually associated with stress and strain, we see that

$$\operatorname{grad}_x u^0 + \operatorname{grad}_y u^1 \tag{4.8}$$

is the expression of the local gradient up to terms in $0(\epsilon)$. Of course, *the expression (4.8) is easily writen when* $u^0(x)$ *is known* (see (4.19) here after). Coming back to our expansion:

$$\left(\dfrac{\partial}{\partial x_i} + \dfrac{1}{\epsilon}\dfrac{\partial}{\partial y_i}\right)p_i^\epsilon = f \Rightarrow \tag{4.9}$$

at various orders of ϵ:

$$\epsilon^{-1}\Big/ -\dfrac{\partial p_i^0}{\partial x_i} = 0 \iff \dfrac{\partial}{\partial y_i}\left[a_{ij}(y)\left(\dfrac{\partial u^0}{\partial x_j} + \dfrac{\partial u^1}{\partial x_j}\right)\right] = 0 \tag{4.10}$$

$$\epsilon^0\Big/ -\dfrac{\partial p_i^0}{\partial x_i} - \dfrac{\partial p_i^1}{\partial y_j} = f \qquad\qquad \text{and so on.} \tag{4.11}$$

We consider the *"mean value on Y"* operator:

(4.12)
$$\tilde{\cdot} = \frac{1}{|Y|} \int_Y \cdot \, dy$$

and we observe that by Y-periodicity:

(4.13)
$$\int_Y \frac{\partial p_i^1}{\partial y_i} \, dy = \int_{\partial Y} p_i^1 \, n_i \, dS = 0$$
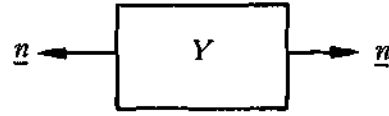

Figure 4.2

and moreover, $\sim$ commutes with $\partial/\partial x_i$. Then, when applied to (4.11):

(4.14)
$$\boxed{-\frac{\partial \tilde{p}_i^0}{\partial x_i} = f}$$

which is a "macroscopic equation" (it does not contain y) of the same form as (4.1).

The question now is if there exists a "constitutive relation"

(4.15)
$$\frac{\partial u^0(x)}{\partial x_i} \longrightarrow \tilde{\underline{p}}^0$$

giving the vector $\tilde{\underline{p}}^0$ as a function of $\nabla_x u^0$. If there exists such a linear relation, we shall write:

(4.16)
$$\tilde{p}_i^0 = a_{ij}^b \frac{\partial u^0}{\partial x_j}$$

where the superscript stands for *"homogenized coefficients"*. In order to obtain (4.16) we use (4.10)

(4.17)
$$\frac{\partial}{\partial y_i}\left[a_{ij}(y)\frac{\partial u^1(x,y)}{\partial y_j}\right] = -\frac{\partial u^0}{\partial x_i}\frac{\partial a_{ij}(y)}{\partial y_j}$$

which is an equation to obtain $u^1(y)$ if u^0 is supposed to be known; the variable x is here a parameter; (4.17) is then an elliptic equation with Y-periodicity "boundary conditions". Then, u^1 exists and is unique up to an additive constant (depending on x). In fact, *this amounts to find*

$$u^1 \in V_Y = \{v; \ v \in H^1_{loc}(R^3) \ Y\text{-periodic}\} \quad \text{with}$$

(4.18)
$$\int_Y a_{ij}(y)\frac{\partial u^1}{\partial y_j}\frac{\partial v}{\partial y_i}\,dy = -\frac{\partial u^0}{\partial x_i}\int_Y \frac{\partial a_{ij}}{\partial y_j} v\,dy \qquad \forall v \in V_Y \,.$$

And we *may* take $\widetilde{V}_Y$ = subspace of V_Y formed by the functions with $\tilde{v} = 0$ instead of V_Y in order to have uniqueness. Moreover, by linearity,

$$(4.19) \qquad u^1 = \frac{\partial u^0(x)}{\partial x_i} \, w^i(y)$$

where w^i is the solution of (4.11) for $\dfrac{\partial u^0}{\partial x_j} = \delta_{ij}$.

Consequently, the mean value of $\underline{p}^0$ is easily obtained

$$(4.20) \qquad \tilde{p}_i^0(x) = \underbrace{\left[a_{ij}(y) + a_{ik}(y) \frac{\partial w_j}{\partial y_k} \right]^{\sim}}_{= a_{ij}^b} \frac{\partial u^0}{\partial x_j}$$

which gives the law (4.16). The resulting a_{ij}^b are independent of $\nabla_x u^0$ and of the additive constant for u^1.

Properties of homogenization. The homogenized coefficients a_{ij}^b are *elliptic* as the $a_{ij}(y)$. Moreover

$$a_{ij}(y) = a_{ji}(y) \Rightarrow a_{ij}^b = a_{ji}^b \ .$$

On the other hand if the given problem conductivity is isotropic $(a_{ij}(y) = a(y)\delta_{ij})$, the homogenized coefficients *are not necessarily isotropic*; this is natural; if the medium is layered, the thermic flow flows the different parts either "in series" or "in parallel" in different directions:
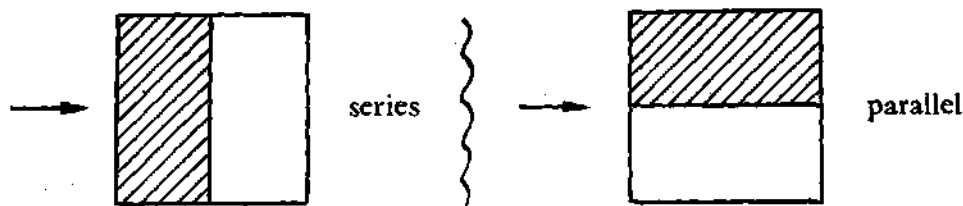

Figure 4.3

5. Examples in mechanics of solids.

In linear elasticity (Sanchez [84]) we have an analogous study with the standard modifications:

Model problem		Elasticity
u	$\longleftrightarrow$	$\underline{u} = (u_1, u_2, u_3)$
$\dfrac{\partial u}{\partial x_i}$	$\longleftrightarrow$	$e_{ij}(\underline{u}) = \dfrac{1}{2}\left(\dfrac{\partial u_i}{\partial x_j} + \dfrac{\partial u_j}{\partial x_i}\right)$
p_i	$\longleftrightarrow$	$\sigma_{ij} = a_{ijkl}\, e_{kl}(\underline{u})$
$a_{ij}(y)$	$\longleftrightarrow$	$a_{ijkl}(y)$
a_{ij}^b	$\longleftrightarrow$	a_{ijkl}^b

(a)

where a_{ijkl} and σ_{ij} are the elastic coefficients and the stress tensor, respectively.

The equations are:

$$(5.1) \qquad -\frac{\partial \sigma_{ij}}{\partial x_j} = f_i \quad ; \quad \sigma_{ij} = a_{ijkl}(y)\, e_{kl}(\underline{u}) \, .$$

The local equation (ϵ^{-1} term) is:

$$(5.2) \qquad -\frac{\partial}{\partial y_j}\left[a_{ijkl}(y)\, e_{kly}(\underline{u}^1)\right] = e_{klx}(\underline{u}^0)\,\frac{\partial a_{ijkl}(y)}{\partial y_j}$$

with Y-periodicity conditions for $\underline{u}^1$, x and t being parameters. Here e_{klx}, e_{kly} denote expressions analogous to (a)$_2$ with partial derivatives in x or y. The equation for the ϵ^0 term is:

$$-\frac{\partial \sigma_{ij}^1}{\partial y_j} - \frac{\partial \sigma_{ij}^0}{\partial x_j} = f_i \;\Rightarrow\; \sim \text{ as in (2.10)}$$

$$(5.3) \qquad -\frac{\partial \tilde{\sigma}_{ij}^0}{\partial x_j} = \tilde{f}_i \quad \text{and from (5.2)} \;\Rightarrow\; \tilde{\sigma}_{ij}^0 = a_{ijkl}^b\, e_{klx}(\underline{u}^0) \, .$$

The case where the strain-stress law is viscoelastic is more interesting (see Sanchez [84], sect. 6.4). If we have a heterogeneous medium with a viscoelastic *instantaneous* relation:

$$(5.4) \qquad \sigma_{ij}(k) = a_{ijkl}(y)\, e_{kl}(\underline{u}) + b_{ijkl}\, e_{kl}\left(\frac{\partial \underline{u}}{\partial t}\right)$$

the local equation is analogous to (5.2) but also contains terms in b_{ijkl} with $\partial/\partial t$; the local problem amounts to find $\underline{u}^1(t)$ with values in V_Y satisfying

$$(5.5) \qquad \int_Y \left(a_{ijkl}(y) + b_{ijkl}(y)\,\frac{\partial}{\partial t}\right)\left(\frac{\partial u_k^1}{\partial y_l} + \frac{\partial u_k^0}{\partial x_l}\right)\frac{\partial v_i}{\partial y_j}\, dy = 0 \qquad \forall \underline{v} \in V_Y$$

and t is no more a parameter. We also give the initial value $\underline{u}^1(0) = 0$ for instance (if not, for large time, the influence of the initial value vanishes). Then, the local equation is an evolution equation which we solve by semigroup theory:

(5.6)
$$\frac{\partial \underline{u}^1}{\partial t} + B\underline{u}^1 = -f_{ik}^0 \frac{\partial u_k^0}{\partial x_i} - f_{ik}^1 \frac{\partial}{\partial t} \frac{\partial u_k^0}{\partial x_k} \Rightarrow$$

$$\underline{u}^1(t) = \int_0^t e^{-B(t-s)} \quad \text{(right hand)} \ (s)\,ds$$

and we arrive at a *homogenized integrodifferential* strain-stress law of the form:

(5.7) $\quad \tilde{\sigma}_{ij}^0(t) = \alpha_{ijkl}\, e_{kl}(\underline{u}^0) + \beta_{ijkl}\, e_{kl}\left(\dfrac{\partial \underline{u}^0}{\partial t}\right) + \displaystyle\int_0^t g_{kl}(t-s)\, e_{kl}(\underline{u}^0)\,ds$.

