Long-wavelength properties of the Kuramoto-Sivashinsky equation

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We study the long-wavelength properties of the one-dimensional Kuramoto-Sivashinsky equation. We determine all the parameters in the effective long-wavelength equation, interpret the phenomenological coefficients in terms of microscopic quantities, and estimate the time and length scales where the behavior crosses over from linear diffusive to that of the full nonlinear equation. We corroborate our analysis by studying variants of the model with more general linear terms.

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I. INTRODUCTION

An important issue in the study of dynamical systems is the extent to which the long-wavelength properties of deterministic systems with many degrees of freedom that are chaotic can be described by coarse-grained models with stochastic noise. Given the notorious difficulty of handling nonlinear partial differential equations in three dimensions, one studies model equations that are simpler but still realistic. The Kuramoto-Sivashinsky (KS) equation [1,2], proposed for chemical waves and flame fronts, affords an excellent testing ground to study this issue. In this paper we use the method pioneered by Zaleski [3] and extend his study of the effective stochastic equation that describes the long-wavelength properties of the KS equation in one dimension. We determine all the parameters in the effective equation and interpret these "macroscopic" phenomenological coefficients in terms of microscopic quantities; we can then show that these imply large values for the length and time scales at which the long-wavelength behavior of the system crosses over from free field to the interacting behavior predicted by the effective equation. The universality of the results are examined by studying variants of the model in which more general linear terms are allowed.

The Kuramoto-Sivashinsky equation in one dimension reads

$$h_t = -h_{xx} - h_{xxxx} - \tfrac{1}{2}h_x^2 , \tag{1}$$

where the subscripts denote partial derivatives. All the coefficients have been fixed by appropriate rescaling, so that the only parameter is the length L of the system, which then plays a role analogous to that of Reynolds number in fluid turbulence. Note that at the linear level the long-wavelength modes are unstable: modes with wave vector k evolve as $e^{\omega_k t}$, where the rate $\omega_k = k^2 - k^4$. The nonlinearity allows the conversion of the growing small-k modes into decaying large-k modes, thus stabilizing the system. For large L the system reaches an asymptotic state that is chaotic and is characterized by a nonzero density of positive Liapunov exponents [2]. The dynamical variable $h(x,t)$ can be interpreted as the height of a one-dimensional interface and the equation displays the corresponding interface symmetry under translation of the interface $h(x) \rightarrow h(x)+c$, where c is a constant. If one sets $u = h_x$, Eq. (1) assumes the form

$$u_t = -u_{xx} - u_{xxxx} - uu_x . \tag{2}$$

Note that this equation possesses a conservation law: $\int_0^L u(x,t)dx$ is invariant under time evolution. The variable $u(x,t)$ in (2) can be interpreted as a one-dimensional velocity field. Indeed, the nonlinear term is the same as that in the one-dimensional Navier-Stokes equation. On the other hand, the nonlinear term in (1) has the same form as that in the equation proposed by Kardar, Parisi, and Zhang (KPZ) [4] as a generic model for interface growth. The latter is given by

$$h_t = \nu h_{xx} + (\lambda/2)h_x^2 + \eta(x,t) , \tag{3}$$

where η is a white-noise term

$$\langle \eta(x,t)\eta(x',t') \rangle = D\delta(x-x')\delta(t-t') . \tag{4}$$

The crucial issue is whether the chaotic fluctuations in the deterministic KS equation (1) simulate the effect of external white noise in an effective KPZ equation (3), thus rendering the long-wavelength properties of the two systems identical. There has been some recent controversy over whether the KS and KPZ equation in fact belong to the same universality class. In work done simultaneously with ours, Sneppen *et al.* [5] show by extensive numerical simulations of Eq. (1) on large systems that the dynamic scaling form for the width as a function of the size of the system L and time t predicted by the KPZ equation is observed; they performed a crossover analysis showing that the intermediate scaling regime where free, diffusive behavior obtains is long. They also determined explicit values of the crossover length and time scales.

