Some examples of Interactive Boundary Layers

P.-Y. Lagrée
CNRS & UPMC Univ Paris 06, UMR 7190,
Institut Jean Le Rond d'Alembert, Boîte 162, F-75005 Paris, France
pierre-yves.lagree@upmc.fr ; www.lmm.jussieu.fr/∼lagree

September 18, 2009

Abstract

In this chapter we present some other examples of IBL, not only in an aerodynamical context.

1 Introduction

In this chapter we focus on some examples which are not only from the aerodynamical field and which may lead to interacting boundary layer.

It means that we have a kind of Prandtl problem, the external boundary of it being linked with the variations of velocity or pressure by an external coupling. To a certain extent the interaction be explained through a retroactive process involving integral concepts as follows: as the variation of pressure is more or less proportional to the variation of the boundary layer thickness, then the increase of boundary layer thickness promotes a rise of pressure, which decreases the velocity, the result is an increase of the boundary layer thickness: the process is self promoting.

First there is the hypersonic problem, this interacting boundary layer flows process was described in Stewartson (1964) [19] with a self induced mechanism involving variations of boundary layer thickness and pressure. The key mechanism in supersonic and hypersonic flows was introduced by Neiland (1969) [14] and Stewartson & Williams (1969) [20]: it is the "triple deck" theory which clarifies the scales and the equations involved in the interaction. rown, Brown Stewartson & Williams (1975) [5] successfully explained the branching solutions calculated in strong hypersonic flows by

Werle *et al* (1973) [22] and the link with Neiland (1970) [15].

Next there is the mixed convection problem. It presented exactly the same features, and Steinrück (1994) [33]) proposed a kind of branching solution and Lagrée [31] showed that this may be reformulated in a triple Deck framework.

We present as well equations for the viscous hydraulic jump. The equations for artery flows are of the same vein.

2 Hypersonic flows

2.1 Self similar flows

Supersonic and hypersonic flows were developed in the 50', there was then over the next 50 years a continuous and sporadic interest linked to the ICBML (Intercontinental Ballistic Missiles) and the space conquest from Moon to Mars via the Space Shuttle. At this early time it was observed that those flows presented some self similar solutions. In case of negligible boundary layer thickness, people were looking to power shock laws. In this case the body may be a power law shape. This came from the observation that if we define from the Mach Number M_∞ and from the local angle of the shock σ and from the slope of the body τ the parameters:

$$K_s = M_\infty \sigma, \quad K = M_\infty \tau$$

they define self similar parameters in the Hypersonic Small Disturbance Theory (Chernyi [8]). The oblique shock wave relation gives for small angles θ and σ :

$$\frac{M_\infty \sigma}{M_\infty \tau} = \frac{\gamma+1}{4} + \sqrt{(\frac{\gamma+1}{4})^2 + \frac{1}{(M_\infty \tau)^2}}$$

the pressure is then

$$\frac{p}{p_\infty} = \frac{2\gamma}{\gamma+1} K_s^2 - \frac{\gamma-1}{\gamma+1} \quad (1)$$

$$= 1 + \gamma \frac{\gamma+1}{4} K^2 + \gamma K \sqrt{(\frac{\gamma+1}{4} K)^2 + 1}. \quad (2)$$

then for moderate Mach number, we recover that the angle of the shock is a Mach Wave ($1/M_\infty$) and the pressure is:

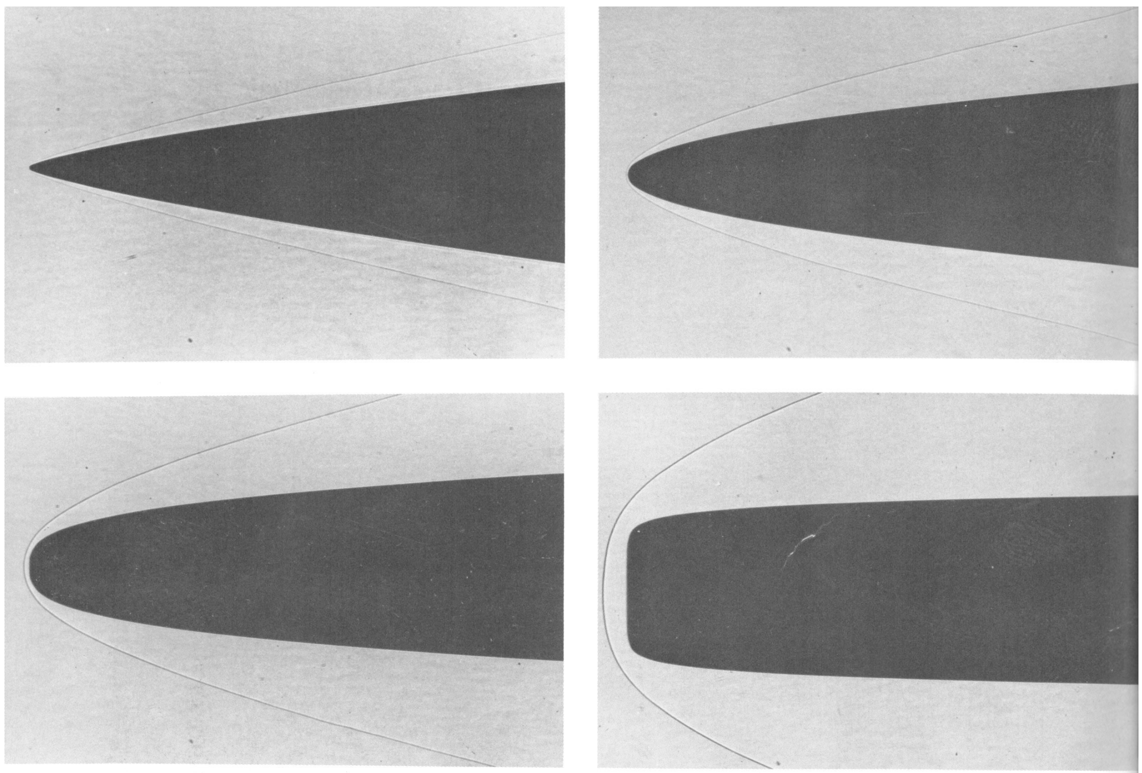
$$\frac{p - p_\infty}{\rho_\infty U_\infty^2} \simeq 1 + K_s$$

and for a large Mach number the body and the shock are proportional:

$$\frac{M_\infty \sigma}{M_\infty \tau} = \frac{\gamma+1}{2},$$

and the pressure

$$\frac{p - p_\infty}{\rho_\infty U_\infty^2} \simeq K_s^2.$$



273. Hypersonic flow past power-law bodies. Shadowgraphs show the bow wave in air at $M=8.8$ for bodies of revolution whose radius varies as a power of axial distance. The exponents are ¾, ½ (a paraboloid of revolution), ⅓, and $^1/_{10}$. *Freeman, Cash & Bedder 1964, courtesy of Aerodynamics Division, National Physical Laboratory*

Figure 1: Self similar Hypersonic flows from Van Dyke "An Album of Fluid Motion".

As only powers appear, self similar solutions are straightforward. See figure 1 for an example extracted from Van Dyke [21].

Those power law were not relevant for all the bodies. One of the self similar solution corresponds to the very special case. This is the blunt body case corresponding to a flat plate with a very small nose.

$$\frac{p - p_\infty}{\rho_\infty U_\infty^2} = F(\gamma) C_d^{2/3} (x/d)^{-2/3}$$

$$r_{shock} = G(\gamma) C_d^{1/3} (x/d)^{2/3}$$

see Guiraud Vallée & Zolver [9], Chernyi [8] and Sedov [18] for details.

2.2 Viscous interaction

For compressible boundary layers, the simple order of magnitude gives:

$$\delta^2 = \frac{\mu L}{\rho U_\infty}$$

taking into account that the viscosity follows in first approximation the Chapman law $\mu = \mu_\infty C \frac{T}{T_\infty}$ and the ideal gaz law: $\rho/\rho_\infty = (pT)/(p_\infty T_\infty)$, then:

$$(\frac{\delta}{L})^2 \sim C(\frac{T}{T_\infty})^2 \frac{p_\infty}{p} \frac{1}{R_\infty}$$

• In the so called "weak hypersonic interaction" pressure remains of order p_∞ and the temperature ratio is of order of order M_∞^2so that:

$$(\frac{\delta}{L}) \sim C^{1/2} (\frac{T}{T_\infty}) \frac{1}{R_\infty^{1/2}} = \frac{\chi_\infty}{M_\infty}$$

were $\chi_\infty = C^{1/2} \frac{M_\infty^2}{R_\infty^{1/2}}$ is the hypersonic parameter. The pressure is then:

$$\frac{p - p_\infty}{\rho_\infty U_\infty^2} \simeq 1 + \chi_\infty$$

• In the so called "strong hypersonic interaction" pressure is now determined bay the equivalent body which is the displacement thickness due to the strong viscous effects.

$$\frac{p}{p_\infty} \simeq K^2 \quad \text{with} \quad K = M_\infty (\delta/L).$$

so wee obtain the displacement thickness as:

$$\frac{\delta}{L} \sim \sqrt{\frac{1}{M_\infty} \chi_\infty}, \quad \frac{p}{p_\infty} \simeq \chi_\infty^2$$

this represents a shock and a boundary layer in $x^{3/4}$.

