The blowup problem for critical wave maps

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The wave map system

Let (N, γ) be a riemmannian manifold and assume $N \hookrightarrow \mathbb{R}^m$ with isometric embedding.
Thus one can define for every point $p \in N$ the tangent and normal space $T_p N$ and $T_p^\perp N$.

Definition : Wave map
Extrinsic definition : $U \ : \mathbb{R} \times \mathbb{R}^n \to N$ is a wavemap if

$$\forall t, x \quad \Box U(t,x) \perp T_{U(t,x)} N.$$

Intrinsic definition (abstract setting) : U is a wave map if it is a critical point of the action $(\eta = \mathrm{diag}(-1, 1 \ldots, 1))$

$$S(U) = \iint \gamma_{ij} \partial_\alpha U^i \partial_\beta U^j \eta^{\alpha\beta} dx^n dt.$$

Euler-Lagrange system :

$$\begin{cases} \forall i = 1 \ldots n, & \Box U^i + \Gamma^i_{jk} \partial_\alpha U^j \partial^\alpha U^k = 0, \\ & (U, U_t)|_{t=0} = (U_0, U_1). \end{cases}$$

Equivariant wave map

Assume $N = [0, R[\times \mathbb{S}^{m-1}$ has polar coordinates (ρ, χ), and the metric writes

$$ds^2 = d\rho^2 + g^2(\rho) d\chi^2,$$

where $g(0) = 0$ and $g'(0) = 1$.
Denote $x = (r, \omega)$ the polar coordinates in $\mathbb{R}^n$. A wave map U is said to be *equivariant* if there exist an angle function χ_0 such that :

$$U(t, r, \omega) = (u(t, r), \chi_0(\omega)) \in N.$$

A CNS to the existence of such U is that χ_0 be a harmonic polynomial.
Let k be the degree of χ_0 (if $k = 1$, U is said to be corotational).
The wave map system is then reduced to a single equation on $u : \mathbb{R}_t \times \mathbb{R}_r^+ \to \mathbb{R}$ (denote $f = g'g$) :

$$\begin{cases} u_{tt} - u_{rr} - \dfrac{n-1}{r} u_r = -\dfrac{k^2 f(u)}{r^2}, \\ (u, u_t)|_{t=0} = (u_0, u_1). \end{cases}$$

- $g = \sin$: case of the sphere $N = \mathbb{S}^2$.
- $g(\rho) = \rho(2 - \rho)$: critical (4D) radial Yang-Mills field.

Elementary properties

Conservation law : energy

$$E(U) = \int |Du(t)|^2 dx^n = \int |U_t|^2 + |\nabla_x U|^2 dx^n = \text{const.}$$

For an equivariant wave map, this write :

$$E(u) = \int \left(u_t^2 + u_r^2 + \frac{k^2 g^2(u)}{r^2} \right) r^{n-1} dr = \text{const.}$$

Scaling

Denote $U_\lambda(t,x) = U(\lambda t, \lambda x)$. Then

$$U_\lambda \text{ wave map } \iff U \text{ wave map}, \qquad \| U_\lambda \|_{\dot H^s}^2 = \lambda^{2s-n} \| U \|_{\dot H^s}^2.$$

$\dot H^{n/2}$ is a critical space.
In particular, for $n = 2$, $E(U_\lambda) = E(U)$. Dimension 2 is the critical dimension.

Finite speed of propagation

The energy is decreasing on incoming cones :

$$\int_{B(x_0,R)} |DU(t_0)|^2 dx^n \leq \int_{B(x_0,R+|\tau|)} |DU(t_0+\tau)|^2 dx^n.$$

In particular, if $(U(t=0), U_t(t=0))$ is supported in $B(x_0,R)$, then $U(t)$ is supported in$B(x_0,R+t)$.
Notation (for equivariant wave maps) :

$$E(u,a,b) = \int_a^b \left(u_t^2 + u_r^2 + \frac{k^2 g(u)}{r^2} \right) r^{n-1} dr.$$

Finite speed of propagation is often used in the form :

$$E(u(t),0,R) \leq E(u(t+\tau),0,R+|\tau|)$$

Local Cauchy theory for general wave map

Results of the type :
Given initial data $(U_0, U_1) \in H^s \times H^{s-1}$, there exist a unique wave map $(U, U_t) \in C^1([0, T), H^s) \times C^0([0, T), H^{s-1})$, where $T = T(\|U_0\|_{H^s}, \|U_1\|_{H^{s-1}})$.