We see that the new integro-differential term takes the local integration of a differential equation into account. This is in fact a realization of a rheological device, differential at the local level but integro-differential after integration.

Other interesting problems appear for *fissured media* (see Sanchez [84]) sect. 6.6 for the case without friction and Leguillon and Sanchez [48] with friction.

In the case without friction, we consider the period Y filled with homogeneous (or inhomogeneous, this only introduces an unessential com-

Figure 4.4

plication) but containing a fissure F as shown in fig. 4.4. The fissure may be either open or closed (or partially open), but the two lips cannot overlap. By choosing the unitary normal $\underline{N}$ in the same direction on both lips, the kinetic condition to be satisfied by the displacement vector $\underline{u}$ (within the small displacement approximation) is
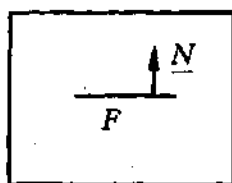
(5.8) $$[\![\underline{u} \cdot \underline{N}]\!] \geqslant 0$$

where the brackets are for the difference of values between the upper and the lower lips in fig. 4.4.

Moreover, the constraint $\underline{\sigma} \cdot \underline{N}$ is zero in points where the fissure is open (i.e., where we have (5.8) > 0) and a normal, compression vector taken equal values on the corresponding points of the upper and lower lips in points

where the fissure is closed (i.e., where we have (5.8) $= 0$).

The equivalent of the expansion (4.5) is now:

(5.9)
$$\underline{u}^\epsilon(x) = \underline{u}^0(x) + \epsilon\,\underline{u}^1(x,y) + \dots$$

and because $\underline{u}^0(x)$ does not depend on y, the asymptotic form of (5.8) is:

(5.10)
$$[\![\underline{u}^1 \cdot \underline{N}]\!] > 0\,.$$

In order to study the local problem, we define the set K_{YF} formed by the vector valued Y-periodic function $\underline{u}^1$ of class H^1 satisfying the kinematic condition (5.10). This is a convex, closed subset of the Hilbert space V_{YF} defined in the same way but without condition (5.10) (i.e., $\underline{u} \cdot \underline{N}$ may take any value on each lip of the fissure). Then, the local problem (analogous to (4.18)) is the variational inequality:

(5.11)
$$\begin{cases} \underline{u}^1 \in K_{YF} \quad \text{such that,} \quad \forall\,\underline{w} \in K_{YF}\,, \\[2mm] \displaystyle\int_{YF} a_{ijkl}[e_{klx}(\underline{u}^0) + e_{kly}(\underline{u}^1)]\,e_{ij}(\underline{w} - \underline{u}^1)\,dy \geq 0 \end{cases}$$

where $e_{klx}(\underline{u}^0)$ are given. This amounts to minimizing the elastic energy under the kinematic constraints given by (5.10) and the Y-periodicity. The first term of the expansion of the stress tensor σ is:

(5.12)
$$\sigma_{ij}^0 = a_{ijkl}[e_{klx}(\underline{u}^0) + e_{kly}(\underline{u}^1)]$$

and, as a consequence of the nonlinear inequality (5.11), *the homogenized strain-stress law*

(5.13)
$$e_{ijx}(\underline{u}^0) \Rightarrow \tilde{\sigma}_{ij}^0$$

is nonlinear. Moreover, it is a hyperelastic law associated with a convex function $W(e)$ such that

(5.14)
$$\tilde{\sigma}_{ij}^0 = \frac{\partial W}{\partial e_{ij}}$$

in fact, W is the value of the energy stored in the period (per unit volume), i.e. for given $e_{ijx}(\underline{u}^0)$, W is

(5.15)
$$W = \frac{1}{2\,|Y|}\int_Y a_{ijkl}[e_{klx}(\underline{u}^0) + e_{kly}(\underline{u}^1)][e_{ijx}(\underline{u}^0) + e_{ijy}(\underline{u}^1)]\,dy$$

where $\underline{u}^1$ is the corresponding solution of (5.11). This function W is convex, of class C^1 (but not C^2 in general: the elastic moduli are in general different in traction or compression).

The case *with friction* is much more difficult and is only solved in the two-dimensional case under the hypothesis that the fissure is a straight segment (as in fig. 4.4). Under this hypothesis it is possible to study processes where the fissure is either open at all its points or closed at all its points. The principal difficulty in this problem is that friction problems have not unique solutions in the case independent of time, and consequently the analogous of the local problem (5.11) do not determine $\underline{u}^1$ when $\underline{u}^0$ is given. Instead of this, it is possible to find $\underline{u}^1$ at any time t_1 if the "history" of $\underline{u}_0$ is known, that is to say if $\underline{u}_0(x, t)$ is known for $t < t_1$. The values of $\underline{u}^1$ and thus of $\tilde{\sigma}^0$ are then obtained by formal integration of small increments starting from an initial state in a somewhat complicated form (the details of which may be seen in Leguillon and Sanchez [48]) involving a "hidden variable" s measuring the eventual sliding of the upper lip with respect to the lower.

There is very much work to do in this direction. The non-elastic behavior is in connection with rheological properties of matter, plasticity and so on (see Suquet [91, 92]). In fact, homogenization is a powerful tool to investigate properties of matter and the relationship between local and macroscopic phenomena. It furnishes an explanation to some unexpected phenomena. For instance, *the failure of some samples of composites may be explained by thermal effects in the history of the material*. Indeed, in *thermoelasticity* the displacement $\underline{u}$ and temperature θ satisfy the equations (for steady processes)

$$
\begin{cases}
- \dfrac{\partial \sigma_{ij}^T}{\partial x_j} = f_i & ; \quad \sigma^T \equiv \sigma(\underline{u}) + \sigma(\theta) \\[2ex]
- \dfrac{\partial}{\partial x_i} \left(K_{ij} \dfrac{\partial \theta}{\partial x_j} \right) = \varphi & ; \quad \sigma_{ij}(\theta) = \delta_{ij}\, \gamma \theta
\end{cases}
$$

and $\sigma(\underline{u})$ is the standard elastic stress tensor, σ^T being for the "total stress tensor". It is easily seen that when heating a composite body with free surface, in general there is no solution with constant temperature and vanishing stress (but of course it exists for a homogeneous body). For a sufficiently strong heating the plasticity or fracture threshold may be reached at some points, and this causes irreversible damage. If the temperature is then brought back to zero, the material is in a deteriorated state. The homogenization of the

thermoelasticity equations is in Francfort [34], but much work is to be done about local properties and non linearities.

The preceeding problem is in connection with the interesting problem of *initial stress*. Even for materials with elastic behavior, if a composite is made after stretching some parts (think about *pre-constraint concrete*), the initial stress is to be added to the deformation one, and this may have an important influence on the reaching of the fracture threshold. In all these problems, the *local* behavior of stress and strain is to be used, of course (see (4.8)).

6. Fluid flow in porous media.

Let us give an account of the simplest problem in this field. In the R^3 space x we consider a periodic structure where each period is hollowed by three tubes forming the region Y_f (the complementary region Y_s being the rigid solid). If ϵY_f denotes the ϵ-homothetic of Y_f prolongated by ϵY-periodicity, we considered as "fluid region" $Q_\epsilon = Q \cap \epsilon Y_f$.
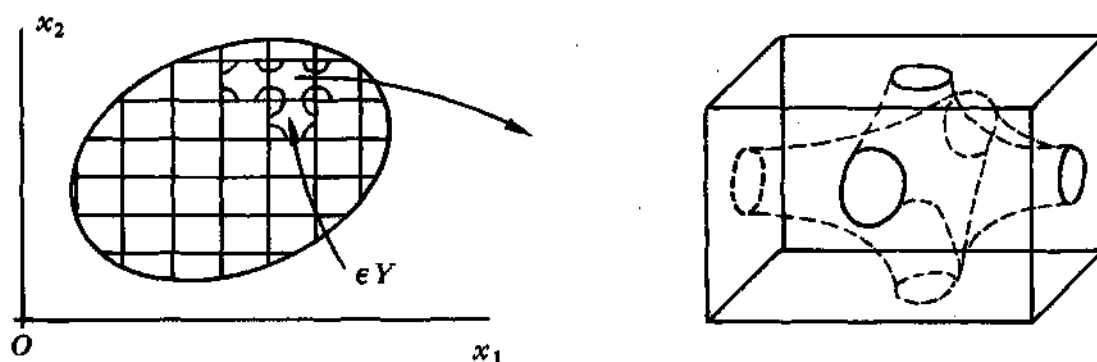

Figure 6.1

In fact, in this problem the domain depends on ϵ, and it is filled with a (homogeneous) viscous fluid. The equation and boundary conditions are:

$$(6.1) \qquad 0 = -\operatorname{grad} p^\epsilon + \Delta \underline{v}^\epsilon + \underline{f} \ ; \quad \operatorname{div} \underline{v}^\epsilon = 0$$

$$(6.2) \qquad \underline{v}^\epsilon \big|_{\partial Q_\epsilon} = 0 \ .$$

It is evident that as $\epsilon \searrow 0$, any point in the fluid is "near the boundary" where the velocity vector is zero (6.2) as a consequence of the no-slip condition for a viscous fluid. Consequently, we have some sort of "boundary

layer effect" and the velocity vector is asymptotically zero (there are other cases if the viscosity coefficient is small). The appropriate asymptotic expansion is

(6.3)
$$\underline{v}^{\epsilon}(x) = \epsilon^2 \underline{v}^0(x,y) + \epsilon^3 \underline{v}^1(x,y) + \dots$$
$$p^{\epsilon}(x) = p^0(x) + \epsilon p^1(x,y) + \dots$$

(6.4)
$$\frac{d}{dx_i} = \frac{\partial}{\partial x_i} + \frac{1}{\epsilon}\frac{\partial}{\partial y_i} \quad ; \quad \Delta = \frac{1}{\epsilon^2}\Delta_y + \frac{1}{\epsilon}\dots$$

and we obtain from (6.1), (6.2) at orders ϵ^0 and ϵ^{-1} :

(6.5)
$$\begin{cases} 0 = -\dfrac{\partial p^1}{\partial y_i} + \Delta_y v_i^0 + \left(f_i - \dfrac{\partial p^0(x)}{\partial x_i}\right) \quad ; \quad \underline{v}^0\big|_{\partial Y_f} = 0 \\[2mm] \text{div}_y\, \underline{v}^0 = 0 \end{cases}$$

with Y-periodicity, which constitute the local system. From $(6.1)_2$ at order ϵ^0 and applying the average operator $\sim$ of (4.12), we have:

(6.6)
$$\text{div}_y\, \underline{v}^1 + \text{div}_x\, \underline{v}^0 = 0 \Rightarrow \text{div}_x\, \tilde{\underline{v}}^0 = 0$$

which is the global equation. The local equation (6.5) is easily studied by using standard modifications of the classical theory of the Stokes problem.