Previously, Yakhot [6] used a perturbative renormalization-group approach and suggested that the KS equation [Eq. (2)] was described at long wavelengths by the stochastic Burgers equation [7]. A constructive approach was initiated by Zaleski in Ref. [3], in which he explicitly eliminated short-wavelength degrees of freedom $u(k)$ with $|k| > \Lambda$, where Λ is an appropriately chosen cutoff and provided numerical evidence that at long wavelengths the system was described by a Burgers equation in the presence of external noise. The latter assumes the following form for $|k|<\Lambda$:

$$u_t(k,t)=-\nu k^2 u(k,t)+g(k,t)+f(k,t)\ , \quad (5)$$

where ν is an effective diffusion constant, and

$$g(k,t)=-ik/2 \sum_{|q|<\Lambda,|k-q|<\Lambda} u(q,t)u(k-q,t) \quad (6)$$

is the k-space expression of the nonlinear term in Eq. (2), with the restriction that wave vectors are smaller than a cutoff Λ and $f(k,t)$ is the random force. The dynamical variable can be obtained in real space by using $u(x,t)=\sum_{|k|<\Lambda}u(k,t)e^{-ikx}$; the sum extends over a restricted range in contrast to the original equation (2), where the values of k would span the entire interval in wave-vector space.

II. METHOD

We summarize the procedure followed by Zaleski to determine the parameters: In order that Eq. (5) be a genuine stochastic equation the "random" force $f(k',t')$ must be uncorrelated with any previous value of the field $u(k)$, i.e.,

$$\langle u(k,t)f(k',t')\rangle=0 \quad (7)$$

for $t<t'$. This condition then serves to fix the value of ν in Eq. (5). The time evolution of modes with $k<\Lambda$ can be written in the form of Eq. (5) with f given by

$$f(k,t)=(\nu+1-k^2)k^2u(k,t) -ik/2 \sum_{|q|\geq\Lambda \text{ or } |k-q|\geq\Lambda} u(q,t)u(k-q,t)\ . \quad (8)$$

In practice, one evolves the equation numerically and evaluates $C_{fu}(t)\equiv\langle f(k,t')u(-k,t'-t)\rangle$ as a function of t, where the angular brackets denote averaging over t'. Numerically one finds that $C_{fu}(t)$ reaches a constant value for $t\geq\tau_x=30$ and small k; this saturation allows one to choose a $\nu(k)$ such that the condition given by Eq. (7) is satisfied for $\tau_x \ll t \ll \tau(k)$, where $\tau(k)$ diverges as $k\rightarrow 0$. We point out that the noise term given above [Eq. (8)] contains both an additive term and a multiplicative term. Numerically, the contribution of the multiplicative piece was found not to be significant (see later). Given $f(k,t)$ one can evaluate its autocorrelation function $g(k,s)=\langle f(k,t)f(-k,t+s)\rangle$, which is found to decay rapidly, thus confirming the validity of the effective equation. Zaleski evaluated ν and found a value of approximately 10 for $\Lambda=0.5$, thus demonstrating that at long wavelengths the effective diffusion constant had become positive. Thus he was able to show by explicit construction that on large scales the KS equation behaves like a stochastic Burgers equation even though he was unable to see the behavior predicted by the stochastic Burgers equation for dynamic scaling phenomena. We have followed this procedure to estimate the crossover length and time scales and discuss their large values.

The numerical integration of Eq. (2) was performed by using two different methods and the results were compared. In the first we used the "approximate solution operator" [8], which is a k-space method; in the other we used a second-order Runge-Kutta method (for the time discretization we used a mesh of size 0.02 to 0.01) with a symmetric discretization of spatial derivatives (on a mesh of unit size). We considered systems of size up to $L=4096$. While the qualitative results were the same, the continuum limit is better approximated by the k-space method. The real-space method alters the effective nonlinear coupling near the zone boundary when the discrete spatial lattice is coarse; it is computationally prohibitive to use a much finer grid.

III. RESULTS

In Fig. 1 we reproduce the spectrum $C_{uu}(k)=\langle u(k,t)u(-k,t)\rangle$, where the average is taken over long times when u is on the asymptotic chaotic attractor. The spectrum is flat for small k, decreases rapidly at large k, and exhibits a hump with a maximum around $k=0.71$; note that the maximally linearly unstable mode lies at $k_0=1/\sqrt{2}=0.707$. The steady-state value of C_{uu} is given by $C_{uu}=A/L$ as $k\rightarrow 0$; we find $A=1.2\pm 0.1$.