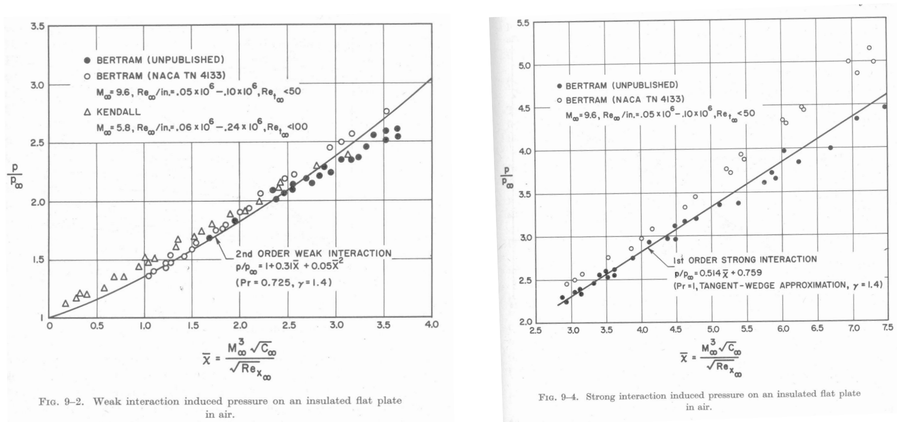


Figure 2: Weak and strong interaction results from Hayes & Probstein [10], pressure as a function of the hypersonic interaction parameter.

2.3 interactive system

Finally the Boundary Layer equations in the case of strong interaction read:

$$\frac{\partial \tilde{\rho}\tilde{u}}{\partial \bar{x}} + \frac{\partial \tilde{\rho}\tilde{v}}{\partial \tilde{y}} = 0, \quad \tilde{\rho}(\tilde{u}\frac{\partial \tilde{u}}{\partial \bar{x}} + \tilde{v}\frac{\partial \tilde{u}}{\partial \tilde{y}}) = -\frac{\gamma - 1}{2\gamma}(\tilde{S} - \tilde{u}^2)\frac{1}{\tilde{p}}\frac{d\tilde{p}}{d\bar{x}} + C\tilde{p}\frac{\partial}{\partial \tilde{y}}(\frac{\partial \tilde{u}}{\tilde{\rho}\partial \tilde{y}})$$

With boundary conditions $\tilde{u}(\bar{x}, 0) = 0$, $\tilde{u}(\bar{x}, \infty) = 1$. The Energy equation reads with the total enthalpy $\tilde{S}$:

$$\tilde{\rho}(\tilde{u}\frac{\partial \tilde{S}}{\partial \bar{x}} + \tilde{v}\frac{\partial \tilde{S}}{\partial \tilde{y}}) = C\tilde{p}\frac{\partial}{\partial \tilde{y}}(\frac{\partial \tilde{S}}{\tilde{\rho}\partial \tilde{y}})$$

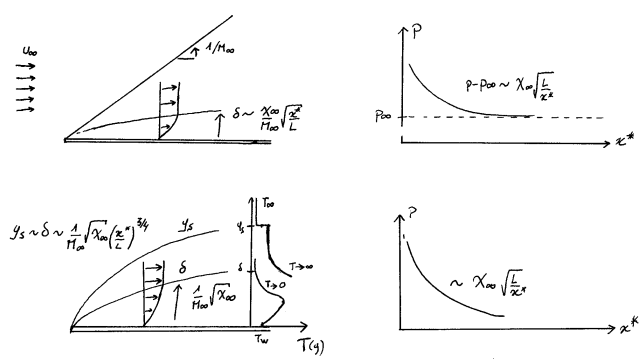


Figure 3: Weak and strong interaction basic solutions Lagrée [11] .

The boundary layer displacement thickness gives the velocity at the edge of the boundary layer as

$$\tilde{v}_e(\bar{x}) = \frac{d}{dx}[\frac{\gamma-1}{2\tilde{p}}\int_0^\infty (\tilde{S}-\tilde{u}^2)\tilde{\rho}\tilde{y}]$$

and the coupling relation is:

$$\tilde{p} = \frac{\gamma+1}{4}\tilde{v}_e^2.$$

This interactive system was believed to be solved with a marching scheme. But, unfortunately it raised difficulties, one observed branching solution when solving numerically for increasing x (see Neiland [15] and figure 6 right thereafter, see Werle et al. [22]... and figure 4).

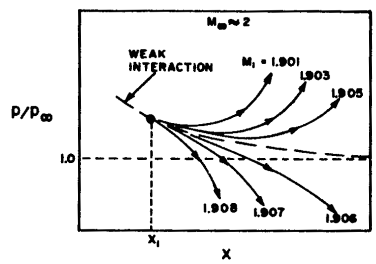


Figure 4: Branching solutions [22]: changing a bit one parameter may cause different solutions while solving the equations with a marching scheme.

2.4 Hypersonic Triple Deck Stewartson

From this system Stewartson [5] obtained the $p = -A$ relation, this comes from the solution of the total enthalpy in the main deck which is obtained from $\tilde{S} = \tilde{S}(\tilde{Y}) + \varepsilon A(x)\tilde{S}'(\tilde{Y})$ and $\tilde{u} = \tilde{U}_B(\tilde{Y}) + \varepsilon A(x)\tilde{U}_B'(\tilde{Y})$. It is substituted in the coupling relation for a short disturbance in longitudinal scale:

$$\tilde{p} = \frac{\gamma+1}{4}(\frac{\gamma-1}{2\tilde{p}}\int_0^\infty (\tilde{S}-\tilde{u}^2)\tilde{\rho}\tilde{y})^2.$$

which gives $-A$ contribution in the integral and the development of $\frac{1}{\tilde{p}}$ gives $-p$ so that the RHS is $(-p-A)'$ with the *ad hoc* scales. The case of very cold walls $s_w << 1$ or "Newtonian flow" (γ close to 1) gives then $p = -A$. The moderate case is then:

$$\mu p = -p' - A', \quad \text{with } \mu \sim (\gamma-1)^2 s_w^6.$$

See Lagrée [12] and Brown et al. [6], Neiland [16] and Lipatov and Neiland for various forms of this interaction.
Then, linearisation of the problem $p = -A$ allows to obtain the exponential solution (Gajjar Smith [49]) e^{Kx} with $K = (-3Ai'(0))^3 = 0.47$

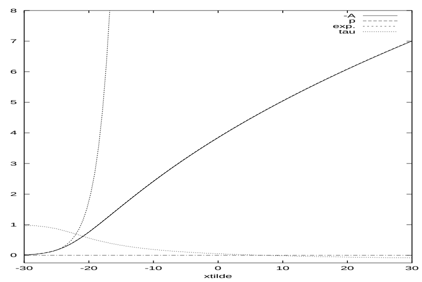


Figure 5: Linearized eigen solution ("exp." is $\exp((-3Ai'(0))^3)\tilde{x}))$, and non linear solution of the self induced ($p = -A$) problem solved with Keller Box and "semi inverse" coupling: pressure (p), displacement (-A) and skin friction (tau).

2.5 Hypersonic Triple Deck Neiland

A generalization of the FS equation may be obtained when the flow is non similar. If f° is a short hand for $\partial_{\bar{x}} f$, we may write the Prandtl equations in introducing $n = \frac{\bar{x}}{\bar{u}_e}\frac{d\bar{u}_e}{d\bar{x}}$ then

$$f''' + ff'' - \beta(S - f'^2) = 2x(f'f'^\circ - f^\circ f''), \quad S'' + fS' = 2x(f'S^\circ - f^\circ S')$$

with $\beta = \frac{\gamma-1}{\gamma}\frac{\bar{x}d\tilde{p}}{\bar{p}d\bar{x}}$. Neiland [15] obtained the branching solutions of this system in writting:

$$f(x, \eta) = f_0(\eta) + \phi(x)f_1(\eta) + ...$$

by substitution, one obtains $F(f_0, f_1)\phi = xd\phi/dx$ whose natural solution is an exponent solution for ϕ. Then Neiland used

$$f(x, \eta) = f_0(\eta) + x^{a+1}f_1(\eta) + ...$$

treating $x^{a+1}f_1(\eta)$ as a small perturbation of the basic self similar flow $f_0(\eta)$ he obtained an eigen value problem and the value of a. The value was large

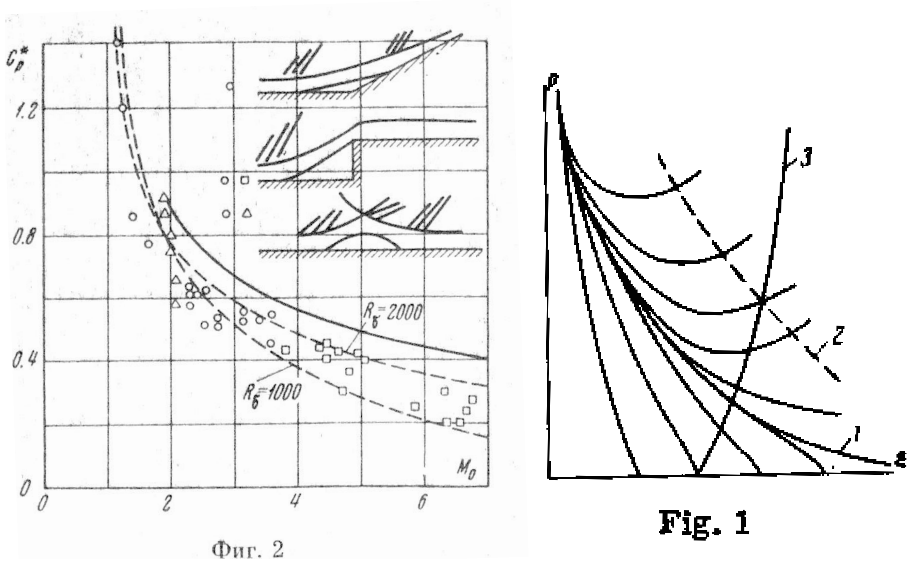


Figure 6: Left: Pressure at separation point far from the cause, showing comparisons with experiments supersonic case Neiland (1969) [14]. Right branching solution for the hypersonic solution by Neiland (1970) [15]

49.6 for $s_w = 1$. Latter Brown and Williams [7] looked at this same problem with the same point of view but do not restrict to $s_w = 1$.

