A Transmission Strategy for Internal Waves of Small Width

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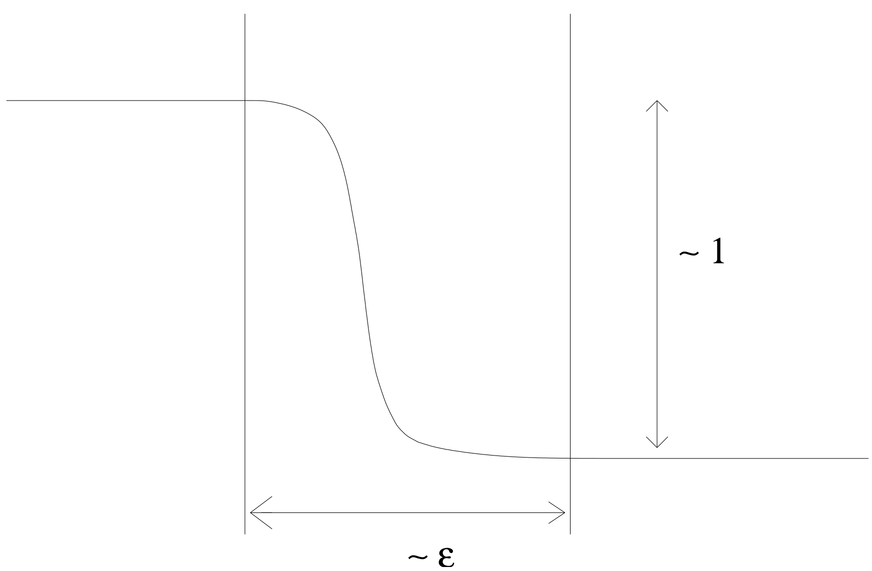

Goals

I. Illustrate through two distinct recent results how rigorous multiscale (in particular short wavelength) aysmptotic analysis works.

This area has progressed a good deal in the last decades. One of the main lessons is that multiscale expansions have proven (so far) better adapted to rigorous analysis than matched asymptotic expansions.

II. A main reason is that to reduce the residuals (errors in the differential equation) in multiscale problems to small enough sizes for estimates requires correctors.

These don't fit neatly in the matched expansions framework.

III. To share some elementary and very revealing computations which may be of interest to the broad spectrum of participants.

I have much enjoyed the experimental side and hope to be able to put at least a little bit in front of that community which may be of use to them.

IV. Each of the two problems involves the choice of an *ansatz* which is not standard. By describing how one arrives at such choices, I hope that you may be guided in your own struggles.

A Cautionary Example from Geometric Optics.

Science discussion of PDE almost invariable involve simplifying the equations.

The criterion is: *Ignore terms which are small compared to others.*

In geometric optics and other multiscale problems, *this can lead to innacurate results.*

Consider the wave packets as $\varepsilon \to 0$,

$$\partial_t u^\epsilon + \partial_x u^\epsilon + u^\varepsilon \;=\; 0\,,$$

$$u^\varepsilon\big|_{t=0} = a(x)\cos(x/\varepsilon)\,, \qquad a \in \mathcal{S}(\mathbb{R}^d)).$$

The exact solution is

$$u^\varepsilon(t,x) \;=\; e^{-t}\, a(x-t)\, \cos((x-t)/\varepsilon)\,.$$

Both $\partial_t u^\varepsilon$ and $\partial_x u^\varepsilon$ are $O(1/\varepsilon)$ while $u^\varepsilon = O(1)$ is negligibly small in comparison.

Dropping this small term yields,

$$\partial_t v^\epsilon + \partial_x v^\epsilon \;=\; 0\,, \qquad v^\varepsilon\big|_{t=0} = a(x)\cos(x/\varepsilon)\,.$$

The exact solution of the simplified equation is

$$v^\varepsilon(t,x) \;=\; a(x-t)\cos\big((x-t)/\varepsilon\big)\,,$$

which misses the exponential decay.

It is a **bad** approximation. (see review of *Wave Motion*)

What Can You Do?

