Non linear instability of the incompressible Euler with gravity equations in the ICF context

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1 Physical context

Ablation front in the inertial confinement fusion of Deuterium and Tritium
Study of the propagation of a defect in the target (which is not a perfect sphere) :
we need to obtain the critical density for the beginning of the thermonuclear reaction, and growth of defects can prevent this
Model presented here :
the ablative Rayleigh-Taylor system in a strip with constant gravity and continuous transition between the high density ρ_a and the vacuum $(\rho \to 0)$;

A (noncomplete)list of references :

J.W. Strutt (Lord Rayleigh) 1883

G. Taylor 1950

S. Chandrasekhar1961

H.J. Kull and S.I. Anisimov 1986

Takabe et al 1996

V. Goncharov 1998

E. Grenier 2000, L 2000

C. Cherfils, L, P.A. Raviart : 2001

J. Garnier, C. Cherfils, P.A. Holstein 2003

S. Cordier, E. Grenier and Y. Guo 2003

Y. Guo and H.J. Hwang 2003

B. Helffer and L 2003, 2007

P. Clavin and L. Masse 2004

B. Desjardins and E. Grenier 2006

R. Poncet 2007

- Mixing of fluids : modeled by a variable density in the medium.
- Presence of gravity.
- Euler equations.
- Energy equation : thermal conduction model.
- Quasi-isobaric model with Fourier law :

$$\mathrm{div}(C_p \rho T \vec{u} + \kappa(T)\nabla T) = 0, \rho T = cst, \kappa(T) = \kappa_0 T^{\nu}, p = \rho T$$

Model system 0 (introducing $Z(\rho) = \frac{\rho_a^{\nu+1}}{(\nu+1)\rho^{\nu+1}}$) :

$$\left\{ \begin{array}{l} \partial_t \rho + \mathrm{div}(\rho \vec{u}) = 0 \\ \partial_t(\rho \vec{u}) + \mathrm{div}(\rho \vec{u} \otimes \vec{u} + pId) = \rho \vec{g} \\ \mathrm{div}(\vec{u} + L_0 V_a \nabla Z(\rho)) = 0. \end{array} \right. \tag{1}$$

Stationary solution $\rho_0(x) = \rho_a \xi(\frac{x}{L_0})$, $u_0(x) = -\frac{V_a}{\xi}$, p_0 solution of $(p - \frac{\rho_a V_a^2}{\xi})' = \rho_a g \xi$, where ξ satisfies

$$\dot{\xi} = \xi^{\nu+1}(1-\xi).$$

Concentrate on ρ_0 only. and forget this system and solution.

2 Euler equations with gravity around a general profile

Profile $(\rho_0(x), \vec{0}, p_0(x))$ with $\nabla p_0 = \rho_0 \vec{g}$. Introduce $k_0(x) = \frac{\rho_0'(x)}{\rho_0(x)}$.
Euler in 2d : $x \in (-\infty, +\infty)$, $z \in [0, a]$.

System of Euler equations in the unknowns
$\vec{u}, T = \frac{\rho_0(x)}{\rho(x,y,t)}, q = \frac{p - p_0(x)}{\rho_0(x)}$:

$$\begin{cases} \partial_t T + (\vec{u}.\nabla)T = k_0 u T \\ \partial_t \vec{u} + (\vec{u}.\nabla)\vec{u} + (T-1)\vec{g} + \nabla q + k_0(x) q \vec{e}_1 = 0 \\ \mathrm{div}\vec{u} = 0 \end{cases} \tag{2}$$

Particular solution $(\vec{0}, 1, 0)$.

Linearized system (operator Emod) :

$$\begin{cases} \partial_t T_1 - k_0 u_1 = 0 \\ \partial_t \vec{u}_1 + T_1 \vec{g} + \nabla q_1 + k_0(x) q_1 \vec{e}_1 = 0 \\ \mathrm{div} \vec{u}_1 = 0 \end{cases} \tag{3}$$

Linearized system with left hand side

$$\begin{cases} \partial_t T_1 - k_0 u_1 = S_0 \\ \partial_t \vec{u}_1 + T_1 \vec{g} + \nabla q_1 + k_0(x) q_1 \vec{e}_1 = \vec{S} \\ \mathrm{div} \vec{u}_1 = 0 \end{cases} \tag{4}$$