- $s > n/2 + 1$, Choquet-Bruhat.
- $s > (n+1)/2$, $n \geq 3$, Klainerman and Machedon 93.
- $s > n/2$, $n \geq 3$, Klainerman and Machedon 95.
- $s > n/2$, $n = 2$, Klainerman and Selberg 01.

Proofs relies on *a priori* estimates using Strichartz inequality and the *null-structure* of the wave map system (algebraic fact).

This kind of results in a critical space gives global well-posedness.

- $n = 1$, $s \geq 1$: global existence (Keel and Tao 98).

Global regularity for small data

Results of the type :
Let X be a critical space : $\|(U_\lambda, U_{t\,\lambda})\|_X = \|(U, U_t)\|_X$.
There exists $\varepsilon > 0$ such that $\|(U_0, U_1)\|_X < \varepsilon$, then the arising wave map $U \in C([0,\infty), X)$ is global in time.

- $X = \dot{B}^{n/2}_{2,1} \times \dot{B}^{n/2-1}_{2,1}$, Tataru 1993.
- $X = \dot{H}^{n/2} \times \dot{H}^{n/2-1}$, equivariant case, Shatah and Tahvildar-Zadeh, 94.
- $X = \dot{H}^{n/2} \times \dot{H}^{n/2-1}$, $N = \mathbb{S}^k$, $n \geq 5$ then $n \geq 2$, Tao 00-01.
- $X = \dot{H}^{n/2} \times \dot{H}^{n/2-1}$, large class of manifolds, $n \geq 2$, Tataru 04.

See also : Klainermann and Rodnianski 01, Shatah and Struwe 02, Nahmod, Stefanov and Uhlenbeck 02, Krieger 03.

Method of proofs : Strichartz estimates, null-structure and geometrical aspects (e.g. $\Gamma^i_{jk} = -\Gamma^k_{ji}$).

Ill posedness and blow up in high dimension

- If $s < \frac{n}{2}$, the wave map system is ill-posed in $H^s \times H^{s-1}$ (D'Ancona and Georgiev 04).

This is a typical result in subcritical spaces. Instantaneous blowup can happen, as well as non uniqueness of strong solutions.

- If $n \geq 3$, smooth initial data can lead to finite time blow up solutions : explicit self-similar equivariant wave maps (Shatah 88).
- If $n \geq 7$, finite time blow up can happen even on negatively curved manifolds (Cazenave, Shatah and Tavildhar-Zadeh 98).

Blow up : energy concentration

In critical spaces, blow up happen if energy concentrates at some point.
From now on, we will focus on the equivariant case.
Blow-up (at time T^*) can only happen at $r = 0$, and as energy decays on cones one must have :

$$\liminf_{t \uparrow T^*} E(u(t), 0, T^* - t) \geq \varepsilon_0.$$

In fact, energy has to concentrate faster.

Theorem (Shatah and Tahvildar-Zadeh 92)

Let u be an equivariant wave map blowing at time T^*. Then there exists $\lambda(t) = o(T^* - t)$ such that

$$\lim_{t \uparrow T^*} E(u(t), 0, \lambda(t)) = 0.$$

This rules out a blowup senario with self-similar rate.

Blowup profile

Theorem (Struwe 03)

Assume u blow-up at $T^* = 0$. Then there exist two sequences $t_n \uparrow 0$ and $\lambda_n = o(t_n)$ such that :

$$u(t_n + \lambda_n t, \lambda_n r) \to Q \quad H^1_{loc}((-1,1) \times \mathbb{R}_r^+),$$

where Q is a non-trivial harmonic map : $\Delta Q = f(Q)/r^2$.

Ideas of proof :

- The energy bound provides a weak limit u^*.
- A subtle reformulation of $\frac{1}{|t|}\int_t^0 \int_0^{|\tau|} u_t^2(\tau, r) r dr d\tau \to 0$ shows that u^* is constant in time, and hence a harmonic map.
- Using the equation, convergence is reinforced to local strong H^1.
- The scaling choice ensure that u^* is non trivial.

Corollary

Assume $g(\rho) > 0$ for all $\rho > 0$. Then any wave map u is global in time.

(The condition is equivalent to the fact that there exist no non trivial harmonic map on N).