I. Plug your approximate solution into the original equations to see how big the errors (a.k.a. residuals) are.

You are very likely to be very dissapointed and maybe upset. This explains why it is rarely done in science and applied math.

Try to construct (small) correctors to reduce the size of the residuals. Not usually easy.

In the rigorous results I will describe the hardest part is typically the construction of correctors.

Often a better *ansatz* helps.

II. Linearize the equations at the approximate solutions to detect sensitivity to perturbations.

III. In the rare event that you achieve these two goals you are not far from proving an error estimate.

Causality

In the best case, a physical theory in dynamics produces a set of PDE so that for general initial data there is one and only one solution satisfying the boundary conditions and that solution depends continuously on the initial data (Hadamard).

Solving physical problems then reduces to finding qualitative and quantitative properties of families of solutions.

The problems from the experiments presented in this meeting are beyond that paradigm. I don't think that there is a single one for which such an exisitence and uniqueness theorem is available.

I describe simpler examples where such theorems are available and where solutions have accurately described singular structure.

The interest is

i. The language in which the results are described,

ii. The structure of the *ansatz* used permitting the constructions with small residual.

Hyperbolic Systems

$$L(t,x,\partial) \;=\; \partial_t + \sum A_j(t,x)\,\partial_j + B(t,x)\,, \qquad \partial_j \;:=\; \frac{\partial}{\partial x_j}\,,$$

A_j, B are $N \times N$ *hermitian symmetric* complex matrix valued functions

$\partial^\alpha_{t,x}\{A_j, B\} \in L^\infty(\mathbb{R}_t \times \mathbb{R}^d_x)$

Theorem. *For any $s \in \mathbb{N}$, and any $u_0 \in H^s(\mathbb{R}^d)$ ($\partial_x^\alpha u_0 \in L^2(\mathbb{R}^d)$ for $|\alpha| \le s$) there is a unique $u \in C(\mathbb{R}\,;\, H^s(\mathbb{R}^d))$ solving*

$$L\,u \;=\; 0, \qquad u|_{t=0} \;=\; u_0\,.$$

Examples. Linearized inviscid Euler, Maxwell, Linear elasticity, linearized MHD, ...

False if replace H^s by C^s or $W^{s,p}$ with $p \neq 2$. The H^s are the natural stability space for multidimensional inviscid wave propagation.

Theorem. *For $Lu + G(u) = 0$ with $G(0) = G'(0) = 0$ then for $s > d/2$ there is an analogous local in time existence result.*

For quasilinear, $s > d/2 + 1$.

Carriers of Singularities

Principal symbol: $L_1(t,x,\tau,\xi) := \tau I + \sum A_j(t,x)\,\xi_j$.

Characteristic variety: $Char(L) := \big\{ \det L_1(t,x,\tau,\xi) = 0 \big\}$

$L_1(\underline{t},\underline{x},\partial)\, e^{i(\tau t + x\xi)} \;=\; 0.$

A d-dimensional surface $\Sigma \subset \mathbb{R}^{1+d}$ can be the carrier of singularity if an only if it is characteristic, meaning that $\det L_1(t,x,\nu) = 0$ for all conormal ν to Σ.

$\Longleftrightarrow$ the conormal variety $\mathcal{N}^*(\Sigma) \subset Char(L)$.

Examples. i. For $u_{tt} - c^2\, u_{xx}$ they are lines with speed $\pm c$.

ii. For the linearized inviscid Euler equations at a constant state hyperplanes are characteristic if and only if their normal velocity is equal to the speed of sound or the velocity of the background flow.

Can always change local coords respecting t so that $\Sigma = \{x_d\} = 0$.

Assumption. 1. $\Sigma := \{x_d = 0\}$ *is a characteristic hypersurface for* L.

2. *On a conic neighborhood of* $\mathcal{N}^*\Sigma$*, Char(L) is a smooth embedded hypersurface* $\tau = \tau(t,x,\xi)$ *in* $\mathbb{R}^{2(1+d)}_{(t,x,\tau,\xi)}$.