Rayleigh equation (1) : apply $\partial_t \partial_z^2$ on the equation on u_1 and use $\partial_t T_1$ and $\partial_z^z q_1 = \partial_t \partial_x u_1$:

$$\begin{aligned} &\partial_t^2 \partial_z^2 u_1 + g k_0(x) \partial_z^2 u_1 + \rho_0^{-1} \partial_x (\rho_0 \partial_x \partial_t^2 u_1) \\ &= \partial_t \partial_z^2 S_1 - g \partial^2 z S_0 - \rho_0^{-1} \partial_x (\rho_0 \partial_t \partial_z S_1). \end{aligned} \tag{5}$$

Rayleigh equation (2) : seek $u_1(t,x,z) = \hat{u}(x) e^{\gamma t} \cos kz$ in the homogeneous equation :

$$-\hat{u}'' - k_0(x)\hat{u}' + (k^2 + \frac{gk^2}{\gamma^2} k_0(x))\hat{u} = 0 \tag{6}$$

Solution

$$(\hat{u}_1 \cos kz, -\hat{u}_1' \frac{\sin kz}{k}, \frac{k_0(x)}{\gamma} \cos kz, -\frac{\gamma}{k^2} \hat{u}_1' \cos kz) e^{\gamma t}.$$

Rayleigh equation (2') :

$$-\frac{d}{dx}(\rho_0 \hat{u}) + (k^2 \rho_0 + \frac{gk^2}{\gamma^2} \rho_0'(x))\hat{u} = 0 \tag{7}$$

Rayleigh equation (2") :

$$-k^{-2} \frac{d^2}{dx^2} \hat{w} + (1 + \frac{g}{\gamma^2} k_0(x) + \frac{1}{2} k^{-2} (k_0' + \frac{1}{2} k_0^2))\hat{w} = 0 \tag{8}$$

Assume $k_0 \geq 0$. From (2'), integrating, one deduces

– $g > 0 \Rightarrow$ no solution such that γ^2 real positive

– if $\gamma^2 > (\max k_0)|g| = \Lambda^2$, no solution

– if $\gamma^2 > |g|k$, no solution

the last equality thanks to

$$(1 - \frac{|g|k}{\gamma^2}) \int \rho_0((\hat{u}')^2 + k^2\hat{u}^2)dx = \frac{|g|k}{\gamma^2} \int \rho_0(\hat{u}' - k\hat{u})^2 dx.$$

3 Semi-classical result

Condition : k_0 has a unique nondegenerate maximum.
This condition is satisfied by $k_0(x) = \frac{\xi'(x)}{\xi(x)} = \xi^\nu(1-\xi)$. Maximum at $\xi = \frac{\nu}{\nu+1}$. This leads to

Proposition
There exists a $h_0 > 0$ such that, for all $k \geq \frac{1}{h_0}$, there exists a sequence $\gamma_n(k)$ such that

$$1 - \frac{|g|}{\gamma_n^2(k)} k_0^{max} + (\frac{|g|}{2\gamma_n^2(k)} |k_0''|)^{\frac{1}{2}} (n + \frac{1}{2}) = 0(k^{-\frac{3}{2}}).$$

This proves that there exists at least one value of γ and one value of k such that $\gamma \in (\frac{\Lambda}{2}, \Lambda)$.

4 Nonlinear result

Under the assumptions k_0 bounded (as well as all the derivatives) and admits at least a nondegenerate maximum, $k_0\rho_0^{-\frac{1}{2}}$ bounded, there exists an initial value for (2) close (in a δ sense) to the stationary solution which departs in a time of order $\ln \frac{1}{\delta}$ of a given finite quantity from this solution.
No need of $\rho_0 \geq \rho_{min} > 0$. Need $\rho_0 > 0$ everywhere.

5 The Duhamel principle

Energy equality :

Note that system (3) leads to the equation on the incompressible quantity $\vec{u}$:

$$\partial^2_{t^2}\vec{u} + k_0 u_1 \vec{g} + \rho_0^{-1}\partial_x(\rho_0 \partial_t q) = \partial_t \vec{S} - S_0 \vec{g}.$$

from which one deduces

$$\frac{1}{2}\int(\rho_0(\partial_t\vec{u})^2 + k_0\rho_0 g u_1^2)dxdz = \frac{1}{2}\int(\rho_0(\partial_t\vec{u}(0))^2 + k_0\rho_0 g u_1(0)^2)dxdz$$
$$= \int_0^t(\int \rho_0(\partial_t\vec{S} - S_0\vec{g})\partial_t\vec{u}dxdz)ds$$

General result :