We consider the space:

$$\underline{V}_Y = \{\underline{u} \; ; \; u_i \in H^1_{\text{loc}}(R^3), \; Y\text{-periodic}, \; \underline{u}\big|_{Y_s} = 0, \; \text{div}\, \underline{u} = 0\}$$

with the scalar product associated with the viscous dissipation in a period:

$$(\underline{v}, \underline{u})_{\underline{V}_Y} = \int_{Y_f} \frac{\partial v_i}{\partial y_j} \frac{\partial u_i}{\partial y_j} \, dy$$

Then, (6.5) amounts to find $\underline{v}^0 \in V_Y$ such that

(6.7)
$$(\underline{v}^0, \underline{u})_{\underline{V}_Y} = \left(f_i - \frac{\partial p^0(x)}{\partial x_i}\right)\int_{Y_f} u_i \, dy \qquad \forall \underline{u} \in \underline{V}_Y$$

where x is a parameter. By linearity, we have the local behaviour of the velocity

(6.8)
$$\underline{v}^0(x,y) = \left(f_k(x) - \frac{\partial p^0(x)}{\partial x_k}\right)\underline{w}^k(y)$$

where $\underline{w}^k$ is the solution of (6.7) for the parenthesis in the right hand side equal to δ_{ik}. It is to be noticed that the p^1 term in (6.8) desappears in the variational formulation (6.7); this is classical because the pressure gives a zero virtual power for the virtual velocities $\underline{w} \in \underline{V}_Y$; in fact:

$$\int_{Y_f} \frac{\partial p^1}{\partial y_i} w_i \, dy = \left(\text{as } \frac{\partial w_i}{\partial y_i} = 0 \right) = \int_{Y_f} \frac{\partial}{\partial y_i} (p^1 w_i) \, dy =$$

$$= \int_{\partial Y_f} n_i p^1 w_i \, ds = 0$$

(as $w_i = 0$ on the solid boundary Γ and by Y-periodicity on the rest of ∂Y_f).

Finally, by taking the mean value of (6.8) we have:

(6.9) $\quad \tilde{v}_i^0 = \left(f_k - \frac{\partial p^0(x)}{\partial x_k} \right) K_{ik} \quad ; \quad K_{ik} = \widetilde{w}_i^k$

Figure 6.2

where K_{ik} is the (symmetric and positive definite) permeability which only depends on the geometric structure of Y_f. (6.7) is "Darcy's law", which, with (6.6) gives the homogenized behaviour.

The theory described in this section only constitutes a first introduction to the very wide domain of fluid flow in porous media, where there are very many unsolved questions. Let us mention acoustic phenomena in porous media (Levy [64], Fleury [32], problems of flow between two neighbouring plates (the so-called Hele Shaw analogy) Bayada [5] and Dridi [27] where integro-differential homogenized equations appear, as in viscoelasticity.

Numerical computations of the local flow (i.e. of the vector $\underline{w}$) should be useful to understand the influence of the form and tortuosity of the pores on the coefficients of the Darcy's law. Of course, non linear terms appear in the local flow for sufficiently fast flows (see Sanchez [84], sect. 7.4. *Computations* on such flows may contribute to the understanding of the *nonlinear Darcy's law, arising of turbulence* at the local level and eventual *failure of Darcy's law as a deterministic law*. Problems of *two fluid phases* in a porous medium are important in the oil recovering industry, but the physics and thermodynamics of the problem, including surface tension and interfaces is not sufficiently comprised for the time being.

Interesting nonlinear phenomena appear when dealing with the flow of a visco-plastic Bingham fluid in a porous solid (see Lions et Sanchez [68] and
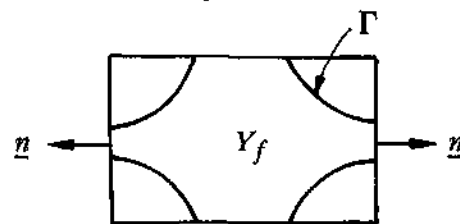
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Lions [67]). As the deformation of the fluid arises only for sufficiently large stress tensor (the fluid behaves as a rigid solid for small stresses), *the corresponding Darcy's law exhibites a threshold*: the average fluid flow is zero for sufficiently small *grad p*, and regions where the fluid is at rest appear.

7. Small vibrations of a solid fluid mixture. Influence of the parameters.

We know that homogenization is an asymptotic method describing the limit behavior as the parameter ϵ associated with the local structure tends to zero. It is clear that in problems containing other small parameters the limit process may be somewhat complicated. We note that the above mentioned problems of boundary layer appear for points near the boundary of the domain. There are very many examples of problems exhibiting very different behaviors according to the relative values of the parameters.

We consider here an elastic body containing pores filled with a compressible viscous fluid. As in Fig. 6.1 we assume that both the fluid and solid regions are connected. Many different situations appear according to the values of the viscosity coefficient η. The equations for the small (linearized) vibrations with the interface conditions of continuity of the displacement and stress are equivalent to the following variational formulation:

Find $\underline{u}^\epsilon(t)$ with values in $\underline{H}_0^1(\Omega)$ such that

$$\int_\Omega \rho^\epsilon(x) \frac{\partial^2 u_i^\epsilon}{\partial t^2} v_i \, dx + a^\epsilon(\underline{u}^\epsilon, \underline{v}) + \eta \, b(\underline{u}^\epsilon, \underline{v}) =$$
$$= \int_\Omega f_i v_i \, dx \qquad V\underline{v} \in \underline{H}_0^1(\Omega)$$

where the left hand side is the virtual power of the inertia forces, and a^ϵ, b^ϵ denote the forms associated with the elastic power (including the compressibility of the fluid) and the dissipation by viscosity; respectively:

$$a^\epsilon(\underline{u}, \underline{v}) = \int_{\Omega_f^\epsilon} a_{ijlm} \frac{\partial u_l}{\partial x_m} \frac{\partial v_i}{\partial x_i} \, dx + \gamma \int_{\Omega_f^\epsilon} \text{div} \, \underline{u} \, \text{div} \, \underline{v} \, dx$$

$$b^\epsilon(\underline{u}, \underline{v}) = 2\eta \int_{\Omega_f^\epsilon} e_{ij}(\underline{u}) e_{ij}(\underline{v}) \, dx$$

and Ω_s^ϵ, Ω_f^ϵ denote the domains occuped by the solid and the fluid.

If η is $O(1)$, i.e. asymptotically independent of ϵ, the appropriate asymptotic expansion is analogous to that of the elasticity or viscoelasticity:

$$\underline{u}^\epsilon(t, x) \simeq \underline{u}^0(t, x) + \epsilon \underline{u}^1(t, x, u) + \ldots$$

where the first term of the right handside is independent of y, i.e. the two phases of the mixture have the same global motion at the first order. The homogenized behavior is viscoelastic, analogous to (5.6) with *one phase*.

On the other hand, if the viscosity is small, $\eta = O(\epsilon^2)$, i.e. $\eta = \nu \epsilon^2$, $\nu = $ cost., the appropriate expansion is

$$\underline{u}^\epsilon(t, x) \simeq \underline{u}^0(t, x, y) + \epsilon \underline{u}^1(t, x, y) + \ldots$$

where $\underline{u}^0$ does depend on y. More exactly, u^0 may be writen

(7.1)
$$\underline{u}^0 = \underline{u}^s(t, x) + \underline{u}^{\text{rel}}(t, x, y)$$

where $\underline{u}^s$ is independent of y and corresponds to the first term of the expansion in the solid, and the term $\underline{u}^{\text{rel}}$, which vanishes in the solid part and in particular on the interface in order to satisfy the no-slip condition, represents the relative displacement of the fluid with respect to the solid. We then have a *two-phases* medium. The relative motion satisfies some Darcy's like integro-differential law with respect to the pressure p in the fluid. The limit behavior is described by $\underline{u}^s$, p and $\underline{u}^{\text{rel}}$ which satisfy equations of the form

$$\tilde{\rho} \frac{\partial^2 u_i^s}{\partial t^2} + \rho^f \frac{\partial^2 \tilde{u}_i^{\text{rel}}}{\partial t^2} - \frac{\partial \tilde{\sigma}_{ij}^T}{\partial x_j} = f_i$$

$$\tilde{\sigma}^T = a_{ijkl}^b e_{klx}(\underline{u}^s) - \alpha_{ij} p$$

$$\delta p + \text{div}_x \tilde{\underline{u}}^{\text{rel}} + \alpha_{ij} e_{ijx}(\underline{u}^s) = 0$$

$$\frac{\partial \tilde{u}^{\text{rel}}}{\partial t} = \int_0^t g_{ki}(t - s) \left(f_i - \frac{\partial p}{\partial x_i} - \rho^f \frac{\partial^2 u_i^s}{\partial t^2} \right) ds$$

which are analogous to the system proposed by Biot, but the Darcy's law is of integro-differential type.