Main Problem. *Describe the behavior of solutions u^ε to*

$$L\, u^\varepsilon + G(u^\varepsilon) \;=\; f^\varepsilon\,, \qquad u^\varepsilon = f^\varepsilon = 0 \quad \text{when} \quad t<0\,.$$

where

$$f^\varepsilon \;=\; F(t,x,x_d/\varepsilon)\,,$$

with $F(t,x,z)$ smooth, compactly supported in x, with limits

$$\lim_{\pm z\to\infty} F(t,x,z) \;=\; \overline{F}^{\pm}(t,x)$$

rapidly achieved.

Define a discontinuous piecewise smooth source,

$$\overline{f}(t,x) \;:=\; \overline{F}^{\pm}(t,x)\,, \quad \text{when} \quad \pm\, x_d > 0\,,$$

As $\varepsilon \to 0$, $f^\varepsilon \to \overline{f}$.

The limit $\varepsilon \to 0$ yields,

$$L\bar{U} + G(\overline{U}) \;=\; \bar{f}\,, \qquad \overline{U} = \bar{f} = 0 \quad \text{for} \quad t<0\,.$$

$\exists\,!$ local in time piecewise smooth solution, $\overline{U} \in L^\infty([0,T_1]\times \mathbb{R}^d)$. Denote by $\overline{U}^{\pm}$ the restriction to $\pm x_d > 0$.

$\overline{U}$ jumps and u^ε does not $\Rightarrow$ the convergence is not uniform.

The problem is to find correctors to $\overline{U}$ to decribe u^ε with error uniformly small.

From the detailed structure of the transition layer in f^ε, predict the details of transition layer for u^ε. By equations without small structures.

Approach. **Step 1.** *Find an ansatz yielding an approximate solution $u^\varepsilon_{\rm approx}$ with small residual.* In our case the residual will have conormal (to Σ) derivatives and ε-derivatives $(\varepsilon\partial_{t,x})^\alpha$ of size $O(\varepsilon^N)$ for all N.

Step 2. *Prove a stability theorem to conclude that the difference between the exact and approximate solutions is $O(\varepsilon^\infty)$.*

We use known stability results, so the key is constructing approximate solutions.

!ATTN! *The obvious ansatz motivated by the cases of wave trains and short pulses yields overdetermined equations for correctors to the leading approximation.*

Even in the linear case.

We use a transmission strategy which has been effective in related problems with layers coming from a vanishing viscosity limit [GMWZ], [Sueur].

The obvious ansatz fails.

Linear **wave packets**: $e^{ix_d/\varepsilon}\Big(a_0(t,x) + \varepsilon a_1(t,x) + \cdots\Big)$

Nonlinear **wave packets**:

$\varepsilon^p\Big(U_0(t,x,x_d/\varepsilon) + \varepsilon U_1(t,x,x_d/\varepsilon) + \cdots\Big), \quad U_j(t,x,\theta)$ pdc.

Short **pulses** $f(t,x,x_d/\varepsilon)$, $f(t,x,\pm\infty) = 0$, $(f(t,x,z))$,
Obvious *ansatz* for nonlinear short pulses

$\varepsilon^p\Big(U_0(t,x,x_d/\varepsilon)+\varepsilon U_1(t,x,x_d/\varepsilon)+\cdots\Big), \quad U_j(t,x,\pm\infty) = 0.$

Internal wave $f(t,x,x_d/\varepsilon)$, $f(t,x,\pm\infty)$ exist.
Obvious *ansatz* for nonlinear internal waves

$\varepsilon^p\Big(U_0(t,x,x_d/\varepsilon)+\varepsilon U_1(t,x,x_d/\varepsilon)+\cdots\Big), \quad U_j(t,x,\pm\infty)$ exist

Choose p not too small. Plug in. Get equations for the U_j which **look like** you can solve them one after the other.