We have already mentioned that supersonic flows presented "upstream influence". It meant that disturbances have an influence far upstream which was a paradox due to the hyperbolic nature of the Euler equations and to the parabolic nature of the boundary layer equations.

2.5.1 Hypersonic Triple Deck Neiland-Stewartson

What is strange is that two points of view gave different results for the same set of equations. Stewartson made a local analysis at a small scale and obtained eigen solution in exponential. Neiland did not change the longitudinal scale, he obtained power law solution with a large value of the exponent.

Stewartson (Brown Stewartson & Williams [5]) reconciled the two points of view in noting the link between the two results (algebraic and exponential). He observed that far upstream near a given X_0, one can do a local study at scale x_3 so that

$$x = x_0 + x_3\tilde{x}$$

the power law behaviour is :

$$(x)^n = e^{nLog(x_0(1+x_3\tilde{x}/x_0))} \simeq e^{nLog(x_0)}e^{x_3\tilde{x}/x_0}$$

so in the local variables algebraic behaviour looks like an exponential.

3 Heat Flows

Flow with temperature are good candidates to coupling in the case of mixed convection. It means that there is a basic stream which imposes forced convection but it means that there is some natural convection as well creating a retroaction. We will see what happens for plumes en jets and in the case of mixed convection.

3.1 The mixed convection problem

Here we consider first the mixed convection problem of an incompressible buoyant (following the Boussinesq approximation) fluid flowing over a semi infinite horizontal flat plate at a constant temperature lower than the incoming flow temperature (see figure 7 for a definition sketch). Obviously, for a given x location, the fluid temperature, by diffusion, increases from the wall value towards that of the free stream. But for a fixed y location the convection induces a longitudinal decrease of the temperature. The outcome is a buoyancy induced stream wise adverse pressure gradient. This gradient brakes the flow, and this creates an interaction between the thermics and the dynamics. This mechanism of mixed convection breakdown has been stated by Schneider & Wasel (1985) [32] (other examples of re-computation with different numerical methods are reviewed by Steinrück (1994) [33]); they showed that this interaction promotes a breakdown of the mixed boundary layer equation: at a relatively small abscissa, the equations are abruptly singular. Instead of a buoyant boundary layer a buoyant wall jet may be studied, the case of adiabatic wall was studied by Daniels (1992) [25] and Daniels & Gargaro (1993) [26], they found the same conclusions. The wall jet problem is solved numerically and asymptotically by Higuera (1997) [51] who notes that the equations are not parabolic as he noted before in the case of the hydraulic jump which is very similar in its behaviour.

3.2 Governing equations of the mixed convection

3.2.1 Equations

We consider an incompressible two dimensional flow past a semi-infinite (heated or cooled) horizontal flat plate (figure 7). The boundary layer equations are obtained from the Navier Stokes counter parts subject to Boussinesq approximation for a large Reynolds number. A re-scaling of the dimensional quantities is carried out with the dynamical boundary layer

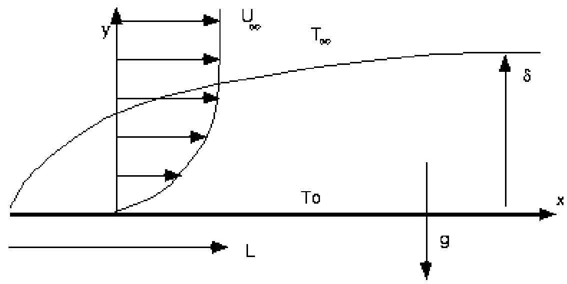


Figure 7: Sketch of the mixed convection boundary layer flow, the temperature of the plate is different from the temperature of the flow. If the plate is cooled, the buoyancy induces an adverse pressure gradient.

scales (with $\delta = Re^{-1/2}$ with $Re = \rho_\infty U_\infty L/\mu$):

$$\begin{array}{ll} u^* = U_\infty u, & v^* = \delta U_\infty v, \\ x^* = Lx, & y^* = \delta L y, \\ p^* = p_\infty + \rho_\infty U_\infty^2 p, & T = T_\infty + (T_0 - T_\infty)\theta, \end{array}$$

the result is the classical system (23- 7) of thermal mixed convection (Schneider & Wasel (1985) [32]), Prandtl number is assumed to be of order unity and hence set, (without to much loss of generality), to one while the Eckert number is assumed sufficiently small to obtain the energy equation as (7)). The remaining parameter is the Richardson number or buoyancy parameter:

$$J = \frac{\alpha g (T_0 - T_\infty) L Re^{-1/2}}{U_\infty^2}, \tag{3}$$

it depends on α the thermal coefficient of expansion of the density in the Boussinesq approximation. The transverse pressure term (6) contains the gravity term, as equation (6) holds for terms greater than $O(\frac{1}{Re})$, we have $|J| >> Re^{-1}$:

$$\frac{\partial}{\partial x} u + \frac{\partial}{\partial y} v = 0, \tag{4}$$

$$u \frac{\partial}{\partial x} u + v \frac{\partial}{\partial y} u = -\frac{\partial}{\partial x} p + \frac{\partial^2}{\partial y^2} u, \tag{5}$$

$$0 = -\frac{\partial}{\partial y} p + J\theta, \tag{6}$$

$$u\frac{\partial}{\partial x}\theta + v\frac{\partial}{\partial y}\theta = \frac{\partial^2}{\partial y^2}\theta, \tag{7}$$

Boundary conditions are:

$$u(x, y = 0) = 0, v(x, y = 0) = 0, \tag{8}$$

$\theta(x, y = 0) = \theta_w$ with $\theta_w = 1$, $u(x, y \longrightarrow \infty) = 1, \theta(x, y \longrightarrow \infty) = 0, p(x, y \longrightarrow \infty) = 0$.

3.2.2 Marching breakdown

The length scale L and the parameter J are independent, it contrasts with the situation in Schneider & Wasel (1985) [32] or in Daniels & Gargaro (1993) [26]. In the "real mixed convection problem with stable stratification flow", the "natural" longitudinal scale is effectively built with Richardson number. It is the length that gives unit Richardson number ($\left|\alpha g(T_0 - T_\infty)L_T U_\infty^{-2}(U_\infty L_T \nu^{-1})^{-1/2}\right| = 1$), so:

$$L_T = \frac{U_\infty}{\nu}\left(\frac{U_\infty^2}{-\alpha g(T_0 - T_\infty)}\right)^2.$$

Note that $J^2 L_T = L$. Schneider & Wasel (1985) [32] (scaled with L_T) showed that this system leads to a singularity when solved with a marching (in increasing x) resolution. They showed that the breakdown occurs for a rather small abscissa. This is the reason why Steinrück (1994) [33] (scaled with L_T) has investigated how the system (23-7) behaves when x tends to 0. In figure 8 are displayed, with symbols, the reduced skin friction from previous works compiled by Steinrück. The curves with numbers show solution of the marching problem with slightly perturbed initial conditions and come from his analysis near $x = 0$. Asymptotic analysis suggests, however, that it is better to consider an intermediate scale L (with $L << L_T$) leading to Blasius boundary layer (with this scales x tends to 0 is the nose effect) with a small thermal perturbation gauged by $|J| << 1$, this means that the Richardson number built with this abscissa is smaller than one. So, we will introduce the triple deck analysis.

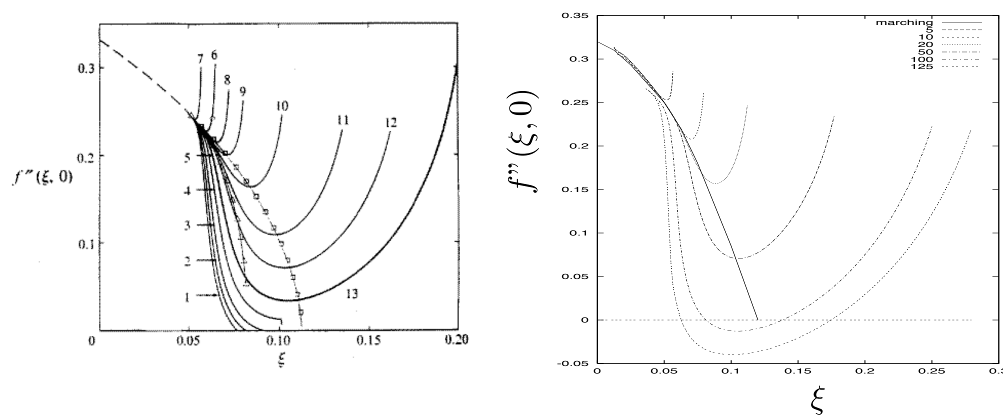


Figure 8: the reduced skin friction $f''(\xi, 0) = \frac{\partial u}{\partial y}(x, 0)\sqrt{x}$ function of $\xi = |J|\sqrt{x}$: compiled and computed by marching computations by Steinrück (JFM 94), the numbered curves show solution of the marching problem with slightly perturbed initial conditions (left). Numerical resolution showing the reverse flow, each curve is associated with different domain length x_{out} (right) by Lagrée [31].

