Existence of characteristic points for blow-up solutions of a semilinear wave equation

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The equation

$$\begin{cases} u_{tt} = \Delta u + |u|^{p-1}u, \\ u(0) = u_0 \text{ and } u_t(0) = u_1, \end{cases}$$

where
$u(t) : x \in \mathbb{R}^N \to u(x,t) \in \mathbb{R}$,
$u_0 \in \mathrm{H}^1_{\mathrm{loc,u}}(\mathbb{R}^N)$ and $u_1 \in \mathrm{L}^2_{\mathrm{loc,u}}(\mathbb{R}^N)$.

$$\|v\|_{\mathrm{L}^2_{\mathrm{loc,u}}(\mathbb{R}^N)} = \sup_{a \in \mathbb{R}^N} \left(\int_{|x-a|<1} |v(x)|^2 dx \right)^{1/2}.$$

$$1 < p \text{ and } p \le p_c \equiv 1 + \frac{4}{N-1} \text{ si } N \ge 2.$$

Rk.: $p_c \equiv 1 + \frac{4}{N-1} < 1 + \frac{4}{N-2}$, the Sobolev critical exponent.

Why is p_c critical ?

If $p = p_c$, then the equation is invariant under the following conformal transformation:

If $U(\xi, \tau)$ is defined by

$$U(\xi,\tau) = (|x|^2 - t^2)^{\frac{N-1}{2}} u(x,t), \quad \xi = \frac{x}{|x|^2 - t^2}, \quad \tau = \frac{t}{|x|^2 - t^2},$$

then U satisfies the same equation as u.

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THE CAUCHY PROBLEM IN $\mathrm{H}^1_{\mathrm{loc,u}}(\mathbb{R}^N) \times \mathrm{L}^2_{\mathrm{loc,u}}(\mathbb{R}^N)$

It is a consequence of:

- the Cauchy problem in $H^1 \times L^2(\mathbb{R}^N)$ (Ginibre and Velo, Lindblad and Sogge, Shatah and Struwe)
- the finite speed of propagation.

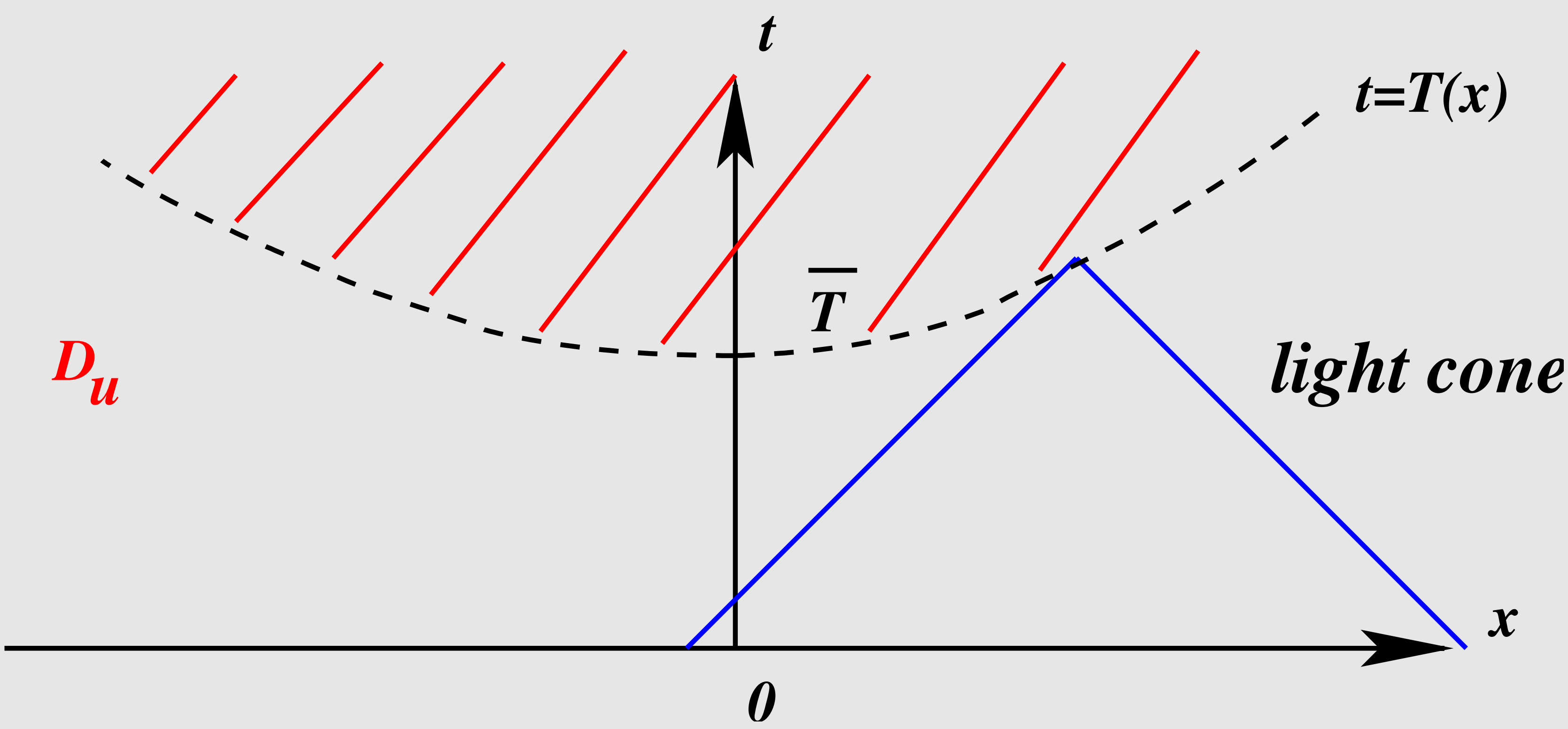
Maximal solution in $\mathrm{H}^1_{\mathrm{loc,u}}(\mathbb{R}^N) \times \mathrm{L}^2_{\mathrm{loc,u}}(\mathbb{R}^N)$