We have also reevaluated the viscosity ν in Eq. (5) and find a value of $\nu=7.5\pm1.5$ in rough agreement with the earlier results of Ref. [3]. For consistency it is clear that the value of ν should be dominated by the $|k|>\Lambda$ modes. We have checked numerically that this holds: as long as Λ falls in the flat region of the spectrum, i.e., the k modes in the hump are integrated over completely, the value of ν is independent of Λ.

Figure 2 shows the spectrum of the noise $f(k)$ divided by k^2. It is flat for small k and rises for $k>0.1$. One can therefore write for small k

$$C_{ff}(k,t-t')=\langle f(k,t)f(-k,t')\rangle \simeq Dk^2[(1/2\tau)e^{-|t-t'|/\tau}]\ , \quad (9)$$

where D denotes the strength of the force-force correlation normalized appropriately and τ is the correlation

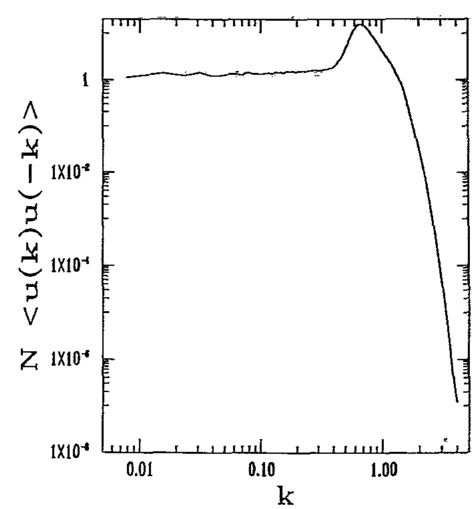

FIG. 1. Spectrum of the velocity field $C_{uu}=\langle u(k,t)u(-k,t)\rangle$ vs k. System size is $L=800$. $N=1024$ is the number of discrete k points.

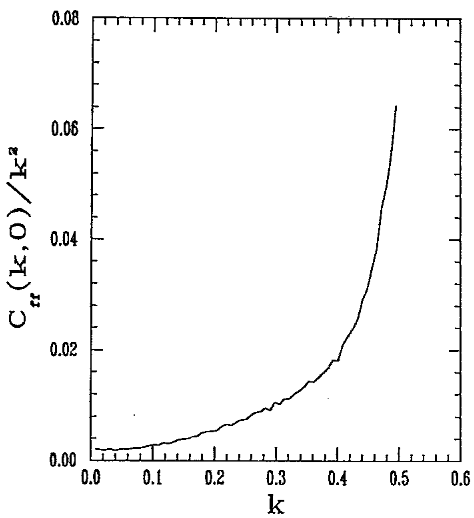

FIG. 2. Spectrum of the effective noise $C_{ff}=\langle f(k,t)f(-k,t)\rangle$ divided by k^2 vs k. System size is $L=800$.

time. The proportionality to k^2 at small k reflects the conservation of total velocity in the original KS equation, and is the required behavior of any stochastic force which would appear in a Burgers equation. We also point out that Eq. (8) for $f(k,t)$ contains contributions from values of k both above and below the cutoff Λ. The contributions to the sum where both $|k|$ and $|k-q|$ are above Λ yield an additive noise term, whereas the remaining part of $f(k,t)$ is multiplicative, i.e., proportional to $u(k,t)$. Again for a bona fide stochastic noise the small-k limit of $\langle f(k,t)f(-k,t)\rangle$ should be dominated by the additive piece. This is indeed the case and is illustrated in Fig. 3, where the additive and multiplicative contributions to the force spectrum are shown separately.

We conclude that if one discards the negligible multi-

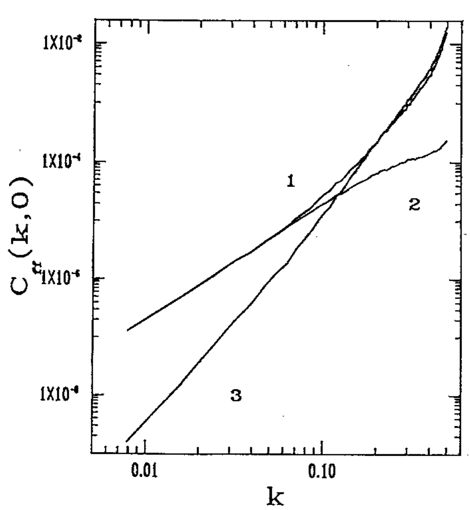

FIG. 3. Additive (2) and multiplicative (3) contributions to the total (1) spectrum of the noise $\langle f(k,t)f(-k,t)\rangle$ for a system of size $L=800$ as a function of k.

plicative contributions both to the value of ν and to $f(k)$ in Eq. (5), the long-wavelength limit of the KS equation effectively behaves as a stochastic Burgers equation with noise correlations given by (9).