The explanation of these phenomena is as follows. For fixed η, the viscosity drives the fluid with the solid and asymptotically there is only one phase. If the viscosity is small, the stress tensor in the fluid reduces to a pressure; the local equation of the type (4.14) becomes

$$\frac{\partial}{\partial y_j} (\delta_{ij} p) = 0 \Rightarrow p = p(x)$$

i.e., for small η the stress in the fluid reduces to a pressure which is asymptotically constant in each period. Then, $grad_x p$ drives a Darcy's like flow with respect to the solid which is of order ϵ^2/η (see (6.3) on account of the fact that the velocity is proportional to η^{-1}). It is then clear that, if $\eta = = 0(\epsilon^2)$, the relative velocity will be of order $0(1)$, in agreement with (7.1). At last, if η is small with respect to 1 but large with respect to ϵ^2, we shall have a stress tensor of the type pressure in the fluid and a negligibly small relative motion; the limit behavior will be *elastic*. We then have

$$\eta = 0(1) \Rightarrow \text{one phase, viscoelastic}$$
$$1 \gg \eta \gg \epsilon^2 \Rightarrow \text{one phase, elastic}$$
$$\eta \leqslant 0(\epsilon^2) \Rightarrow \text{two phases .}$$

We then have a two-parameter problem, and the preceeding scheme may be considered in the framework of matched asymptotic expansions. If we take η as a space-like variable and ϵ as a small parameter, the expansions $\epsilon \to 0$, $\eta = \text{cost.}$ and $\epsilon \to 0$, $\eta = \nu\epsilon^2$, $\nu = \text{cost.}$ are analogous to an outer and an inner expansions. The matching is given by the motion of elastic type. We may perform the change of variables (ϵ, η), (α, β):

$$\alpha = \eta \quad ; \quad \beta = \frac{\epsilon^2}{\eta} = \frac{1}{\nu}$$

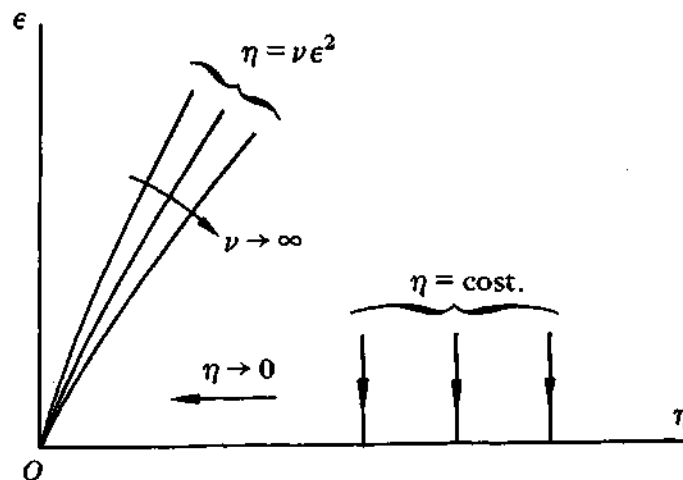and the outer and inner limits become $\alpha = \text{cost.}$ and $\beta = \text{cost.}$

Figure 7.1

In practice, for small values of ϵ and η, we shall compute α and β;

if the corresponding point is near the axis α (resp. β) we shall use the one phase (resp. two-phases) scheme; near the origin, the two schemes agree.
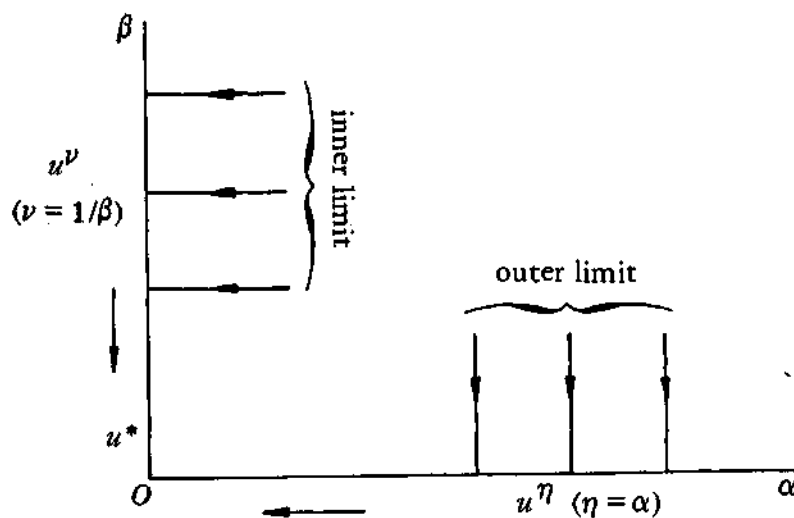

Figure 7.2

The references for these problems are Sanchez [84, chap. 8], Lévy [57], Fleury [33], Sanchez-Hubert [80, 81, 82], Nguetseng [73, 74] and Nguetseng et Sanchez [75].

As *open problems* in this direction we may point out the general motion (large deformations) of a mixture, boundary layers in the preceeding problems for several boundary conditions: clamped body, free surface body, and so on... In this last case the free boundary problem of the seepage of the fluid out of the porous body is completely open.

8. Fluid flow pas an array of small fixed obstacles, Darcy's and Brinkman flows.

The situation of sect. 5 depends strongly on the asymptotic dimensions of the obstacles.

We shall give the asymptotic structure of the solution for the fluid Stokes (linear) flow of a viscous fluid past an array of fixed obstacles (note that the obstacles are supposed to be fixed; this is unrealistic in R^3; the corresponding problem in R^2, flow past an array of bars will be given later) see Lévy [59], [61] Sanchez-Palencia [85] and Geymonat et Sanchez-Palencia [36] for these problems.

We consider in the geometric framework of sect. 6, obstacles $\eta \vartheta$ in the

y variables (i.e. $\epsilon \eta \vartheta$ obstacles in the standard x variables), the dimension of the period being of order ϵ as usual. An important parameter for the description of the asymptotic behaviour is:

$$(8.1) \qquad m = \lim \frac{\epsilon^2}{\eta} \ .$$

(In fact, we have a problem with two small parameters ϵ, η, but the asymptotic behaviour depends on m; one may consider one of the parameters as a function of the other.

If $\epsilon \searrow 0$ and η is fixed ($\eta = 0(1)$), we have the classical homogenization problem of sect. 4 the asymptotic of velocity is:

$$(8.2) \qquad v_i^{\eta} = \epsilon^2 \, w_i^{k\eta}(y)\left(f_k - \frac{\partial p^0}{\partial x_k}\right)$$

and p^0 is defined by $\mathrm{div}_x \, \underline{v}^{\eta} = 0$ i.e.:

$$(8.3) \qquad \frac{\partial}{\partial x_i}\left[K_{ik}^{\eta}\left(f_k - \frac{\partial p^0}{\partial x_k}\right)\right] = 0 \ ;$$
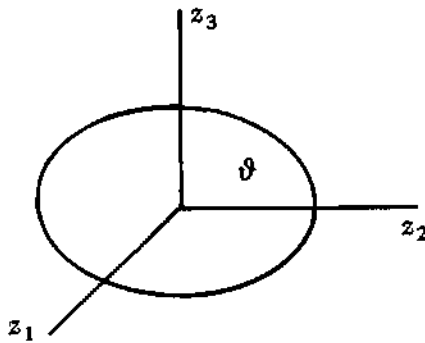

Figure 8.1

where K_{ik}^{η} is the permeability tensor.

Now for the case $\eta \ll 1$, the flow without interaction between different obstacles gives a good approximation. This flow is defined in the following way. In R^3 we define the velocity and pressure fields associated with a unit flow at infinity in the k direction:

$$(8.4) \qquad \begin{cases} -\Delta_z \, \underline{V}^k + \mathrm{grad}_x \, P^k = 0 \\ \qquad \qquad \mathrm{div}_z \, \underline{V}^k = 0 \\ \underline{V}^k \big|_{\partial \vartheta} = 0 \ ; \quad \underline{V}^k \xrightarrow[|z| \to \infty]{} \underline{e}_k \end{cases}$$

and we consider the associated force on the obstacle, T defined by:

$$\underline{T}(\underline{V}^k) = \underline{e}_i \int_{\partial \vartheta} \sigma_{ij} \, n_j \, ds$$

where

$$\sigma_{ij} = -P\delta_{ij} + \left(\frac{\partial v_j}{\partial z_i} + \frac{\partial v_i}{\partial z_j} \right)$$

and the "translation tensor" H of components

(8.5)
$$H_{ij} = T_i(\underline{V}^j)$$

which is symmetric and positive definite. The asymptotic structure of the flow is given by:

(8.6)
$$\underline{v}(x) \simeq f(\epsilon)\, U_k(x)\, \underline{V}^k(z)$$
$$\uparrow$$
$$\text{(centered at the obstacles)}$$

where

(8.7)
$$f(\epsilon) = \begin{cases} \epsilon^2 \eta^{-1} & \text{if } m = 0 \\ 1 & \text{if } m \text{ infinite or finite } \neq 0 \end{cases}$$

and $\underline{U}(x)$ is given by the solution of:

(8.8)   if $m = 0$
$$\begin{cases} grad_x\, p^0 + H \cdot \underline{U} = \underline{f} \\ div_x\, \underline{U} = 0 \; ; \; \underline{U} \cdot \underline{n}\big|_{\partial Q} = 0 \end{cases}$$

(8.9)   if $\underset{\neq 0}{m}$ finite
$$\begin{cases} -\Delta_x\, \underline{U} + grad_x\, p^0 + \frac{1}{m} H \cdot \underline{U} = \underline{f} \\ div_x\, \underline{U} = 0 \; ; \; \underline{U}\big|_{\partial Q} = 0 \end{cases}$$

(8.10)   if $m = \infty$
$$\begin{cases} -\Delta_x\, \underline{U} + grad_x\, p^0 = \underline{f} \\ div_x\, \underline{U} = 0 \; ; \; \underline{U}\big|_{\partial Q} = 0 \, . \end{cases}$$

In fact, the asymptotic behaviour (8.6) contains an order function $f(\epsilon)$, a uniform (i.e. independent of y, z) flow $\underline{U}(x)$ given by (8.8)-(8.10) and the asymptotic structure $\underline{V}$ near the obstacles. Moreover, for $m = \infty$ (i.e. (8.10), obstacles very small) there is no action of the obstacles on the $\underline{U}$ flow, which is the same as in the absence of obstacles. For $m = 0(1)$, (i.e. (8.9)) we have the Brinkman's flow: the global movement $\underline{U}$ takes into account the "force of the obstacles on the fluid". Last, for $m = 0$ (i.e. (8.8)) we have the same as in (8.9) but the global flow is negligibly small with respect to that of the preceeding case, and is given by the order function $f(\epsilon)$.