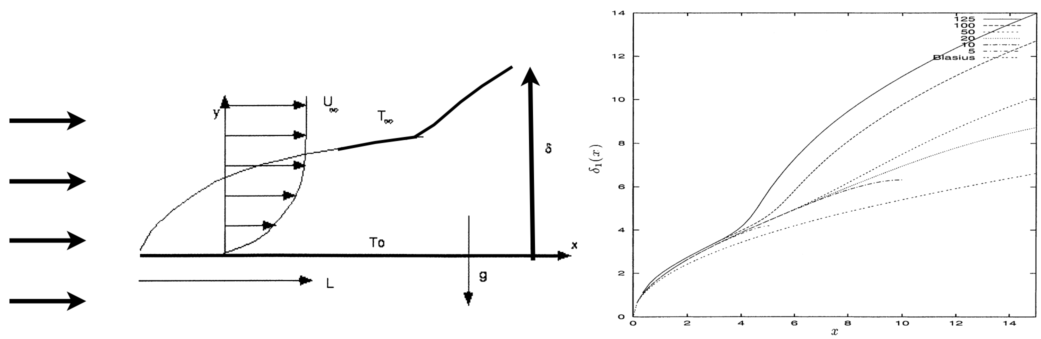


Figure 9: the kind of "mixed thermal convection" hydraulic jump. The observed singular solutions which branch out may then be revisited in the framework of the "triple deck" theory: two salient structures emerge, one in double deck, if $|J| << 1$, and another in single deck, if $|J| = O(1)$. Those two structures are a reinterpretation of Steinrück (94) results. This proves that the marching procedure is not relevant and that an output boundary condition has to be imposed. A numerical simulation of the unsteady version of (1)- (2) is carried out whith the *ad hoc* output boundary conditions: $\partial_x f(x = x_{out}, y) = 0$, (where $f = u, v, \theta, p$) Depending of the size of the computational domain x_{out} (right fig.), the preceding (left fig.) branching solutions are reobtained, some of them correspond to the separation of the boundary layer (as predicted by the triple deck theory).

4 Asymptotic analysis: the triple deck tool

4.1 Small J, with displacement

4.1.1 Main Deck

Here we look for eigen solutions in a boundary layer slightly perturbed by the thermal effect in order to show that system (23-7) is not parabolic in x when the plate is cooled. We use the word "parabolic" for a system of P.D.E. in the sense of a system that can be integrated in marching in x direction from upstream to downstream (with no separation). The basic flow, driven by the free stream uniform velocity, is a classical Blasius boundary layer (thermal and dynamical effects are not coupled). We study how a localized disturbance evolves at the distance L downstream from the leading edge. At this point, the boundary layer thickness is $Re^{-1/2}L$. Pure thermal convection is relevant as long as the transverse gradient from equation (6) is small which implies $1 >> |J|$. So, in this framework, the forced thermal boundary layer is of the same thickness as the dynamic one, and the velocity at station $x = 1$ is the basic Blasius velocity profile (say $U_0(y)$, the transverse variable is then the same as the self similar one) and θ is simply $\theta_0(y) = 1 - U_0(y)$. The choice of L smaller than L_T suggests expanding in powers of a small parameter ε linked to J.

Having defined the "basic state", we follow the classical "triple deck" analysis (Neiland (1969) [14]), Stewartson & Williams (1969) [20], and more precisely Lagrée (1995) [30]): system (23-7) is re-investigated with a smaller longitudinal scale, say x_3L (with $x_3 \ll 1$ and $x = 1 + x_3\bar{x}$), this scale is sufficiently small so that the preceding profiles may be considered as frozen. The reason for this new scale is the fact that near the breakdown point the gradient of the skin friction is infinite at scale 1, so we hope to render it $O(1)$ at this smaller scale. This layer with height δL and length x_3L is in fact the "main deck". Next we suppose that the perturbation of longitudinal speed in the "main deck" is of the order of ε and the pressure of the order of ε^2, where ε is unknown (but depends on δ, J and x_3), so we recover that at these scales the inviscid problem with no longitudinal pressure gradient. The perturbations are then linked by an up to now unknown displacement function of the boundary layer called $-A(\bar{x})$ by Stewartson. In the "main deck", the adimensionalized velocities and temperature up to the order of ε are:

$$u = U_0(y) + \varepsilon A(\bar{x})U_0'(y); \; v = \frac{-\varepsilon A'(\bar{x})U_0(y)}{x_3} \; \& \; \theta = \theta_0(y) + \varepsilon A(\bar{x})\theta_0'(y) \tag{9}$$

For the temperature, as for the speed, there is a matching between the outer limit of the main deck and the inner limit of the upper deck, and likewise for the bottom of the main deck and the top of the lower deck (those decks are defined latter). We see that the temperature behaves as the Stewartson S function (total enthalpy) in hypersonic flows (Brown *et al.* (1975) [5], Brown & Stewartson (1975) [7], Neiland (1986) [16]). This perturbation of temperature gives rise to a transverse change of pressure through the "main deck"; we develop (6) in powers of ε as follows:

$$\frac{\partial}{\partial y}p_0 + \varepsilon\frac{\partial}{\partial y}p_1 + \varepsilon^2\frac{\partial}{\partial y}p_2 + 0(\varepsilon^3) = J(\theta_0(y) + \varepsilon A(\bar{x})\theta_0(y)) + 0(\varepsilon^3) \quad (10)$$

At this stage, for $|J| << 1$ by minor degeneration (*i. e.* to retain the maximum of terms), we put $J = \varepsilon\tilde{J}$, because J is small with $\tilde{J}$ being a reduced Richardson number of the order of 0(1). Looking at each power of ε, we see that the first term is zero (as we supposed in the Blasius Boundary layer); the second one shows that there is a pressure stratification coming from basic temperature profile ($\int_0^\infty \theta_0(y)dy$), it does not depend on $\bar{x}$ at the short scale x_3, and it will appear that such a term can be ignored in the following analysis; the third one integrates (using $\theta_0(\infty) = 0; \theta_0(0) = 1$ by definition) as:

$$p_2(\bar{x}, y \to \infty) - p_2(\bar{x}, y \to 0) = \tilde{J}A(\bar{x})(\theta_0(\infty) - \theta_0(0)) = -\tilde{J}A(\bar{x}),$$

where $p_2(\bar{x}, y \to \infty)$ splices with upper deck and $p_2(\bar{x}, y \to 0)$ with lower deck hitherto both being not defined. The case J of the order of one will is discussed in Lagrée [31].

4.1.2 Lower deck

From the solution (9) we see that the no slip condition is violated: $u \to U_0'(0)(y+\varepsilon A)$, and $\theta \to \theta_0'(0)(y+\varepsilon A)$ as $y \to 0$. So we introduce a new layer of thickness ε (in boundary layer scales), and scale y by $\varepsilon\bar{y}$, so the scale of u is $\varepsilon\bar{u}$ and, by least degeneracy of equation (2), we have $p = \varepsilon^2\bar{p}$ (which is consistent with the matching $\varepsilon^2 p_2(\bar{x}, y \to 0) = \varepsilon^2\bar{p}(\bar{x}, \bar{y} \to \infty)$) and v is of the order of ε/x_3 . The convective diffusive equilibrium gives the relation between x_3 and ε: $x_3 = \varepsilon^3$. The problem of mixed convection near the wall is then:

$$\frac{\partial}{\partial \bar{x}}\bar{u} + \frac{\partial}{\partial \bar{y}}\bar{v} = 0, \quad (11)$$

$$\bar{u}\frac{\partial}{\partial \bar{x}}\bar{u} + \bar{v}\frac{\partial}{\partial \bar{y}}\bar{u} = -\frac{d}{d\bar{x}}\bar{p} + \frac{\partial^2}{\partial \bar{y}^2}\bar{u}, \quad (12)$$

$$\bar{u}\frac{\partial}{\partial \bar{x}}\bar{\theta} + \bar{v}\frac{\partial}{\partial \bar{y}}\bar{\theta} = \frac{\partial^2}{\partial \bar{y}^2}\bar{\theta}, \quad (13)$$

Boundary conditions are no slip at the wall $\bar{\theta}(\bar{x}, 0) = 1$, $A(-\infty) = 0$, and for $\bar{y} \to \infty$, the matchings: $\bar{u} \to U_0'(0)(\bar{y} + A)$, $\bar{p} \to p_2(\bar{x}, y \to 0)$ and $\bar{\theta} \to 1 - U_0'(0)(\bar{y} + A)$. This set of non linear equations is relevant in the "lower deck" of length $x_3 L = \varepsilon^3 L$ and of height $\varepsilon\delta L$ placed at station 1; here, the thermal and the dynamical problem are uncoupled. In this thin layer of small extent, the pressure coming from the main deck is the most dangerous for the velocity and may lead to separation.

4.1.3 The upper deck

Possibility of retroaction with the external flow The perturbations of transverse velocity and pressure at the edge of the main deck introduce a perturbation in the inviscid flow: the upper deck is of size ε^3 in both directions. This perturbation is solved by the standard technique of linearized subsonic perfect fluid, this gives the Hilbert integral (the new pressure displacement relation):

$$\frac{1}{\pi}\int \frac{-A'}{\bar{x} - \xi} d\xi - p_2(\bar{x}, y \to 0) = -\tilde{J}A(\bar{x})$$

and the usual gauge (Smith (1982) [4]): $\varepsilon = \delta^{-1/4} = Re^{-1/8}$ (so $J = Re^{-1/8}\tilde{J}$) and this gives the lower limit for $x_3 = Re^{-3/8}$ in the preceding §. The effect of the temperature is to add a new term proportional to the displacement function A, it may be interpreted as a hydrostatic pressure variation.