Denote $E_0 = \min\{E(Q) | Q \text{ harmonic}\}$.

Corollary

Let u be a wave map which blow ups at time $T = 0$. Then u concentrates an energy of at least E_0:

$$\lim_{t \uparrow 0} E(u(t), 0, |t|) \geq E_0.$$

In particular, $E(u) > E(Q)$.

In fact there is the following more precise result for wave maps with energy less or equal than E_0.

Theorem (R.C., Kenig, Merle 07)

Let $k = 1$ or 2 and u be a wave map with energy $E(u) \leq E_0$. Then we have the dichotomy :

- *u scatters (in the sense that there exist $U_\pm \in \dot{H}^1$ such that $\| U(t) - e^{it\sqrt{\Delta}} U_\pm \|_{\dot{H}^1} \to 0$ as $t \to \pm\infty$).*
- *u is a harmonic map (and is constant in time).*

From now on, we assume g vanishes at some point $C^* > 0 : g(C^*) = 0$. To fix ideas, we set $g = \sin$.
Hence, there exists a harmonic map Q, which joins 0 to $C^* = \pi$.

We study long time dynamics around Q (and the family Q_λ).

Previous results regarding long time dynamics

- Numerical simulations : $g = \sin$, $k = 1$, Bizoń, Chmaj and Tadeusz 01.
- Formal blowup and blowup rate : 4D Yang-Mills (k=2), Bizoń, Ovchinnikov and Sigal 03.
- Q is strongly instable in the energy space (for all g, k)

$$H = \left\{ u \,\middle|\, \|u\|_H^2 = \int \left(u_r^2 + \frac{u^2}{r^2} \right) rdr < \infty \right\}.$$

Theorem (R.C. 05)

Given $\lambda_0 > 0$, there exist a sequence of finite energy wave maps u_n and of time t_n such that

- $(u_n, u_{n\,t}) \to (Q, 0)$ *in* $H \times L^2$,
- $(u_n(t_n), u_{n\,t}(t_n)) \to (Q_{\lambda_0}, 0)$ *in* $H \times L^2$.

Ideas of proof : regularization of self-similar wave map and finite speed of propagation.

Theorem (Rodnianski and Sterbenz 06)

Let $k \geq 4$, $g = \sin$. There exist $\varepsilon > 0$ and initial data (u_0, u_1), with energy $E(u_0, u_1) \leq E_0 + \varepsilon^2$ such that the following holds. The arising wave map u blows up in finite time T^*, and there exists $\lambda(t) > 0$ such that for $t \in [0, T^*)$,

$$\|(u, u_t) - (Q_{\lambda(t)}, 0)\|_{H \times L^2} \lesssim \varepsilon \quad \textit{and} \quad \lambda(t) = C_0(1 + O(\sqrt{\varepsilon}))\frac{\sqrt{|\ln(T^* - t)|}}{T^* - t}.$$

where C_0 is an explicit constant.

Theorem (Krieger, Schlag and Tataru 06)

Let $k = 1$, $g = \sin$. Fix $\nu > \frac{1}{2}$, $\lambda(t) = t^{-1-\nu}$, and a large integer N. Then there exist a wave map u defined on $[-t_0, 0)$ such that :

$$u(t, r) = Q_{\lambda(t)}(r) + u^\epsilon(t, r) + \epsilon(t, r),$$

with $E(u^\epsilon(t), 0, |t|) \lesssim (t\lambda|t|)^{-2} \ln^2 |t|$ and

$$\|(\epsilon, t\epsilon_t)\|_{H^{1+\nu-}_{\text{loc}} \times H^{\nu-}_{\text{loc}}} \lesssim t^N, \quad E(\epsilon(t), 0, |t|) \lesssim t^N \quad \textit{as} \quad t \uparrow 0.$$

Proof

$$\partial_{tt} u - \partial_{rr} u - \frac{1}{r}\partial_r u = -k^2 \frac{\sin 2u}{2r^2}. \qquad \text{(Wave map)}$$

Denote $R = \frac{d}{d\lambda} Q_\lambda \big|_{\lambda=1} = r\partial_r Q = k \sin Q$ and the rescaled family $R_\lambda = r\partial_r Q_\lambda = k \sin Q_\lambda$.

$R, R_\lambda \in L^2 = L^2(rdr)$ if and only if $k \geq 2$.