In the *two dimensional case*, the analogous of the flow (8.4) does not exist (Stokes' paradox). In this case $(8.4)_3$ must be replaced by:

$$(8.11) \qquad \underline{V}^k \simeq \underline{\varrho}_k \log |z| \qquad (|z| \to \infty) .$$

Moreover, $f(\epsilon)$ in (6.7) must be replaced by:

$$(8.12) \qquad f(\epsilon) = \begin{cases} \epsilon^2 & \text{if } m = 0 \\ [\log \eta^{-1}]^{-1} & \text{if } m \text{ is infinite or finite } \neq 0 \end{cases}$$

and the asymptotic behaviour (8.6) becomes:

$$(8.13) \qquad \underline{v} \simeq [f(\epsilon) \log \eta^{-1}] \, U_k(x) \, \underline{V}^k(z) .$$

9. Suspensions of solid particles in a fluid.

We pointed on that the periodicity of the microstructure is an important hypothesis in homogenization. In fact, *local periodicity* is sufficient, i.e. each cell is almost the same as the neighbouring ones, but it may be very different from the far located cells. Now, if we have in a fluid medium particles in suspension, the geometric structure, even if it was periodic at the initial time, undertakes large deformations, i.e. the structure depends on the motion itself. Of course, a periodic structure is not a very realistic scheme for a suspension, but the theoretical results may be used as a model for other problems. On the other hand, it may be seen that the local periodicity of a structure is preserved by large deformations.
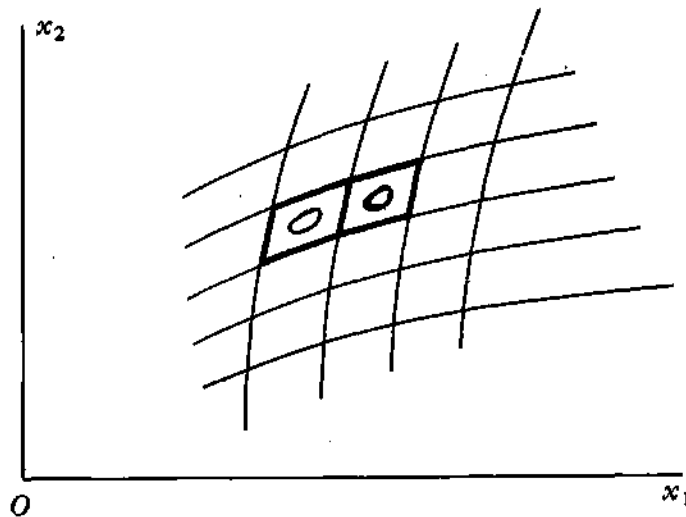

Figure 9.1

We consider (Lévy and Sanchez [65], Lévy [55], Sanchez [89]) the

equations of the fluid motion

$$\begin{cases} \operatorname{div} \underline{v}^\epsilon = 0 \\[2mm] \rho_0 \left(\dfrac{\partial v_i^\epsilon}{\partial t} + v_j \dfrac{\partial v_i}{\partial x_j} \right) = \dfrac{\partial \sigma_{ij}^\epsilon}{\partial x_j} + F_i \\[2mm] \sigma_{ij}^\epsilon = -p^\epsilon \delta_{ij} + 2\mu e_{ij}(\underline{v}^\epsilon) \end{cases}$$

as well as the resultant and moment equations for each particle, and an asymptotic expansion

(9.1)
$$\begin{cases} \underline{v}^\epsilon(t,x) = \underline{v}^0(t,x) + \epsilon \underline{v}^1(t,x,y) + \dots \\[2mm] p^\epsilon(t,x) = p_0(t,x) + \epsilon p^1(t,x,y) + \dots \end{cases}$$

with terms which are Y-periodic functions of y, the period Y depending on t and the large scale variable x, in a fashion to be defined later. The asymptotic process leads to homogenized equations of the form

(9.2)
$$\begin{cases} \tilde{\rho} \left(\dfrac{\partial v_i^0}{\partial t^2} + v_j^0 \dfrac{\partial v_i^0}{\partial x_j} \right) - \beta_{ijlm} \, v_j^0 \, e_{lm}(\underline{v}^0) = \dfrac{\partial \tilde{\sigma}_{ij}^0}{\partial x_j} + \tilde{F}_i \\[2mm] \operatorname{div} \underline{v}^0 = 0 \end{cases}$$

where the average stress tensor $\tilde{\sigma}^0$ is associated with an *anisotropic viscosity which depends on the form of the period at the considered time.* On the other hand, we see that *the inertia terms contain an extra term* with the coefficients β which depend on the microstructure. This correction of the inertia terms is due to the local structure of the velocity field, and even appears in inviscid flows.

As for the microstructure, it depends on a finite number of parameters for each cell (i.e. depending on the large scale variable x). For instance, we may take the vectors which are edges of the period, and the six parameters defining the position of the particle. It may be shown (the fact that $\underline{v}^0$ is independent of y and that $\underline{v}^1$ is Y-periodic plays an essential role here) that the geometric structure evolves in time keeping its locally periodic character. The equations giving the variation in time of the set $\mathscr{S}$ of parameters take the form

$$\frac{d\mathscr{S}}{dt} = \alpha(\mathscr{S}) : \nabla \underline{v}^0$$

which are of the form given by Hinch and Leal [39] on the basis of a phe-

nomenological study of the problem.

In the preceeding study we did not find any evolution of the concentration of solid particles or global motion of particles under the action of the given forces $\underline{F}$. In fact, *sedimentation phenomena* are very slow and the terms $\underline{v}^0 + \epsilon \underline{v}^1$ are more important than them. Sedimentation phenomena may only be apparent if the terms $\underline{v}^0$ are zero. This may happen if the initial and boundary values for $\underline{v}^0$ are zero; in addition, if the boundary values for $\underline{v}^0$ are zero, but not the initial values, the viscous dissipation gives (see Sanchez [89]) a decay of the velocity to zero as $t \to +\infty$. In fact this decay is in certain cases exponential and the corresponding motion may be considered to be zero after some time. In any case, *if* $\underline{v}^0 = 0$, *lower order terms may be studied, and sedimentation appears.* This sedimentation implies a gravity driven motion in each period which implies relative velocity of havy portions with respect to light portions in the direction of the applied gravity. On the other hand, *this motion implies in general rotation of the particles* and *not only translational motion* through the fluid.

It is then seen (see Sanchez [89] for details) that the appropriate expansion takes the form

$$\begin{cases} \underline{v}^\epsilon = \epsilon \underline{v}^1(t,x) + \epsilon^2 \underline{v}^2(t,x) + \dots \\ p^\epsilon = p^0(x) + \epsilon p^1(t,x,y) + \epsilon^2 p^2(t,x,y) + \dots \end{cases}$$

which is analogous to (9.1) but for "slow motions", with factor ϵ, and of course a term $p^0(x)$ which is associated with the fluid at rest. The term $\underline{v}^2$ writes

$$\underline{v}^2 = \underline{v}^{2\,\mathrm{del}} + \underline{v}^{2g}$$

where $\underline{v}^{2\,\mathrm{def}}$ is associated with the deformation $\underline{v}^1$, and $\underline{v}^{2g}$ with the gravity forces. This second term is responsible for sedimentation i.e. migration of particles with respect to the fluid; but it should be noticed that this term may be masked by $\underline{v}^{2\,\mathrm{del}}$ if the later is sufficiently large.

There are very many *open problems* in this direction. The evolution of the microstructure, which is necessary to compute the macroscopic flow is not sufficiently known: much work is necessary (working out examples, computing coefficients of functions of the microstructure and so on) to understand the influence of the micro-location of particles and the location of the applied forces (see Lévy [55] for problems with couples). See Nunan [77] for some numerical results on homogenized viscosity coefficients. The problem of the *boundary layers* near a solid wall deserves attention: there is

very much work in this direction and of course also on the influence of the boundary layers on the general flow. On the other hand, our knowledge ot the evolution of the microstructure is far from complete, in particular the question of the eventual shocks of particles seems open. Compressibility effects would also be interesting to deal with. It is to be noticed that very many features of suspension theory also appear in mixture theory of fluids; in this case some equations are simpler because the complicated questions of the rigid-body velocity field are not present, but the deformation of the microstructure is more complicated because deformations of drops may involve infinitely many parameters.