Retroaction only in the boundary layer Consideration of (9) shows that another (but equivalent) choice of ε could have been made: $\varepsilon = |J|$. With this choice, $x_3 = |J|^3$, and the preceding relation reads:

$$\frac{|J|^{-4} Re^{-1/2}}{\pi}\int \frac{-A'}{\bar{x} - \xi} d\xi - p_2(\bar{x}, y \to 0) = -(|J|/J)A(\bar{x}).$$

This choice implies that we concentrate on thermal effects rather than on perfect fluid effects, if $|J| \sim Re^{-1/8}$ (note that $Re^{-1/8} >> Re^{-1/2}$), the three terms are of the same magnitude (as seen in the preceding paragraph). Now, if $|J| >> Re^{-1/8}$ (or $\tilde{J}$ bigger than one) there is no interaction of the boundary layer with the external perfect fluid, the thermal effect is dominant and the pressure displacement relation degenerates in the form:

$$p_2(\bar{x}, y \to 0) = \bar{p}(\bar{x}) = -A(\bar{x}), \tag{14}$$

for a cold wall ($J < 0$), and in the form:

$$p_2(\bar{x}, y \to 0) = \bar{p}(\bar{x}) = A(\bar{x}), \tag{15}$$

for a hot one ($J > 0$), where in both cases $Re^{-1/8} \ll |J| \ll 1$. This shows that the upper deck is not necessary for the interaction to take place, the same phenomenon exists in free convection hypersonic flows (Brown *et al.* (1975) [5] or Neiland (1986) [16] and Brown Cheng & Lee (1990) [6]) for cold wall.

4.1.4 The fundamental problem of mixed convection on "double deck" scales with displacement

Finally, the mechanism relevant for the problem of infinitely small mixed convection is without external perfect fluid retroaction, the whole process of interaction takes place in the "main deck". This is a double deck interaction. We write here the final re-scaled problem (in order to avoid $U_0'(0)$). With scales:

$$\begin{array}{ll} x = L + |J|^3 \, (L/U_0'(0))\tilde{x}, & y = |J| \, ((U_0'(0))^{-2} L/Re^{1/2})\tilde{y} \\ t = |J|^2 \, (L/U_\infty)\tilde{t} & \\ u = |J| \, ((U_0'(0))^{-1} U_\infty)\tilde{u}, & v = (|J|^{-1} \, ((U_0'(0))^{-2} U_\infty Re^{-1/2})\tilde{v}, \\ p = J^2((U_0'(0))^{-2} \rho U_\infty^2)\tilde{p} & \end{array}$$

(and $Re^{-1/8} \ll |J| \ll 1$), the final "canonical problem of infinitely small mixed convection" is:

$$\frac{\partial}{\partial \tilde{x}}\tilde{u} + \frac{\partial}{\partial \tilde{y}}\tilde{v} = 0, \tag{16}$$

$$\frac{\partial}{\partial \tilde{t}}\tilde{u} + \tilde{u}\frac{\partial}{\partial \tilde{x}}\tilde{u} + \tilde{v}\frac{\partial}{\partial \tilde{y}}\tilde{u} = -\frac{d}{d\tilde{x}}\tilde{p} + \frac{\partial^2}{\partial \tilde{y}^2}\tilde{u}, \tag{17}$$

Boundary conditions are: no slip at the wall ($\tilde{u} = \tilde{v} = 0$ in $\tilde{y} = 0$), no displacement far upstream ($\tilde{A} = 0$ in $\tilde{x} \to -\infty$), the matching $\tilde{y} \to \infty, \tilde{u} \to \tilde{y} + \tilde{A}$ and the coupling relation (hot wall, $sign(J) = 1$, cold wall $sign(J) = -1$):

$$\tilde{p} = sign(J)\tilde{A}. \tag{18}$$

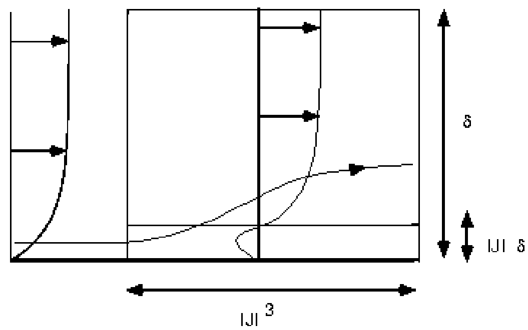


Figure 10: the two final layers involved: the boundary layer itself and a thin wall layer.

The introduction of time changes only the "lower deck" by the adjunction of the $\partial\tilde{u}/\partial\tilde{t}$ term (Smith (1979)[3]). Figure 10 displays a rough sketch of the double deck structure.

4.1.5 Resolution

The eigen value solution System (16-18) admits the Blasius solution $\tilde{u} = \tilde{y}$ as the basic one. Invariance by translation in space and time suggests linearized solutions of the form:

$$\tilde{u} = \tilde{y} + ae^{i(k\tilde{x}-\omega\tilde{t})}f'(\tilde{y}), \tilde{v} = -ikae^{i(k\tilde{x}-\omega\tilde{t})}f(\tilde{y}), \ \& \quad \tilde{p} = ae^{i(k\tilde{x}-\omega\tilde{t})},$$

were $a << 1$. After substitution, f verifies an Airy differential equation with the variable $\eta = (ik)^{1/3}\tilde{y}$, so classically we find:

$$-f'(\infty) = \frac{(ik)^{1/3}}{Ai'(-i^{1/3}\omega/k^{2/3})}\int_{-i^{1/3}\omega/k^{2/3}}^{\infty} Ai(\zeta)d\zeta. \qquad (19)$$

Cold wall, eigen value and comparison with Steinrück In the case of cold wall, the coupling ($\tilde{p} = -\tilde{A}$) gives $1 = -f'(\infty)$, and a stationary exponentially growing solution may be obtained: $\omega = 0$, $ik = \Lambda = (-3Ai'(0))^3 \simeq 0.47$. We recover the same behavior as in hypersonic flows (Brown *et al.* (1975) [5] and Gajjar & Smith (1983) [49]), in the birth of hydraulic jumps (Bowles & Smith (1992) [48]) and in supersonic pipe flows (Ruban & Timoshin (1986) [17]). Λ is called the Lighthill eigenvalue, it shows that there is upstream influence, for example the preceding solution is the linearization of what happens far upstream of the separating point. The occurrence of eigen functions states that system (23-7) is not parabolic.

We have proved that the perturbation grows like $e^{(-3Ai'(0))^3\tilde{x}}$. It may be compared with Steinrück's result: he showed that the system (23-7) scaled longitudinally by L_T admits near the origin eigen function growing like $\exp(\frac{\lambda_0^+}{\xi_0^4}\xi)$ where $\lambda_0^+ = 2U_0'(0)\ (-\ 3Ai'(0))^3$, (formula 2.29 from [33] or A.15 from [34], with $Pr = 1$, $U_0'(0) = f''(0) = 0.3321$ and $\int_0^\infty Ai(\zeta)d\zeta = 1/3$) where $\xi = (x/L_T)^{1/2}$ and where ξ_0 is the place where the flow is perturbed. If we substitute λ_0^+, ξ and ξ_0 in the exponential, bearing in mind $L/L_T = J^2$, and $|J| \ll 1$, and ξ_0 is $(L/L_T)^{1/2}$ *i.e.* $|J|$), we rewrite it with our variables, and develop with the first power of $|J|$:

$$e^{\frac{\lambda_0^+}{\xi_0^4}\xi} = \exp(\frac{\lambda_0^+}{|J|^3}(1+|J|^3\,(1/U_0'(0))\tilde{x})^{1/2}\,) \sim \exp(|J|^{-3}\,\lambda_0^+ + \lambda_0^+(1/U_0'(0))\tilde{x}/2))$$

so, factorizing $\exp(|J|^{-3}\,\lambda_0^+)$ and substituting the value of λ_0^+, we recover the exponential growth with $\tilde{x}$:

$$\exp((-3Ai'(0))^3)\tilde{x}).$$

So the conclusion is that the triple deck theory (which is a theory in the limit of small J at $x = 1$) is equivalent to Steinrück's result (with only a different choice of scales: L_T instead of L so $J = 1$ and x is small).

4.2 Thermal Jets

Case of vertical plate El Hafi [27] , Exner Kluwick [28]...

4.3 Jet and Plumes

In this case the system is:

$$\frac{\partial}{\partial x}u+\frac{\partial}{\partial y}v = 0, \tag{20}$$

$$u\frac{\partial}{\partial x}u+v\frac{\partial}{\partial y}u = J\theta+\frac{\partial^2}{\partial y^2}u, \tag{21}$$

$$u\frac{\partial}{\partial x}\theta+v\frac{\partial}{\partial y}\theta=\frac{\partial^2}{\partial y^2}\theta, \tag{22}$$

Boundary conditions are far away : $u(x,y\longrightarrow\infty)=0,\theta(x,y\longrightarrow\infty)=0$ $\frac{\partial}{\partial y}\theta(x,y=0)=0$ $\frac{\partial}{\partial y}u(x,y=0)=0$ with a given first profile, here Poiseuille. In the case of $J>0$ the solution is very simple and one goes from the Poiseuille profile to the jet profile, and then to a jet profile to a plume profile.