The equations for the corrector U_1 are overdetermined for pulses and internal waves. Have parts determined by

$$\partial g/\partial z \;=\; f\,, \quad g(-\infty) = a, \quad g(+\infty) = b$$

Necessary condition

$$b - a = \int_{-\infty}^{\infty} f(z)\,dz.$$

Even for linear problems, generically violated.

Good news.

We find approximate solutions with error $O(\varepsilon^N)$ for all N.

So we know what the solution looks like.

We know that the preceding ansatz for U_1 is incorrect for pulses and internal waves.

For the case of pulses where the leading term is known accurate [AR], we provide approximations of order ε^∞.

These are new even in the linear case.

Next question.

Push the asymptotics to longer times $t \sim 1/\varepsilon$. We want to understand the long time behavior of boundary layers and so far have not got it. Work in progress.

Assumption 2 and symmetry $\Longrightarrow \dim\ker L_1(t,x,\tau(t,x,\xi),\xi)$ is constant for (t,x,ξ) in a conic neighbhd of $\xi = (0,0,\ldots,1)$.

In particular $\dim\ker A_d(t,x',0) = k$ is constant on Σ.

By a t,x-dependent orthogonal change of basis can assume that

$$A_d(t,x',0) \;=\; \begin{pmatrix} 0_{k\times k} & 0_{k\times N-k} \\ 0_{N-k\times k} & \mathcal{A}(t,x') \end{pmatrix},$$

$$\det \mathcal{A}(t,x') \;\geq\; \delta > 0\,.$$

Define the spectral projector

$$\underline{\pi} \;:=\; \begin{pmatrix} I_{k\times k} & 0_{k\times N-k} \\ 0_{N-k\times k} & 0_{N-k\times N-k} \end{pmatrix}$$

and the **group velocity** (:= ray velocity)

$$\mathbf{v}(t,x') \;:=\; -\nabla_\xi \tau(t,x',x_d=0,\tau=0,\xi'=0,\xi_d=1)$$

Since τ vanishes on $\mathcal{N}^*\Sigma$ and is homogeneous of degree 1, it follows that $\mathbf{v}$ is tangent to $\Sigma := \{x_d = 0\}$.

The differential operator $\underline{\pi} L(t,x',0,\partial)\underline{\pi}$ is essentially a directional derivative,

$$\underline{\pi}\, L(t,x',0,\partial)\, \underline{\pi} \;=\; \underline{\pi}\big(\partial_t + \mathbf{v}(t,x').\partial'_x\big) \;+\; \text{lower order terms}\,.$$

This transport operator is the centerpiece of the description of wavefront propagation.

The analogous transport operator for internal waves, $\mathbb{H}$, is

$$\mathbb{H} = \underline{\pi}\,\big(\,\partial_t + \mathbf{v}(t,x').\nabla_{x'} + \partial_d\tau(t,x',0;0,\ldots,0,1)z\partial_z\,\big) \\ + \text{lower order terms}$$

If coordinates are chosen so that the hyperplanes $x_d = \text{const.}$ are all characteristic then the $z\partial_z$ term is not present.

In $\pm z \geq 0$, define $\tilde{F}_0^{\pm}(t,x',z)$ with $\tilde{F}^{\pm}(t,x',\pm\infty) = 0$ by

$$\widetilde{F}_0^{\pm}(t,x',z) \;:=\; F(t,x',x_d=0,z) - \overline{F}^{\pm}(t,x',x_d=0)\,.$$

Denote by $\mathcal{Z} := (\partial_t, \partial_1, \ldots, \partial_{d-1}, \phi(x_d)\partial_d)$ the standard conormal derivatives tangent to $\{x_d = 0\}$.