10. Composite plate theory.

We only give some indications about the difficult and interesting problem of heterogeneous plates; the reader is refered to Caillerie [13, 14, 15] for an explicit treatment. We mention that in a heterogeneous elastic plate classical homogenization must be modified on account of the fact that there is no periodicity of the microstructure in the direction normal to the plate. Thus, periodicity conditions must be replaced by other conditions (Neumann, for instance, if the plate is free) on the surfaces. But the main difficulty in heterogeneous plate theory come from the fact that, if the plate is not symmetric, the traction forces in its plane induce flexural deformation of the plate.
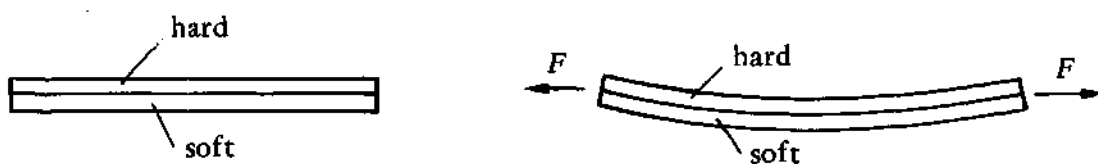

Figure 10.1

As a matter of fact flexion and traction are coupled and the "homogenized" behavior is not that of the classical theory of plates. In fact, the asymptotic behavior is given by the displacements:

$$\begin{cases} u_1 = \epsilon v_1 - x_3 \dfrac{\partial v_3}{\partial x_1} \quad ; \quad u_2 = \epsilon v_2 - x_3 \dfrac{\partial v_3}{\partial x_2} \\ u_3 = v_3 \end{cases}$$

where v_1, v_2, v_3 are functions of the coordinates x_1, x_2 in the plane of the

plate and x_3 is normal to it. The equilibrium equations are

$$\frac{\partial N_{\alpha\beta}}{\partial x_\beta} = 0 \quad ; \quad \frac{\partial^2 M_{\alpha\beta}}{\partial x_\alpha \, \partial x_\beta} = 0 \qquad (\alpha = 1,2)$$

where the stretching stresses $\underline{N}$ and the bending moments $\underline{M}$ are defined by the relations

$$\begin{cases} N_{\alpha\beta} = A^{1\,1}_{\alpha\beta\,\gamma\delta} \, e_{\gamma\delta}(\underline{v}) - A^{1\,2}_{\alpha\beta\,\gamma\delta} \, \dfrac{\partial^2 v_3}{\partial x_\gamma \, \partial x_\delta} \\[4mm] M_{\alpha\beta} = A^{2\,1}_{\alpha\beta\,\gamma\delta} \, e_{\gamma\delta}(\underline{v}) - A^{2\,2}_{\alpha\beta\,\gamma\delta} \, \dfrac{\partial^2 v_3}{\partial x_\gamma \, \partial x_\delta} \end{cases}$$

which involve the "homogenized coefficients" A depending on the microstructure.

Very many problems are open in plate theory, in particular layers near the boundary of the plate for various support conditions, and the singularities of the stress associated with them. Some (far from complete!) results may be seen in Sanchez [87]; we give some indications about them here after.

11. Boundary layers in elasticity problems.

Let us consider a free boundary $y_3 = 0$ of a composite elastic solid such that a boundary of the period Y may be taken to be also $y_3 = 0$. The asymptotic expansion far from the boundary is classically:

$$(11.1) \qquad \underline{u}^\epsilon(x) = \underline{u}^0(x) + \epsilon \, \underline{u}^1(x,y) + \dots$$
$$y = x/\epsilon$$

The corresponding expansions for strain and stress are

$$(11.2) \qquad e^\epsilon_{ij}(x) = e^0_{ij}(x,y) + \epsilon \, e^1_{ij}(x,y) + \dots$$

$$(11.3) \qquad \sigma^\epsilon_{ij}(x) = a_{ijlm} \, e^\epsilon_{lm} = \sigma^0_{ij}(x,y) + \epsilon \, \sigma^1_{ij}(x,y) + \dots$$

where

$$(11.4) \qquad \begin{cases} e^0_{ij}(x,y) \equiv e_{ijx}(\underline{u}^0) + e_{ijy}(\underline{u}^1) \\[3mm] \sigma^0_{ij} = a_{ijlm} \, e_{lm}(\underline{u}^0) \end{cases}$$

with

(11.5) $\qquad u^1 = e_{krx}(\underline{u}^0)\,\underline{w}^{kr}(y) + \text{cost}$

where $\underline{w}^{kr}$ are the Y-periodic solutions of the local problems

(11.6) $\quad -\dfrac{\partial}{\partial y_j}\{a_{ijlm}(y)(\delta_{kl}\delta_{mr} + e_{lmy}(\underline{w}^{kr}))\} = 0$

and the homogenized coefficients are

(11.7) $\quad a^b_{ijkr} = \{a_{ijlm}(\delta_{kl}\delta_{mr} + e_{lmy}(\underline{w}^{kr}))\}^{\sim}$

where the tilde $\sim$ denote average on Y.
Then $\underline{u}^0$ is the solution of the homogenized
equation (11.8) and the boundary condition
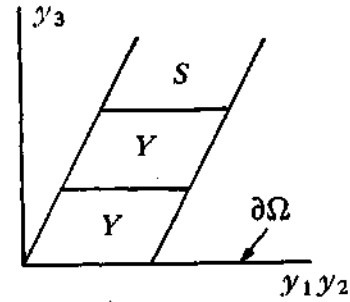(which we write in (11.9) for the free boundary of Fig. 11.1.

Figure 11.1

(11.8) $\qquad -\dfrac{\partial \tilde{\sigma}^0_{ij}}{\partial x_j} = f_i \quad ; \quad \tilde{\sigma}_{ij} = a^b_{ijlm}\,e_{lmx}(\underline{u}^0)$

(11.9) $\qquad \tilde{\sigma}^0_{3i} = 0 \qquad \text{on } \partial\Omega \;.$

REMARK 11.1. In (11.9), $\sim$ denotes the mean value on Y; in fact it is also the mean value on the face Γ of the period (see Fig. 11.2) or on any section of the period $y_3 = c$ which is independent of c. Indeed, the local equation for u^1 is
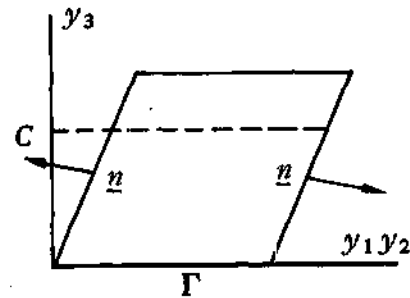
Figure 11.2

$$-\dfrac{\partial \sigma^0_{ij}(x,y)}{\partial y_j} = 0$$

and integrating by parts in the region of Y between $y_3 = 0$ and $y_3 = c$ (Fig. 11.2) we have

(11.10) $\qquad \displaystyle\int_{Y \cap \{y_3 = c\}} \sigma^0_{i3}\,dy_1\,dy_2 = \int_{\Gamma} \sigma^0_{i3}\,dy_1\,dy_2$

(nothe that the integrals on the lateral faces cancel by Y-periodicity). $\qquad\blacksquare$

Now we study of the *boundary layer*.
We introduce the complementary term $\underline{u}^{1c}$:

(11.11) $\qquad \underline{u}^\epsilon(x) = \underline{u}^0(x) + \epsilon(\underline{u}^1(x,y) + u^{1c}(x,y)) + 0(\epsilon^2)$

which satisfies (11.12)-(11.15):

(11.12) $\quad -\dfrac{\partial}{\partial y_j}(a_{ijlm}(y)\,e_{lmy}(\underline{u}^{1c})) = 0 \qquad$ in S

(11.13) $\quad \underline{u}^{1c}(x,y) \qquad$ is S-periodic in y

(11.14) $\quad grad_y\,\underline{u}^{1c} \xrightarrow[y_3 \to +\infty]{} 0$

(11.15) $\quad a_{3ilm}\,e_{lmy}(\underline{u}^{1c}) = -\sigma^0_{3i} \equiv -a_{3ilm}(e_{lmx}(\underline{u}^0) + e_{lmy}(\underline{u}^1))$.

The existence and uniqueness of $\underline{u}^{1c}$ defined up to a constant (i.e. only depending on x_1) vector of R^3 follows from the Lax-Milgram theorem by noticing that it is equivalent to find $\underline{u}^{1c} \subset V$ such that

(11.16) $\quad \displaystyle\int_S a_{ijlm}\,e_{lmy}(\underline{u}^{1c})\,e_{ijy}(\underline{v})\,dy = \int_\Gamma \sigma^0_{3j}\,v_j\,dy_1\,dy_2 \qquad \forall \underline{v} \in V$

where V denote the Hilbert space of the S periodic vectors (defined up to a constant additive vector) with finite

$$\|\underline{v}\|^2_V = \int_S e_{ijy}(\underline{v})\,e_{ijy}(\underline{v})\,dy \ .$$

We note, in particular that, by virtue of (11.9) and Remark 11.1, the right side of (11.16) takes the same value for $\underline{v}$ or $\underline{v}$ + *constant*.

We note that the preceeding study only shows that $grad_y\,\underline{u}^{1c} \in \underline{L}^2(S)$; this amounts to saying that in some generalized sense it tends to zero as $y_3 \to \infty$; in fact this is true exponentially as was proved by Tartar (see Lions [67] sect. I.10.4) and Dumontet [28].

12. Singularities in elliptic non smooth problems.

Singularity theory, i.e. lack of regularity of the solutions is a mathematical theory which is independent of homogenization. Nevertheless, in composite media there are very many situations where singularities appear at the microscopic level, i.e. the local gradient, given by (4.8) or analogous expressions take infinite values at some points. This is a widely open research domain; the references for the mathematical theory are given here after; some physical or computational results may be seen in Anquez [1], Bogy [8, 9],

Davet and Destuynder [23], Dempsey and Synclair [25, 26], Raju and Crews [78], Wang [97], Zwiers [101].