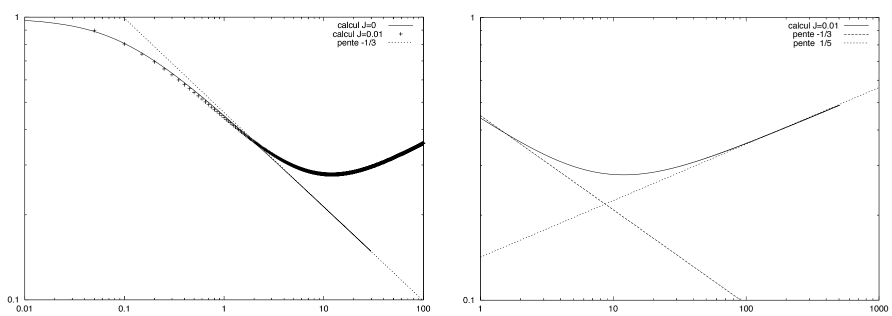


Figure 11: Velocity at the center of the Jet. Left the centerline velocity decreases from Poiseuille to the Bickley jet profile in $x^{-1/3}$ and then increases again. Right, the new increase corresponds to the plume solution with a centerline velocity in $x^{1/5}$.

5 Shallow water equations

5.1 Saint Venant Equations

5.2 Hydraulic jump in a viscous laminar flow

Exactly the same kind of interaction may be observed with a liquid layer (Higuera [50] and [51]). In this case the $\frac{\partial}{\partial y}p$ is constant, it is the inverse of the Froude number (say S), the pressure is then hydrostatic:

$$\frac{\partial}{\partial x}u + \frac{\partial}{\partial y}v \quad = \quad 0, \qquad (23)$$

$$u\frac{\partial}{\partial x}u + v\frac{\partial}{\partial y}u \quad = \quad -S\frac{d}{dx}h + \frac{\partial^2}{\partial y^2}u, \qquad (24)$$

with boundary conditions : $u = v = 0$ at $y = 0$, and $\frac{\partial}{\partial y}u = 0$, $v = u\frac{dh}{dx}$ at the interface h. Plus a given first profile solution. But if of course this problem seems to be parabolic it is not so that an output boundary condition has to be included.

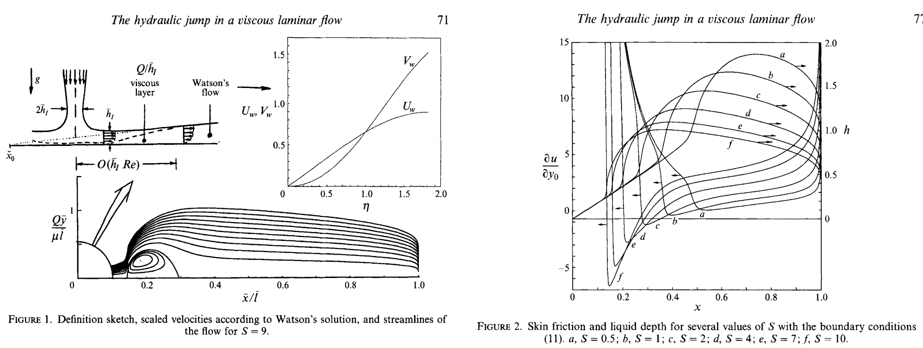


Figure 12: The hydraulic jump as an interacting problem.

6 Blood Flows

6.1 Equations

Same long wave or thin layer approximation may be done for the viscous flow in arteries. The flow is supposed axi symmetrical $\underline{u} = u(x,r), v(x,t)$ and the radius of the vessel is written $R(x,t)$. then

$$\frac{\partial u}{\partial x} + \frac{\partial}{r\partial r}(rv) = 0, \tag{25}$$

and as again the pressure is constant in every section $p(x,t)$:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial r} = -\rho^{-1}\frac{\partial p}{\partial x} + \nu\frac{\partial}{r\partial r}(r\frac{\partial u}{\partial r}), \quad \frac{\partial p}{\partial r} = 0, \tag{26}$$

with $v(x,R,t) = \frac{\partial R}{\partial t}$ et $u(x,R,t) = 0$ at the wall.

6.2 Classical Integral equations

By integration of the incompressibility equation (25), taking into account the velocity of the wall $v(x,R,t) = \partial R/\partial t$, and defining the flux

$$Q = \int_0^R 2\pi u r dr \quad S = \pi R^2. \tag{27}$$

so that

$$\frac{\partial S}{\partial t} + \frac{\partial Q}{\partial x} = 0. \tag{28}$$

The conservative formulation

$$\frac{\partial ru}{\partial t} + \frac{\partial ru^2}{\partial x} + \frac{\partial ruv}{\partial r} = -\rho^{-1}r\frac{\partial p}{\partial x} + \nu\frac{\partial}{\partial r}(r\frac{\partial u}{\partial r}), \quad \frac{\partial p}{\partial r} = 0. \tag{29}$$

gives:

$$\frac{\partial Q}{\partial t} + \frac{\partial}{\partial x}(\int_0^R 2\pi u^2 r dr) = -S\rho^{-1}\frac{\partial p}{\partial x} + 2\pi\nu[r\frac{\partial u}{\partial r}]_R. \tag{30}$$

To go on, one has to do some other hypothesis to link Q and $Q_2 = (\int_0^R 2\pi u^2 r dr)$ and the skin friction $\nu[r\frac{\partial u}{\partial r}]_R$. We have to close the equations

6.3 Integral equations with displacement

Here we adapt Von Kármán integral methods (from aerodynamics Schlichting (1987) [2]) to the system (**??**-**??**). The key is to integrate the equations with respect to the variable η from the centre of the pipe to the wall ($0 \le \bar{\eta} \le 1$ with $\eta = r/R$). So, we introduce $\bar{U}_0$, the velocity along the axis of symmetry, a kind of loss of flux $\bar{q}$, and $\bar{\Gamma}$ as follows:

$$\bar{U}_0(\bar{x},\bar{t}) = \bar{u}(\bar{x},\bar{\eta}=0,\bar{t}), \quad \bar{q} = \bar{R}^2(\bar{U}_0 - 2\int_0^1 \bar{u}\bar{\eta}d\bar{\eta}) \;\&\; \bar{\Gamma} = \bar{R}^2(\bar{U}_0^2 - 2\int_0^1 \bar{u}^2\bar{\eta}d\bar{\eta}). \tag{31}$$

We note that $\bar{q}$ is like the flux difference between a perfect fluid profile and the real one; it is analogous to the displacement thickness δ_1 well known in aerodynamics. $\bar{\Gamma}$ is nearly analogous to the energy displacement thickness δ_2. In aerodynamics the shape factor H links δ_1 and δ_2. Our new unknown functions are q, R and U_0, and we now establish their P.D.E. of evolution. Once again in establishing the fluid motion equation, we suppose that ε_2 is not necessarily too small and $\alpha = O(1)$. The transverse integration of the incompressibility relation (**??**) with the help of the boundary conditions (**??**) gives:

$$\frac{\partial \bar{R}^2}{\partial \bar{t}} + \varepsilon_2 \frac{\partial}{\partial \bar{x}}(\bar{R}^2\bar{U}_0 - \bar{q}) = 0, \quad \bar{R} = 1 + \varepsilon_2 \bar{h}. \tag{32}$$

If we integrate (**??**), with the help of the boundary conditions (**??**), we obtain the equation for $q(x,t)$:

$$\frac{\partial \bar{q}}{\partial \bar{t}} + \varepsilon_2(\frac{\partial}{\partial \bar{x}}\bar{\Gamma} - \bar{U}_0\frac{\partial}{\partial \bar{x}}\bar{q}) = -2\frac{2\pi}{\alpha^2}\tau, \qquad \tau = (\frac{\partial \bar{u}}{\partial \bar{\eta}})|_{\bar{\eta}=1} - (\frac{\partial^2 \bar{u}}{\partial \bar{\eta}^2})|_{\bar{\eta}=0}. \tag{33}$$

From the same equation (**??**) (and from (**??**)), evaluated on the axis of symmetry (in $\eta = 0$), we obtain an equation for the velocity along the axis $U_0(x,t)$:

$$\frac{\partial \bar{U}_0}{\partial \bar{t}} + \varepsilon_2 \bar{U}_0 \frac{\partial \bar{U}_0}{\partial \bar{x}} = -\frac{\partial \bar{p}}{\partial \bar{x}} + 2\frac{2\pi}{\alpha^2}\frac{\tau_0}{\bar{R}^2}, \qquad \tau_0 = (\frac{\partial^2 \bar{u}}{\partial \bar{\eta}^2})|_{\bar{\eta}=0}. \tag{34}$$

The two previous relations introduced the values of the friction in $\eta = 0$, the axis of symmetry: $((\frac{\partial^2 \bar{u}}{\partial \bar{\eta}^2})|_{\bar{\eta}=0})$ and the skin friction in $\eta = 1$, at the wall: $((\frac{\partial \bar{u}}{\partial \bar{\eta}})|_{\bar{\eta}=1})$. Information has been lost here, so we need a closure relation between $(\bar{\Gamma}, \tau, \tau_0)$ and $(\bar{q}, \bar{R}, \bar{U}_0)$. As there are so far no ambiguities, we remove the bars over the adimensionnalized symbols.