Linearized operator around Q_λ :

$$H_\lambda = -\partial_{rr} - \frac{1}{r}\partial_r + \frac{k^2}{r^2} \cos 2 Q_\lambda.$$

Critical setting : $H_\lambda R_\lambda = 0$.

Initial data :

$$\begin{cases} u|_{t=0} = Q + v_0 \quad \text{where} \quad \langle v_0, R \rangle = 0, \\ u_t|_{t=0} = \dfrac{\varepsilon}{\|R\|_{L^2}} R + v_1, \end{cases}$$

$$\text{with} \qquad \|v_0\|_{H^{2,1}} + \|v_1\|_{H^{2,1}} \lesssim \varepsilon^{3/2}.$$

Modulation theory

(cf. Weinstein 83).
Decomposition of u

$$u(t,r) = Q_{\lambda(t)}(r) + v(t,r),$$

with

$$\int \left(u_t^2 + v_r^2 + \frac{v^2}{r^2} \right) rdr \lesssim \varepsilon^2, \quad \langle v, R_\lambda \rangle = 0.$$

This defines v uniquely.

As a by product, one gets $\left| \frac{\dot\lambda}{\lambda^2} \right| \lesssim \varepsilon$ for all time.
Also notice $\lambda(0) = 1$ and $\dot\lambda(0) = \frac{\varepsilon}{\|R\|_{L^2}} + O(\varepsilon^{3/2})$.

Equation on v and λ :

$$\begin{cases} v_{tt} + H_\lambda v = -\ddot{Q}_\lambda + \mathcal{N}(v), \\ \ddot\lambda - 2\dfrac{\dot\lambda^2}{\lambda} = \lambda^2 \dfrac{d}{dt}\left(\dfrac{\dot\lambda}{\lambda^2} \right) = \dfrac{\lambda^3}{\|R\|_{L^2}^2} \left(2\langle v_t, \dot{R}_\lambda \rangle + \langle v, \ddot{R}_\lambda \rangle + \langle \mathcal{N}(v), R_\lambda \rangle \right). \end{cases}$$

where $\mathcal{N}(v) = \dfrac{k^2 \sin 2Q_\lambda}{2r^2}(1 - \cos 2v) + \dfrac{k^2 \cos(2Q_\lambda)}{r^2}(v - \dfrac{1}{2}\sin(2v))$.

H_λ positive, with one embedded eigenvalue (0) associated to R_λ.

Decomposition of $H_\lambda = A_\lambda A_\lambda^*$ where

$$A_\lambda = -\partial_r + \frac{k}{r}\cos Q_\lambda, \quad A_\lambda^* = -\partial_r + \frac{1}{r} + \frac{k}{r}\cos Q_\lambda.$$

Super symetric companion of H_λ

$$\tilde{H}_\lambda = A_\lambda^* A_\lambda = -\partial_{rr} - \frac{1}{r}\partial_r + \underbrace{\frac{k^2+1}{r^2} + \frac{2k}{r^2}\cos Q_\lambda}_{V_\lambda(r)}.$$

V_λ is positive and *space-time* repulsive, with some uniformity in λ.

$$V_\lambda \geq \frac{(k-1)^2}{r^2}, \qquad -\partial_r V_\lambda \geq \frac{2(k-1)^2}{r^3}, \qquad -\partial_t V_\lambda = \frac{\dot\lambda}{\lambda} \cdot \frac{2k^2}{r^2}\sin^2 Q_\lambda.$$

Also :

$$[\partial_t, A_\lambda] = \dot\lambda F_{1\lambda}, \qquad [\partial_{tt}, A_\lambda] = \ddot\lambda F_{2\lambda} + \frac{\dot\lambda^2}{\lambda} F_{3\lambda},$$

for some explicit functions F_1, F_2, F_3.