We now proceed to check the consistency of the procedure and deduce the crossover length and time scales. It is straightforward to derive for the linear part of the effective Eq. (5) with the noise given above the relation

$$\langle u(k,t)u(-k,t)\rangle=(A/L)\frac{1}{(1+\nu\tau k^2)} . \tag{10}$$

Because of the existence of a fluctuation-dissipation theorem for the one-dimensional Burgers equation [7], the nonlinear term $g(k)$ does not contribute to static quantities at very long times. This statement holds true in the long-wavelength limit for the noise that arises from the integrated degrees of freedom because the additional terms are irrelevant in the renormalization-group sense. Thus the asymptotic equal-time correlation function in the $k\to 0$ limit takes the simple form

$$\langle u(k)u(-k)\rangle=A/L\equiv D/2\nu L . \tag{11}$$

We have checked that the product $L\langle u(k)u(-k)\rangle$ is independent of system size L for sizes from $L=400$ to $L=3200$. For $L=800$, for which N, the number of points within L, is equal to 1024 in the simulations, we have determined the decay time τ to be approximately 7.0 ± 1.0 in agreement with the results of Ref. [3]. Using this value, we estimate D from the small-k limit of the equal-time force-force correlation function $C_{ff}(k,0)$ and find $D=17.9$ when the cutoff $\Lambda=0.5$. With $\nu=7.5$, we thus find consistency with relation (11).

When the cutoff Λ is reduced from a value of 0.5 to a value of 0.25, we therefore expect the ratio D/ν to be unchanged. This is the case. While the value of $C_{ff}(k,0)$ is slightly smaller, τ also changes correspondingly, yielding essentially the same value for D. As mentioned earlier, ν is essentially unchanged when Λ is decreased.

With the values of D and ν obtained from the coarse-graining procedure, we can determine the crossover length and time scales using the results of a recent paper by Krug, Meakin, and Halpin-Healy [9]. In order to do this we revert to the interface representation; apart from an obvious change in the noise-noise correlations due to the transformation [the k^2 factor is missing in $C_{ff}(k,0)$], the correspondence is trivial. Comparing the temporal growth of the width of the linearized KPZ equation

$$w_0(t)\sim[\langle(h-\langle h\rangle)^2\rangle]^{1/2}=\left[\frac{D}{(2\pi\nu)^{1/2}}t^{1/2}\right]^{1/2} \tag{12}$$

with the asymptotic behavior of the complete equation

$$w(t)=c(A^2\lambda t)^{1/3} , \tag{13}$$

one obtains the crossover time t_c, when the system starts to exhibit nontrivial behavior. The universal value of c has been obtained accurately in Ref. [9] to be 0.63. This yields $t_c\approx 252\nu^5/D^2\lambda^4\approx 18\,700$. This compares with the value of 7000 obtained by Sneppen *et al.* [5]. The cross-

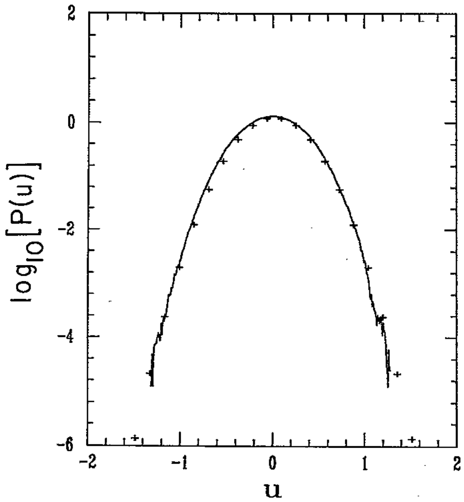

FIG. 4. Logarithm of the probability distribution $P(u(k))$ of the velocity field vs $u(k)$ for values of k below the cutoff.