6.4 The wall

Mostly used is a kind of elastic wall as:

$$\mu\frac{\partial^2 R}{\partial t^2} + \eta\frac{\partial R}{\partial t} + \kappa(R - R_0) = p - p_0 \qquad (35)$$

this is a set of viscoelastic strings. But often only a string like behavior is considered:

$$p - p_0 = \kappa(R - R_0) \qquad (36)$$

6.4.1 Womersley

There exist an exact solution of (29) without non linear term and with a given oscillating wave pressure: $p = pe^{i\omega t - ikx}$ with $k = \omega/c$:

$$\frac{\partial u}{\partial t} = +\frac{ikp}{\rho} + \nu\frac{\partial}{r\partial r}(r\frac{\partial u}{\partial r}), \qquad (37)$$

so with (25):

$$u = \frac{p}{\rho c}(1 - \frac{J_0(i^{3/2}\alpha r/R)}{J_0(i^{3/2}\alpha)})e^{i\omega t - ikx}, \qquad (38)$$

$$v = \frac{i\omega p}{\rho c^2}(\frac{r/R}{2} - \frac{i^{3/2}J_1(i^{3/2}\alpha r/R)}{\alpha J_0(i^{3/2}\alpha)})e^{i\omega t - ikx}. \qquad (39)$$

Defining the Womersley number:

$$\alpha = R\sqrt{\frac{\omega}{\nu}}. \qquad (40)$$

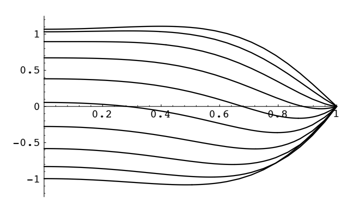


Figure 13: Womersley Profiles during one half period.

Those profiles are used to close the system.
This allows to build an integral method for the flow.

6.4.2 Closure

As in aerodynamics, the previous system of equations is not closed: we have lost details of the velocity profile in the integration process. Therefore, we have to imagine a velocity profile and deduce from it relations linking Γ, τ and τ_0 and q, U_0 et R. These relations are found from the radial dependence of u. Pohlhausen's idea, explained in Schlichting (1987) [2] or Le Balleur (1982) [1], consists in postulating an *ad hoc* velocity distribution in η which fits the boundary conditions and "looks like" observed profiles. Here the most simple idea is to use the profiles from the analytical linearized solution given by Womersley (1955) [43] for the case with no transverse pressure variation that we have already seen. This solution in complex form ($i^2 = -1$) is rewritten as:

$$U_{Womersley} = (F_W(x,t) + iG_W(x,t))(j_r(\alpha\eta) + ij_i(\alpha\eta)), \qquad (41)$$

where F_W, G_W, j_i and j_r are real functions defined as follows:

$$(F_W(x,t)+iG_W(x,t)) = \frac{kp}{c}(1-\frac{1}{J_0(i^{3/2}\alpha)})e^{i2\pi(t-x/c)}, \quad (j_r+ij_i) = \left(\frac{1-\frac{J_0(i^{3/2}\alpha\eta)}{J_0(i^{3/2}\alpha)}}{1-\frac{1}{J_0(i^{3/2}\alpha)}}\right).$$

Thus, we will assume that the velocity distribution in the following has the same dependence on η. It means that we suppose that the fundamental mode imposes the radial structure of the flow. The real velocity is:

$$u = 1/2\,((F+iG)(j_r+ij_i) + cc) = (Fj_r - Gj_i), \qquad (42)$$

where $F(x,t)$ and $G(x,t)$ are now real unknown functions that we want to find and cc is the conjugate complex. We immediately see that $U_0(x,t) = F(x,t)$ (because $j_r(0) = 1$ and $j_i(0) = 0$) and that if we compute q with (42) we obtain $G(x,t)$ as:

$$G(x,t) = \frac{q/R^2 - U_0 + U_0 2\int_0^1 j_r\eta d\eta}{2\int_0^1 j_i\eta d\eta}. \qquad (43)$$

The two functions F and G are only functions of (U_0, R, q) and we keep the Womersley radial dependence.

6.4.3 The coefficients of closure

The velocity at any radius η (42) and (43) may be written with the value of the velocity at the centre U_0 ,the radius R, and the loss of flux q. Next, by integration, we obtain Γ as a function of (U_0, R, q) and, by derivation, we obtain τ and τ_0 as functions of (U_0, R, q):

$$\Gamma = \gamma_{qq}\frac{q^2}{R^2} + \gamma_{qu}qU_0 + \gamma_{uu}R^2U_0^2, \quad \tau = \tau_q\frac{q}{R^2} + \tau_u U_0 \quad \tau_0 = \tau_{0q}\frac{q}{R^2} + \tau_{0u}U_0. \tag{44}$$

This closes the problem. The coefficients $((\gamma_{qq}, \gamma_{qu}, \gamma_{uu}), (\tau_q, \tau_u), (\tau_{0q}, \tau_{0u}))$ are only functions of α. They involve combinations of integrals and derivatives of the Bessel function. For example we have (if $\int f$ is a shorthand for $\int_0^1 f(\eta)d\eta$ and $\partial_\eta f_{\eta=0}$ an other for $\frac{\partial f}{\partial \eta}(0)$):

$$\begin{aligned}\gamma_{uu} \quad = \quad & 1 - \int j_i^2/(\int j_i)^2 - (2\int j_r j_i)/\int j_i - \int j_r^2 + \\ & +(2\int j_i^2 \int j_r)/(\int j_i)^2 + (2\int j_i j_r \int j_r)/\int j_i - \\ & -(\int j_i^2(\int j_r)^2)/(\int j_i),\end{aligned}$$

$$\tau_{0u} = \partial_\eta^2 j_{r\eta=0} + \partial_\eta^2 j_{i\eta=0}/\int j_i - (\partial_\eta^2 j_{i\eta=0}\int j_r)/\int j_i.$$

These coefficients are nearly constant for $\alpha < 5$. For α small we obtain from the preceding computations:

$$((\frac{-6}{5}, \frac{11}{5}, \frac{-2}{15}), (24, -12), (-12, 4)), \tag{45}$$

so, we recover the values for the Poiseuille profile at small frequency. The fact that those coefficients are nearly constant makes the model robust. For $\alpha \to \infty$ (in practice, $\alpha > 12$ is enough) we find from asymptotic behaviour of Bessel functions and from the preceding computations the asymptotic form of the coefficients:

$$((\frac{-\alpha}{4\sqrt{2}}, 2, -\frac{\sqrt{2}}{2\alpha}), (\alpha^2/2, -\alpha\sqrt{2}), (0, 0)).$$

One can easily show that this is coherent with Wormesley's solution in the limit of large α. We note that for $\alpha \to \infty$ and $\varepsilon_2 = 0$, the wave solution for q is

$$q = \frac{\sqrt{2}}{\alpha\pi}(1-i)e^{2i\pi(t-x/x)}, \quad c = \sqrt{\frac{k}{2}(1 - \frac{\sqrt{2}}{\alpha}(1-i) + O(\alpha^{-2})}.$$

Now equations (**??**), (32), (33) and (34) with the closure (44) define a set of four monodimensional equations linking the pressure p, the velocity along the axis U_0, the loss of flux q and the variation of the radius h.

6.4.4 Remarks

1- The main difference from other integral methods ([40] , [41], [42], [46], [45], or [44] ...) in our approach is the introduction of an auxillary partial differential relation (33) obtained from an aeronautical analogy. Instead of q, Γ and U_0 authors mainly use Q, Q_2 and U_0:

$$Q = \int_0^R 2\pi u r dr \qquad Q/\pi = U_0 R^2 - q$$

$$Q_2 = \int_0^R 2\pi r u^2 dr \qquad Q_2/\pi = U_0^2 R^2 - \Gamma$$

If we substract (32) from (33) we obtain the classical system of two equations:

$$2\pi R\frac{\partial R}{\partial t} + \varepsilon_2 \frac{\partial}{\partial x}(Q) = 0,$$

$$\frac{\partial Q}{\partial t} + \varepsilon_2 \frac{\partial}{\partial x}(Q_2) = -\pi R^2 \frac{\partial p}{\partial x} + \pi \frac{2\pi}{\alpha^2}(\frac{\partial u}{\partial \eta})|_{\eta=1}$$

Often, the relation for Q_2 is written as $Q_2 = \frac{Q^2}{\pi R^2}$ (in this case the radial variation of the profile is neglected: flat profile) or $Q_2 = \frac{4Q^2}{3\pi R^2}$ (parabolic profile: see equation (45)). Note, that we have instead a third differential equation to link Q_1 and Q_2. The effect of the skin friction ($\tau_1 = \frac{2\pi}{\alpha^2}(\frac{\partial u}{\partial \eta})|_{\eta=1}$) is often estimated by $\tau_1 = -\frac{8\pi}{\alpha^2}\frac{Q}{\pi R^3}$, true for a Poiseuille flow only ((45) again). It may be replaced by an unsteady relation (deduced from unsteady Poiseuille flow) such as:

$$T_\tau \frac{\partial \tau_1}{\partial t} + \tau_1 = -\frac{8}{\alpha^2}(Q + T_Q \frac{\partial Q}{\partial t} + ...)$$

See Yama et al (1995) [44] for the derivations and values of coefficients T_τ and T_Q. We do not claim that our description is better, but for a sinusoidal input we find again (at any frequency) the Womersley linear solution. Our profiles are realistic in the sense that they present overshoots in the core and back flow near the wall. This is not the case when the closure is simply $\tau_1 = -\frac{8\pi}{\alpha^2}\frac{Q}{\pi R^3}$ or in the case of very peculiar profiles chosen by Belardinelli & Cavalcanti (1992) [36].

2- We noted that the coefficients vary little with α, this shows that our model is very robust: it is easy to see that equations (32) and (33) are invariant under the rescaling $t \to t/\Omega$, $/\sqrt{\Omega}$, and $c \to c$, if τ is taken constant (independant of α). This explains why methods based on Poiseuille coefficients are robust too.