Most of the solutions of problems in mathematical physics are given by variational problems in spaces of the kind H^1 of Sobolev, i.e. they exist and are unique in spaces of functions having square integrable first order derivatives. This is a very poor regularity, and such solutions may be singular at some points; more precisely, *grad u* (where u is the considered solution) may tend to infinity at some points.

Physically speaking, such solutions are meaningless at the vicinity of such singularities: in fact, the smallness hypotheses for linearization are not fulfilled. Then, such *singularities show that new phenomena* (non linearities, qualitative modification of the medium, qualitative modification of the medium, etc.) *may appear.* An example is the *lightning rod.* The singularities of *grad u* (u is a harmonic function, the electric potential) at the point 0 provokes ionization of the air, which becomes conducting near 0.

Figure 12.1

In elasticity theory, infinite values of grad u i.e. singularities of strain and stress provoke modifications of the elastic behavior: depending on the nature of the material, it may become plastic or a fracture may appear.

The study of singularities is well developped for second order elliptic equations in R^2 but there is much to do concerning problems in R^3 and elliptic systems, in particular the elasticity system. Fortunately, some problems in boundary layers are in fact in R^2, and we have at our disposal some (but not all) tools to study them. The principal references on these problems are Grisvard [37, 38], Lemrabet [51, 52], Kondratiev [41, 42], Sanchez [87], Sovin [90], and for numerical computation, Lelièvre [49, 50], and Leguillon et Sanchez-Palencia [47].

Let us consider an elliptic problem of the form

$$(12.1) \qquad -\frac{\partial}{\partial x_j}\left(a_{ij}(x)\frac{\partial U}{\partial x_i}\right) = f$$

in a domain Ω of R^2 with appropriate boundary conditions. Under suitable smoothness hypotheses about the coefficients and $\partial\Omega$, classical regularity theory holds. In particular, if f belongs locally to the H^m space (m real > 0) and the boundary conditions are homogeneous, the solution belongs to
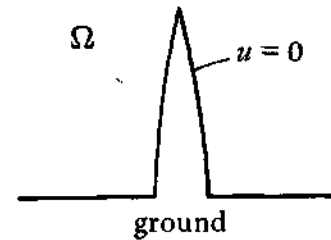
H^{m+2} in any subdomain D' included in D. Moreover, if Ω is bounded, an inequality of the type

(12.2) $$\| U \|_{H^{m+2}(\Omega)} \leq C(\| f \|_{H^m(\Omega)} + \| U \|_{H^m(\Omega)})$$

holds for $m \geq 0$, with a constant C which depends only on Ω, m and the coefficients of the equation.

An analogous situation holds if the coefficients of (1.1) are piecewise smooth, having a discontinuity line Γ where (1.1) is considered in the distribution sense (here the brackets denote "jump"):
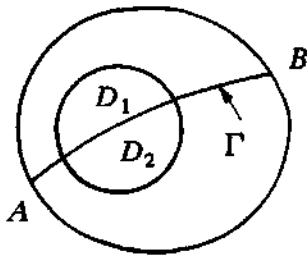

Figure 12.2

(12.3) $[U] = 0$, $\left[a_{ij} \dfrac{\partial U}{\partial x_j} n_i \right] = 0$ on Γ

then, if $f \in H^m$, the solution U belongs to H^{m+2} on each side of Γ (Fig. 12.2) (of course on Γ itself the solution is not of class H^n, $n \geq 2$ as the first derivatives are not continuous across Γ) (see Ladyzhenskaya et Ouralceva [45] sect. III.16) *in regions where Γ is smooth*, buth not (as we shall see later) at points as A, B in Fig. 12.2. For instance, in problems with layers, singularities may appear at the intersection of layers with the boundary $\partial \Omega$.

A different situation appears if the coefficients are smooth (constant, say) but the boundary $\partial \Omega$ does not, in particular if it has *angular points*. In such a case, the local regularity depends on the angle ϕ of the domain.

Figure 12.3

For instance, let consider the Laplace equation with Neumann condition:

(12.4) $$- \Delta U = 0 \quad ; \quad \frac{\partial U}{\partial n} = 0 .$$

Searching for solutions of the form $(r, \theta$ = polar coordinates):

(12.5) $$U(x_1, x_2) = r^\alpha u(\theta)$$

we obtain

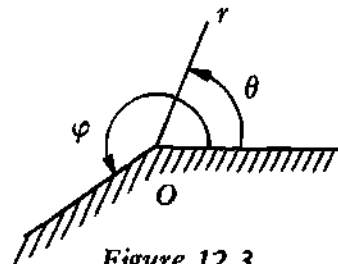(12.6) $u = A \cos \alpha \theta$ for $\alpha = 0 , \pm \dfrac{\pi}{\phi} , \dfrac{2\pi}{\phi} , \ldots$

Of course, *grad U* behaves as $r^{\alpha-1}$; we are interested by *solutions exhibiting a singularity as* $r \to 0$, *i.e.* *grad* $U \to \infty$ *as* $r \to 0$ and this amounts to Re $\alpha - 1 < 0$. On the other hand, if the solution exists according to a variational problem in $H^1(\Omega)$, *grad* $u \in \underline{L}^2(\Omega)$ and this implies Re $\alpha > 0$. We see from (1.7) that *such singular solutions exist if* $\phi \in (\pi, 2\pi)$, *i.e. if the domain is not convex, but they do not exist if* $\phi \in (0, \pi)$, *i.e. if* Ω *is convex.* A picture of the flux lines (i.e. lines tangent to *grad U*) furnishes some insight on the physical phenomenon: for a convex (resp. non-convex) domain the flux lines spread out (resp. push to each other) as shown in Fig. 12.4, 12.5.
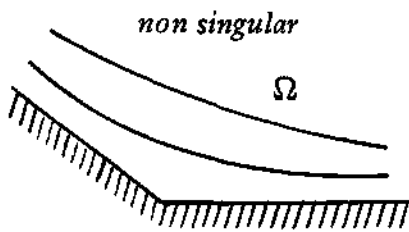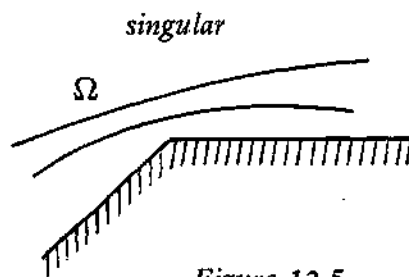

Figure 12.4

Figure 12.5

We now consider the *singularities at the boundary for transmission problems.*

We now consider the case where the interface Γ in the transmission problem (12.1), (12.3) touches $\partial\Omega$. We shall see that the convexity criterion for the Laplace equation (Fig. 12.6) *becomes now a convexity with respect to the refracted fluxes.*

Let us consider to fix ideas, the *transmission problem (12.1) with piecewise constant coefficients*, the interface conditions across a line Γ of discontinuity of the coefficients being of course (12.3). Moreover, we consider *Neumann boundary conditions*
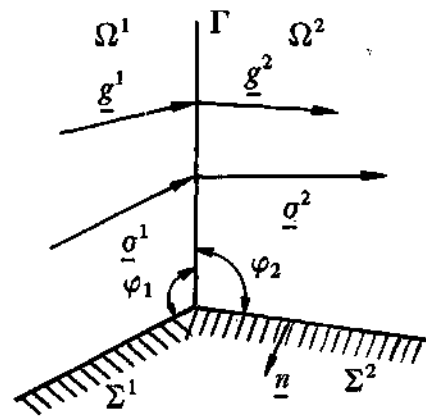

Figure 12.6

$$(12.7) \qquad a_{ij} \frac{\partial U}{\partial x_i} n_j = 0 \qquad \text{on } \partial\Omega.$$

We are studying the vicinity of a point 0 where Γ intersects $\partial\Omega$ (Fig. 12.6). Let Ω^1 and Ω^2 be (in the vicinity of 0) the two subdomains

where the coefficients are constant. It will prove useful writing the equation and boundary conditions in terms of the vectors gradient g and the flux $\underline{\sigma}$, defined by:

$$(12.8) \qquad g_i = \frac{\partial U}{\partial x_i} \quad ; \quad \sigma_i = a_{ij} \frac{\partial U}{\partial x_j} \equiv a_{ij} g_j$$

then (12.1), (12.7) become:

$$(12.9) \qquad \begin{cases} \operatorname{div} \underline{\sigma} = 0 \ , \quad (\underline{\sigma} \equiv a \underline{g}) \quad \text{in} \ \Omega \\ \sigma_i n_i = 0 \quad \text{on} \ \partial\Omega \end{cases}$$

and of course the transmission conditions (12.3) become (the first is obtained by differentiation of the first (12.3) along Γ):

$$(12.10) \qquad [g_t] = 0 \quad ; \quad [\sigma_n] = 0$$

where the indexes t, n denote "tangential" and "normal" components to Γ.

Solutions with constant gradient on each of the regions Ω^1, Ω^2 *are associated with* g *and* $\underline{\sigma}$ *taking constant values* $g^i, \underline{\sigma}^i$ *in* Ω^i, $i = 1, 2$. *We shall say that* $g^2, \underline{\sigma}^2$ *are the "refracted" of* $g^1, \underline{\sigma}^1$. To construct such solutions, we give arbitrarily either g^1 or $\underline{\sigma}^1$ (the other is then obtained by

$$(12.11) \qquad \sigma_i = a_{ij} g_j$$

with the values of a_{ij} on Ω^1). Then, the two relations (12.10) and the two (12.11) with the values of a_{ij} on Ω^2 furnish uniquely the refracted vectors $g^2, \underline{\sigma}^2$.