Main Theorem. *Define in* $\{\pm x_d \geq 0\} \times \{\pm z \geq 0\}$ *the principal profile*

$$U_0^\pm \ := \ \overline{U}^\pm(t,x) \ + \ \widetilde{U}_0^\pm(t,x',z)\,,$$

where $\widetilde{U}_0^\pm\big(t,x',z\big) \in H^\infty\big([0,T_2]\times\mathbb{R}^{d-1}\times\mathbb{R}\big)$ *is determined as the local solution of,*

$$(I-\underline{\pi})\widetilde{U}_0^\pm = 0\,,$$

$$\mathbb{H}\,\widetilde{U}_0^\pm \ + \ \underline{\pi}\Big(G(\overline{U}_0^\pm|_{x_d=0} + \widetilde{U}_0^\pm) - G(\overline{U}_0^\pm)\Big) \ = \ \underline{\pi}\widetilde{F}_0^\pm$$

$$\widetilde{U}_0^\pm\big|_{t<0} \ = \ 0\,.$$

Then $u^\varepsilon - U_0(t,x,x_d/\varepsilon) = O(\varepsilon)$ *in the sense that if* ε *is sufficiently small then* u^ε *exists on* $[0,T_2]$ *and* $\forall\beta$,

$$\Big\|\big(\mathcal{Z},\varepsilon\partial_d\big)^\beta\Big(u^\varepsilon - U_0(t,x,x_d/\varepsilon)\Big)\Big\|_{L^\infty([0,T_2]\times\mathbb{R}^d_\pm)} = O(\varepsilon)$$

Remark. We construct approximations of accuracy $O(\varepsilon^\infty)$.

The transmission strategy.

A hint that the moment condition should not be a fatal stumbling block comes from the following remark.

In $U_0(t,x,z)$ *one makes the substitution* $z = x_d/\varepsilon$. *In* $x_d > 0$ *only the limit at* $z = \infty$ *counts and in* $x_d < 0$ *only the limit at* $z = -\infty$ *counts.*

One never really needs **both** $z = \pm\infty$ *limits.*

To capitalize on this, it is natural to split the problem according to the two sides $\pm x_d > 0$.

The initial value problem for u^ε is equivalent to the transmission problem

$$L\,u^\varepsilon + G(u^\varepsilon) \;=\; f^\varepsilon \quad \text{in} \quad \{x_d \neq 0\}\,,$$

$$\Big[(I - \underline{\pi})u^\varepsilon\Big]_{x_d=0} \;=\; 0\,.$$

Square brackets indicate the jump.

The *ansatz* for u^ε has profiles for each half space.

A preliminary version is

$$u^\varepsilon \;=\; U^\varepsilon(t,x,x_d/\varepsilon)$$

where, $U^\varepsilon(t,x,z)$ is compactly supported in x with asymptotic expansions

$$U^\varepsilon(t,x,z) \;\sim\; \sum_{j=0}^{\infty} \varepsilon^j\, U_j^\pm(t,x,z)\,, \text{ in } \{\pm x_d \geq 0\} \times \{\pm z \geq 0\}$$

$$U_j^\pm(t,x,z) \;=\; \overline{U}_j^\pm(t,x) + \widetilde{U}_j^\pm(t,x,z)\,,$$

with $\widetilde{U}_j^\pm$ rapidly decreasing as $\pm z \to \infty$.

We do **not** require that $\widetilde{U}^\pm \to 0$ when $z \to \mp\infty$. In fact, $\widetilde{U}^\pm$ is not even defined at such points.

At the heart of our analysis is a sort of calculus of such expansions. The first remark it that, without loss of generality, the $\tilde{U}_j$ parts can be taken independent of x_d.

Because of the rapid decrease, $\widetilde{U}_j(t,x,x_d/\varepsilon)$ is essentially supported in an ε neighborhood of $x_d = 0$.

Taylor expansion in x_d yields

$$\widetilde{U}_j^{\pm}(t,x',x_d,z) \ \sim\ \sum_{k=0}^{\infty} \frac{x_d^k}{k!}\, \partial_{x_d}^k \widetilde{U}_j^{\pm}(t,x',0,z)\,.$$

Replacing x_d by εz yields an equivalent profile whose z dependent parts depend only on t, x', z and not on x_d.