Further decomposition v and final equation

We write

$$v = w_0 + w, \quad \text{such that} \quad \langle w_0, R_\lambda \rangle = 0 \quad \text{and} \quad H_\lambda w_0 = \ddot{Q}_\lambda.$$

This can be solved explicitly !

$$w_0 = \frac{\dot{\lambda}^2}{4\lambda^4} \left(-\|rR\|_{L^2}^2 R_\lambda + (r^2 R)_\lambda \right).$$

Equation on $W = A_\lambda w$:

$$\begin{aligned} &\partial_{tt} W + \tilde{H}_\lambda W \\ &= -A_\lambda(w_{0\,tt}) + [\partial_{tt}, A_\lambda] w + A_\lambda \mathcal{N}(v) \\ &= \underbrace{\partial_t(A_\lambda \partial_t w_0) + [\partial_t, A_\lambda](\partial_t w_0)}_{\text{main source}} + \underbrace{2\partial_t([\partial_t, A_\lambda] w) - \partial_{tt}(A_\lambda) w + A_\lambda \mathcal{N}(v)}_{\text{non-linear error terms}}. \end{aligned}$$

Morawetz estimate on $\partial_{tt} + \tilde{H}_\lambda$.

$\partial_{tt}\psi + \tilde{H}_\lambda \psi = \partial_t G + H.$ (linear wave equation with $\tilde{H}_\lambda$ operator)

Let $0 < \delta \ll 1$, denote $L = \partial_t + \partial_r$, $\Lambda\psi = \frac{1}{\lambda}\frac{(\lambda r)^\delta}{1+r^\delta}\left((L\psi)^2 + \frac{\psi^2}{r^2}\right)$ and

$$\mathbf{E}_{t,\delta}(\psi) = \sup_{0\le s\le t}\int (\Lambda\psi) r dr + \int_0^t \int \frac{1}{r}(\Lambda\psi) r dr ds,$$

$$\mathbf{F}_{t,\delta}(G,H) = \int_0^t \int \frac{(\lambda r)^\delta}{\lambda}\left((\partial_r G)^2 + \varepsilon(\lambda G)^2 + H^2\right) r^2 dr ds + \sup_{0\le s\le t}\int \frac{1}{\lambda}\frac{(\lambda r)^\delta}{1+r^\delta} G^2 r dr$$

Then, under the assumptions $\dot\lambda \ge 0$ and $\dot\lambda/\lambda^2 \lesssim \varepsilon$, we have

$$\mathbf{E}_{t,\delta}(\psi) \lesssim \frac{1}{\delta}\left(\mathbf{E}_{0,\delta}(\psi) + \mathbf{F}_{t,\delta}(G,H)\right), \qquad \text{(Morawetz)}$$

where the implicit constant does not depend on λ, δ and ε.

Proof : Use of an adequate multiplier on the equation, integrations by parts and repulsive properties of $\tilde{H}_\lambda$.

From the Morawetz estimate and under the bootstrapping assumptions :

$$\dot\lambda \geq 0, \qquad t \in [0, \varepsilon^{-5}] \tag{1}$$

$$\left| \ddot\lambda - 2\frac{\dot\lambda^2}{\lambda} \right| \leq 2C\varepsilon \frac{\dot\lambda^2}{\lambda} + 2C \left(\varepsilon^4 + \sup_{0 \leq s \leq t} \frac{\dot\lambda^4}{\lambda^7} \right) \lambda^2, \tag{2}$$

$$\int_0^t \frac{\dot\lambda^4}{\lambda^7} \leq 2C\varepsilon. \tag{3}$$

We get :

$$\mathbf{F}_{t,\delta}(\text{Main terms}) \lesssim \varepsilon^4 + \varepsilon \sup_{0 \leq s \leq t} \frac{\dot\lambda^4}{\lambda^7}.$$

$$\mathbf{F}_{t,\delta}(\text{Error terms}) \lesssim (1 + t^\delta)\varepsilon^2 \mathbf{E}_{t,\delta}(W) + \varepsilon \sup_{0 \leq s \leq t} \frac{\dot\lambda^4}{\lambda^7},$$

and we obtain the main estimate on W :

$$\mathbf{E}_{t,\delta}(W) \lesssim (1 + t^\delta)\varepsilon^2 \mathbf{E}_{t,\delta}(W) + \varepsilon^4 + \varepsilon \sup_{0 \leq s \leq t} \frac{\dot\lambda^4}{\lambda^7}. \tag{4}$$

Going back to the equation on λ

$$\|R\|_{L^2}^2 \frac{d}{dt}\left(\frac{\dot\lambda}{\lambda^2}\right) = 2\partial_t(\langle v, \dot R_\lambda\rangle \lambda) - 2\langle v, \dot R_\lambda\rangle \dot\lambda - \langle v, \ddot R_\lambda\rangle\lambda + \langle \mathcal{N}(v), R_\lambda\rangle\lambda.$$