Now, coming back to Fig. 12.6, let us suppose that $\underline{\sigma}^1$ and the refracted $\underline{\sigma}^2$ are respectively parallel to the portions of $\partial\Omega$ in contact with Ω^1, Ω^2 (denoted by Σ^1, Σ^2). In this case, the Neumann boundary condition (i.e. the second of (2.9)) is satisfied. We then have the analogous, for equation (1.1), of the solution of constant gradient parallel to a straight boundary for the Laplace equation. We may gess (and this is proved in Sanchez [87]) that the presence of singularities is associated with non-convexity with respect to the line formed by $\underline{\sigma}^1, \underline{\sigma}^2$. Precisely:

PROPOSITION 12.1. *In the framework of this section (in particular Fig. 12.6), the Neumann problem (12.1), (12.7) has (resp. has not) a singularity at the point 0 of Fig. 12.6 (i.e. there exists a solution of the form (12.5) with $0 < \operatorname{Re}\alpha < 1$) if when constructing a flux vector $\underline{\sigma}^1$ parallel to the portion*

of $\partial\Omega$ *adjacent to* Ω_1, *pointing to* 0 *(see Fig. 12.7 a) and b)), the refracted vector* $\underline{g}^2$ *is inside (respectively out of)* Ω.
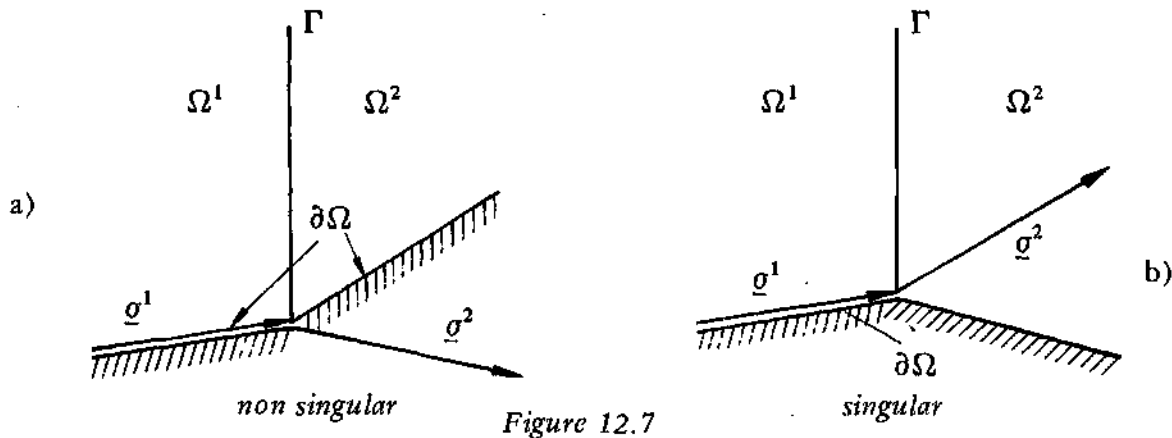

a)

non singular

Figure 12.7

singular

b)

Analogous rules hold for Dirichlet or mixed (Neumann and Dirichlet on two adjacent segments) problems (see Sanchez [87]). On the other hand, the corresponding problem for the elasticity system seems to be much more involved, and simple criteria as the preceeding one are not available. But the numerical method of the following section works in somewhat general problems.

We now give a general method for computing singularities.

When singularities appear, the general form *(roughly speaking)* of the solution is

$$(12.12) \qquad U(x_1, x_2) = c r^\alpha u(\theta) + U^{\text{regular}}(x_1, x_2)$$

where α *and* $u(\theta)$ *depend on the local geometry and coefficients* of the problem, and the coefficient c and the regular part $U^{\text{regular}}(x_1, x_2)$ depend on the other data of the problem. The knowledge of α for a given problem shows if wether or not a singularity exists. Moreover, if $u(\theta)$ is known, the solution (12.12) may be computed in an accurate way by using a standard finite element discretization plus a special finite element in the vicinity of 0. This finite element is constructed to describe the singularity with not very important perturbation of the voids of the discretized matrix (see Lelièvre [49], [50].

The problem of finding α and $u(\theta)$ may be reduced to some implicit eigenvalue problem, and may be solved by numerical methods (at least theoretically, for the real singular values α).

We now explain the method for an elliptic equation, but it is useful for general systems (with two independent variables x_1, x_2 of course). To fix ideas, we consider the problem of Fig. 12.6, where the domain Ω is an angle $\omega = \phi_1 + \phi_2$ in the vicinity of 0. Moreover, the boundary conditions are of the Neumann type and the coefficients depend only on θ in the vicinity of 0. The sesquilinear form associated with the problem is

$$(12.13) \qquad \int_\Omega a_{ij} \frac{\partial U}{\partial x_j} \frac{\partial V}{\partial x_i} dx$$

we take as Ω the angle

$$(12.14) \qquad \Omega = \{r, \theta \; ; \; r \in (0, \infty) \; ; \; \theta \in (0, \omega)\} \; .$$

In order to search for solutions of the form $r^\alpha u(\theta)$ which do not belong to $H^1(\Omega)$ we take

$$(12.15) \qquad \begin{cases} U(x_1, x_2) = r^\alpha u(\theta) \; ; \; u \in H^1(0, \omega) \\ V(x_1, x_2) = \phi(r) v(\theta) \; ; \; v \in H^1(0, \omega) \; ; \; \phi \in \mathscr{D}(0, \infty) \end{cases}$$

and the homogeneous equation with Neumann boundary conditions become

$$(12.16) \qquad 0 = \int_0^\infty r \, dr \int_0^\omega a_{ij} \frac{\partial(r^\alpha u)}{\partial x_j} \frac{\partial(\phi v)}{\partial x_i} d\theta$$

which after the change

$$\frac{\partial}{\partial x_1} = \cos\theta \frac{\partial}{\partial r} - \frac{\sin\Theta}{r} \frac{\partial}{\partial \theta}$$

$$\frac{\partial}{\partial x_2} = \sin\theta \frac{\partial}{\partial r} + \frac{\cos\theta}{r} \frac{\partial}{\partial \theta}$$

and after integrating with respect to θ, becomes:

$$0 = \int_0^\infty (F(\alpha, u, v) r^\alpha \phi' + \Psi(\alpha, u, v) r^{\alpha-1} \phi) dr \;, \qquad \forall \phi \in \mathscr{D}(0, \infty)$$

or integrating by parts in r:

$$0 = \int_0^\infty (-\alpha F + \Psi) r^{\alpha-1} \phi \, dr$$

which amounts to

(12.17) $$0 = -\alpha F(\alpha, u, v) + \Psi(\alpha, u, v) \equiv b(\alpha, u, v)$$

which defines a sesquilinear form b (depending on α) for $u, v \in H^1(0, \omega)$. The problem reduces to find the values of α such that a non zero $u \in \in H^1(0, \omega)$ exists satisfying

(12.18) $$b(\alpha, u, v) = 0 \qquad Vv \in H^1(0, \omega).$$

This is an implicite eigenvalue problem as it amounts to find the values of α for which zero is an eigenvalue of the operator $B(\alpha)$ associated with the form b.

In order to compute the singular values α, we discretize (by finite elements for instance) and use a finite dimensional basis $v^1, \ldots, v^m$ of the discretized space $H^1(0, \omega)$. The searched values α are those for which the matrix with coefficients

(12.19) $$b_{st} = b(\alpha, u^s, u^t)$$

is singular. For real α, as $0 < \alpha < 1$, it suffices to compute the determinant of the matrix for several α and to obtain by interpolation *the values for which it vanishes*. When the value α is known, the corresponding (discretized) $u(\theta)$ *is the corresponding eigenvector* which may be obtained by the inverse iteration method, for instance.

13. Other references and open problems.

Problems about non homogeneous media with holes or cracks are considered in Krasucki [43], Léné and Leguillon [53, 54], Lions [66], Sanchez [84], sect. 6.6, 6.7, 6.8. Problem with inclusions located near a surface, undulated boundaries, flow past a grid, and associated questions may be seen in Conca [22], Nguetseng et Sanchez [76], Sanchez [83]. Spectral and scattering problems are considered in Codegone [17, 18], Kesevan [40]. Problems in electromagnetism where dealt with in Codegone and Negro [19].

There are very many problems involving several small parameters. In addition to the above mentioned in sect. 7 and 8, we mention problems with small concentration inclusions; this problem is related with the Einstein approximation of the viscosity of suspensions: see Cioranescu et Murat [16], Lévy [56], Sanchez [88]. Problems with narrow but elongated inclusions are dealt in Caillerie [10, 11, 12], Marchenko and Khruslov [70]. Acoustic vibra-

tions in the suspensions may be seen in Fleury [31], Lévy [58, 62] and Lévy and Sanchez [63].

Exact bounds for the homogenized elasticity coefficients are considered in Francfort and Murat [35].

Stochastic distribution of inclusions are dealt with in Bensoussan [6], chap. 2 and Attouch [2].

Nonlinear problems and bifurcations may be seen in Duvaut [29, 30], Luborski and Telega [69], Mignot [71].

To conclude this survey, we mention *two widely open problems.*

The first one is the buckling of periodic structures made of elastic bars. Local buckling may appear for deformations with a local period different from that of the structure (Fig. 13.1).
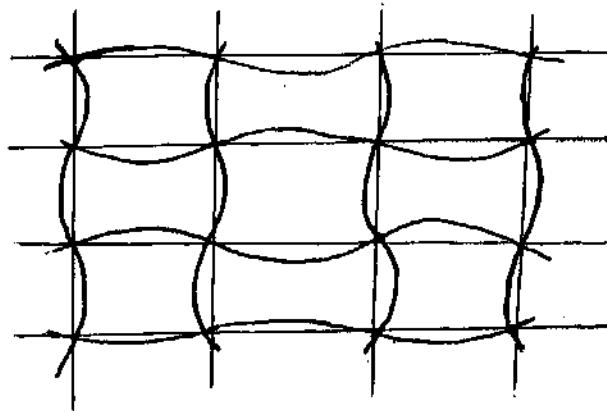

Figure 13.1

The second one is an asymptotic description of the stress field for elastic bodies with "round corners" for instance with boundaries formed by two segments joined by a small radius arch.

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