This leads to the final form for the *ansatz*

$$u^\varepsilon = U^\varepsilon\Big(t,x,\frac{x_d}{\varepsilon}\Big), \qquad U^\varepsilon(t,x,z) \ \sim\ \sum_{j\ge 0} \varepsilon^j U_j^{\pm}(t,x,z)$$

$$U_j^{\pm}(t,x,z) \ =\ \overline{U}_j^{\pm}(t,x) + \widetilde{U}_j^{\pm}(t,x',z)$$

$\widetilde{U}_j^{\pm}$ is independent of x_d and rapidly decreasing as $\pm z \to \infty$.

Proposition 2.1. *If a family u^ε has an asymptotic expansion of this form, then the profiles $\overline{U}_j^{\pm}$ and $\widetilde{U}_j^{\pm}$ are uniquely determined.*

A different way to generate smoothed sources f^ε is to take a standard mollification of the piecewise smooth source $\overline{f}$.

Suppose that $j(t,x)$ is smooth compactly supported in $t \geq 0$ with $\int j\,dt\,dx = 1$.
Define $j^\varepsilon(t,x) = \varepsilon^{-d-1} j(t/\varepsilon, x/\varepsilon)$. Denote by J^ε the operator which is convolution with j^ε.

Suppose that $\overline{f}$ is piecewise smooth and compactly supported on on $\{t \leq T\} \times \mathbb{R}^d$ with jumps on $\{x_d = 0\}$.

Proposition 2.2. *With the hypotheses of the preceding paragraph, $f^\varepsilon := J^\varepsilon \overline{f}$ has an asymptotic expansion of the above form.*

Proposition 2.3. *The set of families u^ε which have expansions of the above form is invariant under smooth change of coordinates*

$$\big(\tilde t, \tilde x\big) = \big(\tilde t(t,x), \tilde x(t,x)\big)\,, \qquad \big(t,x\big) = \big(t(\tilde t,\tilde x), x(\tilde t,\tilde x)\big)$$

which map the half spaces $\pm x_d > 0$ to the corresponding halfspaces $\pm \tilde x_d > 0$.

Proposition 3.1. *If u^ε has an expansion of the above form and u^ε satisfies the transmission condition exactly then $Lu^\varepsilon + G(u^\varepsilon)$ has an expansion of the same form*

$$Lu^\varepsilon + G(u^\varepsilon) \;=\; W^\varepsilon(t,x,x_d/\varepsilon) \;\sim\; \sum_{j=-1}^{\infty} \varepsilon^j\, W_j(t,x,x_d/\varepsilon)$$

$$W_j(t,x,z) \;=\; \overline{W}_j^{\pm}(t,x) + \widetilde{W}_j^{\pm}(t,x',z).$$

Remark. If the transmission condition were not exactly satisfied there would be $\delta(x_d)$ terms on the left from ∂_d applied to a jump.

Summary. Once the transmission problem form of the *ansatz* and the basic calculus is in hand, the proofs are interesting in detail but go smoothly.

There are no moment conditions, and one constructs an infinitely accurate approximate solution,

$$L(u_{approx} - f^\varepsilon) \ = \ \text{conormal infinitely small.}$$

The stability,

$$L^{-1}\big(\text{conormal infinitely small}\big) = \text{conormal infinitely small},$$

is known since the 80s.

The key is the *ansatz*. The key discovery step is the fact that for the obvious *ansatz*, $U(t, x, x_d/\varepsilon)$, the two limits $U(t, x, \pm\infty)$ never occur for the same point (t, x) with $x_d \neq 0$. There is a natural association of $x_d > 0$ with $z > 0$ and $x_d < 0$ with $z < 0$ which leads to the transmission problem approach.

This description of layers is a flexible idea which should serve in other problems. Also the conormal spaces of tangential regularity.

Ideas to Remember

For multiscale problems you need correctors to get a small residual.

Hypersurface singularities and layers are characterictic.

Sobolev spaces with only tangential derivatives give good spaces to describe layers, and, have good mapping properties by the solution of differtential equations. Such questions were much studied in the 80s.

Separate *ansatz* for the two sides.