Using the main estimate on W (4) together with elliptic properties of A_λ allows to bootstrap assumption (2).
Upon integration in time we get

$$\left[\|R\|_{L^2}^2 \frac{\dot\lambda}{\lambda^2} - 2\langle v, r\partial_r R_\lambda\rangle \dot\lambda\lambda\right]_0^t = -\int_0^t \left(C_* \frac{\dot\lambda^4}{\lambda^7} + \mathcal{E}(s)\right) ds, \qquad (5)$$

where $C_* = \frac{5}{8}\|rR\|_{L^2}^2$ is explicit (comes from terms purely in w_0) and $\mathcal{E}$ is an error term (terms in w) satisfying

$$\mathcal{E}(t) \lesssim \varepsilon^4 + \varepsilon \sup_{0\le s\le t} \frac{\dot\lambda^4}{\lambda^7}.$$

Notice that

$$\left|\langle v, r\partial_r R_\lambda\rangle \dot\lambda\lambda\right| \le \dot\lambda\lambda \left\|\frac{v}{r}\right\|_{L^2} \lambda^{-3}\|r^2\partial_r R\|_{L^2} \le \varepsilon \frac{\dot\lambda}{\lambda^2}.$$

It remains to study the ODE (5) to complete the bootstrap argument and to show blowup. We rewrite it in the more manageable form

$$(\|R\|_{L^2}^2 + O(\varepsilon))\dot{\lambda} = \underbrace{(\|R\|_{L^2}^2 \dot{\lambda}(0) + O(\varepsilon^2))}_{\sim \varepsilon \|R\|_{L^2}} \lambda^2 - \lambda^2 \int_0^t \left(C_* \frac{\dot{\lambda}^4}{\lambda^7} + \mathcal{E}(s) \right) ds. \quad (5)$$

Consider the quantity $\dfrac{\dot{\lambda}^4}{\lambda^7}$, fix a large constant C and define $T^\dagger$ such that

$$\forall t \in [0, T^\dagger], \quad \frac{\dot{\lambda}^4}{\lambda^7} \leq C\varepsilon^4, \quad \text{and} \quad \frac{\dot{\lambda}^4}{\lambda^7}(T^\dagger) = C\varepsilon^4 > 0.$$

- Initially, $\dot{\lambda}^4 \lambda^{-7} \sim \varepsilon^4$ so that $T^\dagger > 0$.
- $T^\dagger$ is finite and satisfies $T^\dagger \lesssim \varepsilon^{-1}$, otherwise $\frac{\dot{\lambda}}{\lambda^2} \geq \frac{\varepsilon}{2} \|R\|_{L^2}$ on that interval, which leads to $\lambda \to \infty$ and $\dot{\lambda}^4 \lambda^{-7} \gtrsim \lambda \to \infty$.

- Compute

$$\frac{d}{dt}\left(\frac{\dot\lambda^4}{\lambda^7}\right) = \frac{\dot\lambda^5}{\lambda^8} + O\left(\varepsilon\frac{\dot\lambda^2}{\lambda^3} + \varepsilon^4 + \sup_{0\le s\le t}\frac{\dot\lambda^4}{\lambda^7}\right)\frac{\dot\lambda^3}{\lambda^5}.$$

  Then at time $T^\dagger$, $\frac{d}{dt}\left(\frac{\dot\lambda^4}{\lambda^7}\right) \ge (1 - C\varepsilon)\frac{\dot\lambda^5}{\lambda^8} > 0$, and this estimate bootstraps to show that $\frac{\dot\lambda^4}{\lambda^7} \ge C\varepsilon^4$ and is an increasing function.
- From this one shows $\lambda(t) \to \infty$ as $t \uparrow T^*$ for some $T^* \lesssim \varepsilon^{-4} \ll \varepsilon^{-5}$ (finite time blowup).
- From this analysis, assumption (1) is also bootstrapped : on $[0, T^\dagger]$ as well as on $[T^\dagger, T^*)$.
- Plugging this information again in (5) shows bootstrap assumption (3).
- Going back carefully through the previous steps allows to compute the blow up rate

$$\lambda(t) = C_0(1 + O(\varepsilon))\frac{\sqrt{|\ln(T^* - t)|}}{T^* - t}.$$

  (where C_0 is an explicit constant).