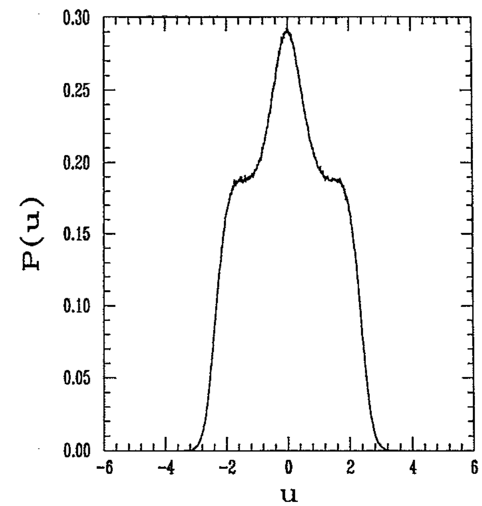

FIG. 5. Probability distribution of the velocity field $P(u(k))$ vs $u(k)$ on a linear plot for values of k including those in the hump region of Fig. 1. System size is $L=800$.

over length can be estimated analogously using Ref. [9] to be $L_c \approx 152\nu^3/D\lambda^2 \approx 3600$, to be compared with 2500 in Ref. [5]. The difficulty in obtaining ν accurately combined with the rapid variation of the crossover length and time scales with ν lead to not insignificant uncertainties. Nevertheless, we believe that the larger crossover values are characteristic of the true continuum limit. Indeed, our second-order Runge-Kutta integration procedure, which is not as accurate in reproducing the continuum limit (for the chosen space discretization), yields smaller crossover values. The extra terms introduced by the discretization procedure are presumably irrelevant around the KPZ fixed point and therefore, one expects the overall behavior to be universal.

The difficulty in observing KPZ behavior in the interface model can be traced to the fact that the diffusion constant in the effective equation has a large value. We expect the diffusion constant to be of the order of $l_{\text{eff}}^2/\tau_{\text{eff}}$; the characteristic length l_{eff} in the problem is given by $2\pi/k_0 \approx 8.9$, which corresponds to the maximum in the spectrum (see Fig. 1) and is the scale of the cellular structures in the system. The characteristic time scale t_{eff} is determined by the inverse of the largest Liapunov exponent $\lambda_{\max}$, characterizing the chaotic behavior. We have determined $\lambda_{\max}$ numerically using standard methods and found a value of approximately 0.1. One expects the decorrelation time for the effective noise term to be set by $1/\lambda_{\max}$ and the value of 7.0 obtained for τ is consistent with this expectation. Using these values for the effective time and length scales we estimate $\nu \approx (8.9)^2 0.1 = 7.9$, which is in rough agreement with our calculation. Thus the large diffusion constant can be attributed to the large scale of the cellular structures.

We have carried out similar calculations for different models, where the linear term has the form $a_2k^2 - a_4k^4 - a_6k^6 - a_8k^8$ in order to ascertain the universality of the results and the applicability of the qualitative arguments. Results similar to the ones described above are obtained in a model where the fourth-order derivative in Eq. (1) is replaced by a sixth-order one [3]. What is worth pointing out is that the values of D and ν become huge. For a cutoff of $\Lambda = 0.25$, we find (in agreement with Ref. [3]) $\nu = 20 \pm 2.5$; in addition we have determined D to be approximately 310. This enhanced value is due to the much larger local fluctuations of $u(x,t)$ in this model and the concomitant larger Liapunov exponent. However, the relation (11) is still well satisfied: $D/2\nu \approx 7.8$, while $N\langle u(k,t)u(-k,t)\rangle \approx 7.74$. In addition, the correlation time τ in this case is 3.38, a factor of 2.18 smaller than the original model; correspondingly the evaluation of the largest Liapunov exponent yields 0.22, a factor of 2.2 larger. Similar reasonable agreement was found for the $k^2 - k^8$ model. Varying the model by trying different coefficients a_2, a_4, a_6, a_8 in order to find a small value for ν so that the crossover time could be made smaller was unfruitful; the original KS equation yielded the smallest value of the crossover time scale.

We have also considered the probability distribution $P(u(k))$ of the velocity field for Eq. (2) (cf. Fig. 4). For small k's below the cutoff, the distribution is, as expected [7], Gaussian, with a width given by $\langle u(k)u(-k)\rangle$. As soon as values of k are included which belong to the region of the hump in Fig. 1, the distribution ceases to be Gaussian and acquires the broad background shown in Fig. 5.

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