The solution of

$\frac{1}{2}\frac{d}{dt}(\int(\rho_0(\partial_t\vec{u}_N)^2 - g\frac{\rho_0'}{\rho_0}\rho_0(u_N)^2)dxdz) = g(t, x, \partial_t\vec{u}_N)$

with initial condition $\partial_t\vec{u}_N(0), \vec{u}_N(0)$, with the assumption

$|g(t, x, \partial_t\vec{u}_N)| \leq K(t)||\rho_0^{\frac{1}{2}}\partial_t\vec{u}_n||_{L^2}$

where K is a positive increasing function for $t \geq 0$ satisfies the inequalities

$$||\rho_0^{\frac{1}{2}}\vec{u}_N||^{\frac{1}{2}} \leq [C_1 + \int_0^t \sqrt{K(s)e^{-\Lambda s}}ds]e^{\frac{\Lambda}{2}t}$$
$$||\rho_0^{\frac{1}{2}}\partial_t\vec{u}_N|| \leq [C_1 + \int_0^t \sqrt{K(s)e^{-\Lambda s}}ds]^2e^{\Lambda t}$$

where C_1 depends on the initial data.

Result for the linear instability with mixing of modes :
Let $T_1(t), \vec{u}_1(t)$ be the solution of the linearized Euler system. There exists a constant C_s depending only on the characteristics of the system, that is of k_0 and $|g|$, such that

$$||\rho_0^{\frac{1}{2}} T_1(t)||_{H^s} + ||\rho_0^{\frac{1}{2}} \vec{u}_1(t)||_{H^s} \leq C_s (1+t)^s e^{\Lambda t} (||\rho_0^{\frac{1}{2}} T_1(0)||_{H^s} + ||\rho_0^{\frac{1}{2}} \vec{u}_1(0)||_{H^s}).$$

6 Weakly non linear approximate solution of order N

We use the Grenier construction, called the weakly non linear expansion (or solution in the physics solutions) :
$(T^N, u^N, v^N, q^N) = (1, 0, 0, 0) + \sum_{j=1}^{N} \delta^j U_j$
with $U_j(x, z, 0) = 0$ for $j \geq 2$ and U_1 solution of the linearized system (with k, a growth rate $\gamma(k) \in (\frac{\Lambda}{2}, \Lambda)$ and $\hat{u}$). Obviously
$(Emod)(U_j) = S(U_1, ..., U_{j-1})$.
Goal : identify a time T, two constants C_0 and M_0 such that
$||U_j||_{L^2} \leq M_0 (C_0)^j e^{j\gamma(k)t}, t < T$
$\Rightarrow$ the sequence $\sum_{j=1}^{N} \delta^j U_j$ converges normally in L^2 for
$t < \frac{1}{\gamma(k)} \ln \frac{1}{C_0 \delta}$.

Counting terms in the left hand side and estimates :

This result relies on a recurrence, in which one must control the behavior in N of the left hand term of the equation on U_N.

Time derivative $\partial_t \vec{S}$: N

Number of terms (using the **quadratic** nonlinearity) : N

Behavior of $K_N(s)$: $N^2 e^{N\gamma(k)s}$

$\Rightarrow \int_0^t \sqrt{K_N(s)e^{-\Lambda s}}ds \leq \frac{2N}{N\gamma(k)-\Lambda} e^{\frac{N\gamma(k)-\Lambda}{2}t}$.

and for $\gamma(k) > \frac{\Lambda}{2}$ and $N \geq 2$ bounded independantly on N.

Rk : not possible to do so when **cubic** nonlinearity.

7 Nonlinear solution

Consider a solution $(T, \vec{u}, q)$ of the Euler system (2) such that $(T, \vec{u}, q)(0) = (1, 0, 0) + \delta U_1(0)$.
Denote by $(\tilde{T}, \vec{v}, \tilde{q})$ the difference $(T, \vec{u}, q) - U^N$.
Moser estimates and usual energy equalities $\Rightarrow$
control on $\rho_0^{\frac{1}{2}}\vec{v}$, $\rho_0^{-\frac{1}{2}}T$, $\rho_0^{-\frac{1}{2}}q$
Use the nonlinear equation and write as a source term
$k_0 u = (k_0 \rho_0^{-\frac{1}{2}})\rho_0^{\frac{1}{2}} u$: one gets estimates on $\tilde{T}$, $\vec{v}$ and $\rho_0^{-1}\tilde{q}$.

End of the proof :

$\vec{u} = \vec{v} + \delta \vec{u}_1 + \delta^2 \sum_{j=2}^{N} \delta^{j-2} U^j$

First term controlled by ϵ

Last term controlled by $\delta^2 C$

Leading term is the second one : can be of order $\frac{1}{2}$.