6.5 Interactive problem

The interactive problem is mainly solved in an 1D description ([40], [45]). As people solved linearised systems at first, it was clear that those equations reduce to wave equations and need two boundary conditions one at the entrance, the other at the output. Some solutions of the full interacting problem have been done by [38] and [37]

7 Conclusion on Interactive problems

No definite conclusion will be given, we insist on the fact that some thin layer flows must be solved with the good set of boundary conditions. The supersonic paradox of the "free interaction" is in fact present in a lot of flows (hypersonic, mixed convection, artery, supercritical...).

References

[1] Le Balleur J.C. (1982): "Viscid- inviscid coupling calculations for 2 and 3D flows", VKI lecture series 1982-02.

[2] Schlichting H. (1987): "Boundary layer theory", Mc Graw Hill.

[3] F.T. Smith (1979): "On the non parallel flow stability of the blasius boundary layer", Proc. Roy. Soc. Lond., A366, pp. 91- 109.

[4] F. T. Smith (1982): "On the high Reynolds Number Theory of Laminar Flows", IMA Journal of Applied Mathematics, vol 28, pp. 207-281.

Hypersonic

[5] S.N. Brown, K. Stewartson & P.G. Williams (1975): "Hypersonic self induced separation", The Phys. of Fluids, vol 18, No 6, June.

[6] S. N. Brow H. & K. Cheng Lee (1990): "Inviscid viscous interaction on triple deck scales in a hypersonic flow with strong wall cooling", J. Fluid Mech., vol 220, pp. 309- 307.

[7] S.N. Brown & K. Stewartson P.G. (1975): "A non uniqueness of the hypersonic boundary layer", Q.J. appl Math, vol XXVIII, Pt 1 pp. 75- 90

[8] Chernyi (1961): "Introduction to hypersonic flow" , Academic Press

[9] Guiraud, J. P., Vallee, D., Zolver, R. 1965. In Basic Developments in Fluid Dynamics. ed. M, Holt, 127-247, New York : Academic. 447 pp.

[10] Hayes & Probstein (1959):"Hypersonic flow theory" Academic Press .

[11] P.-Y. Lagrée (1992): "Structures interactives Fluide Parfait/ Couche limite en hypersonique, Variations autour du thème de la triple couche", Thèse de l'Université Paris VI,

[12] P.-Y. Lagrée (1991): "Influence de la couche d'entropie sur la longueur de séparation en aérodynamique hypersonique, dans le cadre de la triple couche II", C.R. Acad Sci Paris, t 313, Série II, p. 999-1004, 1991.

[13] P.-Y. Lagrée (1990): "Influence de la couche d'entropie sur l'échelle de la région séparée en aérodynamique hypersonique", C.R. Acad Sci Paris, t 311, Série II, p. 1129- 1134, 1990.

[14] V. Ya Neiland (1969): "Theory of laminar boundary layer separation in supersonic flow ", Mekh. Zhid. Gaz., Vol. 4, pp. 53-57.

[15] V. Ya Neiland (1970): "Propagation of perturbation upstream with interaction between a hypersonic flow and a boundary layer", Mekh. Zhid. Gaz., Vol. 4, pp. 40-49.

[16] V. Ya Neiland (1986): " Some features of the transcritical boundary layer interaction and separation", IUTAM London SN Brown edi. Springer

[17] A. I. Ruban & S. N. Timoshin (1986): "Propagation of perturbations in the boundary layer on the walls of a flat chanel", Fluid Dynamics, No 2, pp. 74-79.

[18] L.I. Sedov: 1959 "Similarity and Dimensional Methods in Mechanics." Infosearch Ltd., London., 1959,
"Similitude et dimensions en mécanique", éditions Mir, 1977.

[19] Stewartson K. (1964) "the Theory of Laminar Boundary Layers in Compressible Fluids", Oxford University Press, London (1964).

[20] K. Stewartson & P.G. Williams (1969): "Self - induced separation",Proc Roy Soc, A 312, pp 181-206.

[21] Milton Van Dyke (1982) "An Album of Fluid Motion." Stanford The Parabolic Press.

[22] M. J. Werle, D. L. Dwoyer a W. L. Hankey (1973) "Initial conditions for the hypersonic-shock/ boundary- layer interaction problem", AIAA J., vol 11, no 4, pp525- 530.

[23] **Thermal Flows**

[24] F. J. Higuera (1997): "Opposing mixed convection flow in a wall jet over a horizontal plate" J. Fluid Mech., vol 342, pp. 355-375.

[25] P.G. Daniels (1992): "A singularity in thermal boundary- layer flow on a horizontal surface", J. Fluid Mech., vol 242, pp. 419- 440.

[26] P.G. Daniels & R.J. Gargaro (1993): " Buoyancy effects in stably stratified horizontal boundary- layer flow", J. Fluid Mech., Vol. 250, pp. 233- 251.

[27] M. El Hafi (1994): "Analyse asymptotique et raccordements, étude d'une couche limite de convection naturelle", thèse de l'Université de Toulouse.

[28] Exner, A ; Kluwick, A (2001). "On thermally induced separation of laminar free-convection flows" Physics of Fluids, Volume 13, Issue 6, pp. 1691-1703 (2001).

[29] P.-Y. Lagrée (1994): "Convection thermique mixte à faible nombre de Richardson dans le cadre de la triple couche", C. R. Acad. Sci. Paris, t. 318, Série II, pp. 1167- 1173.

[30] P.-Y. Lagrée (1995): "Upstream influence in mixed convection at small Richardson Number on triple, double and single deck scales", symp.: Asymptotic Modelling in Fluid Mechanics, Bois, Dériat, Gatignol & Rigolot (Eds.), Lecture Notes in Physics, Springer, pp. 229-238.

[31] P.-Y. Lagrée (2001): "Removing the marching breakdown of the boundary layer equations for mixed convection above a horizontal plate", International Journal of Heat and Mass Transfer, Vol 44/17, pp. 3359-3372.

[32] W. Schneider & M.G. Wasel (1985): "Breakdown of the boundary layer approximation for mixed convection above an horizontal plate", Int. J. Heat Mass Transfert , Vol. 28, No 12, pp. 2307-2313.

[33] H. Steinrück (1994): "Mixed convection over a cooled horizontal plate: non-uniqueness and numerical instabilities of the boundary layer equations", J. Fluid Mech., vol 278, pp. 251-265

[34] H. Steinrück (1995): "Mixed convection over a horizontal plate: self similar and connecting boundary- layer flows", Fluid Dynamic Res, 15, pp 113- 127.

[35] **Blood Flows**

[36] Belardinelli E. & Cavalcanti S. (1992): "theorical analysis of pressure pulse propagation in arterial vessels", J. Biomechanics vol 25 pp. 1337-1349.

[37] Canić, S.; Hartley, C.; Rosenstrauch, D.; Tambaca, J. Guidoboni, G.; Mikelić, A. (2006) "Blood Flow in Compliant Arteries: An Effective Viscoelastic Reduced Model, Numerics, and Experimental Validation" Annals of Biomedical Engineering, Vol. 34, No. 4, April 2006 (2006) pp. 575–592

[38] P.-Y. Lagrée (2000): "An inverse technique to deduce the elasticity of a large artery ", European Physical Journal, Applied Physics 9, pp. 153-163

[39] Ling S.C. & Atabek H.B. (1972): "a non linear analysis of pulsatile flow in arteries" J.F.M. vol 55 p 3 pp.

[40] Olufsen MS, Peskin CS, Kim WY, Pedersen EM, Nadim A, Larsen J. (2000): "Numerical simulation and experimental validation of blood flow in arteries with structured-tree outflow conditions." Ann Biomed Eng. 2000 Nov-Dec;28(11):1281-99.

[41] Pedley T. J. (1980): "The Fluid Mechanics of Large Blood Vessel", Cambridge University press.

[42] Pythoud F., Stergiopulos N. & Meister J.-J. (1996): "Separation of arterial pressure waves into their forward and backward running components", J. Biom. Eng., august, vol 118, pp. 295- 301.

[43] Womersley J. R. (1955): "Oscillatory Motion of a Viscous Liquid in a Thin- Walled Elastic Tube", Philosophical Magazine, volume 46, pages 199-221.

[44] Yama J.R., Mederic P. & Zagzoule M. (1995): "Etude comparative des différentes approximations de la contrainte de cisaillement à la paroi"Archives of Physiology and Biochemestry, vol 103, N° 3, Jully p C164.

[45] Zagzoule M. & Marc- Vergnes J.-P (1986) "a global mathematical model of the cerebral circulation in man", J. Biomechanics Vol 19, No 12, pp. 1015-1022.

[46] Zagzoule M. J. Khalid- Naciri & J. Mauss (1991): "Unsteady wall shear stress in a distensible tube", J. Biomechanics Vol 24, No 6, pp. 435-439.

[47] **Saint Venant**

[48] R.I. Bowles & F.T. Smith (1992): "The standing hydraulic jump: theory, computations and comparisons with experiments", J. Fluid Mech., vol 242, pp. 145-168.

[49] J. Gajjar & F.T. Smith (1983): "On hypersonic self induced separation, hydraulic jumps and boundary layer with algebraic growth", Mathematika, 30, pp. 77-93.

[50] F. J. Higuera (1994): "The hydraulic jump in a viscous laminar flow" J. Fluid Mech., vol 274, pp. 62-92.

[51] F. J. Higuera (1997): "The circular hydraulic jump " Phys. Fluids 9 (5), May 1997 pp. 1476-1479.

The web page of this text is:
`http://www.lmm.jussieu.fr/~lagree/COURS/CISM/`

The last version of this file is on:
`http://www.lmm.jussieu.fr/~lagree/COURS/CISM/ex_CISM.pdf`