- either it exists for all $t \in [0, \infty)$ (**global solution**),
- or it exists for all $t \in [0, \bar{T})$ (**singular solution**).

Existence of singular de solutions

It's a consequence of ODE techniques and the finite speed of propagation (see Levine, Antonini and Merle).

Singular solutions: the maximal influence domain



The blow-up set $t \longrightarrow T(x)$ is 1-Lipschitz (**finite speed of propagation**).

Remark : $\bar{T} = \inf T(x)$ is the **blow-up time**. For all $x \in \mathbb{R}^N$, there exists a "local" blow-up time $T(x)$.

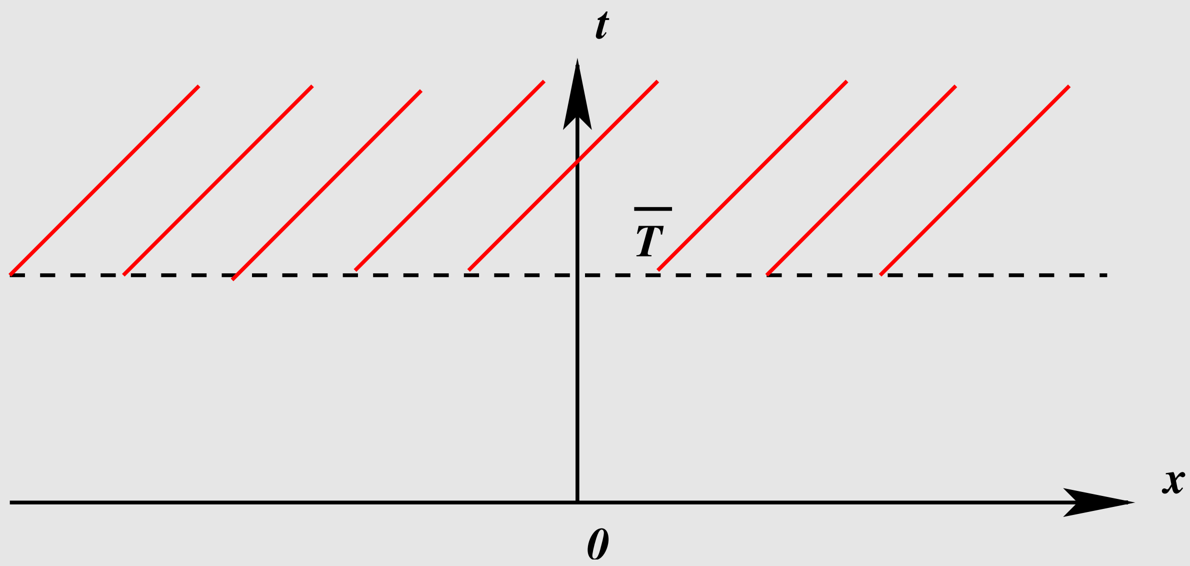
Remark: A comparison with the semilinear heat equation

$$u_t = \Delta u + |u|^{p-1}u$$

with

$$1 < p < 1 + \frac{4}{N-2} \text{ if } N \geq 2.$$

The singular solution is a maximal solution in $C_0(\mathbb{R}^N)$ which exists on $[0, \bar{T})$ where $\bar{T}$ is **the blow-up time** (and the only one). The solution cannot be extended beyond $\bar{T}$.



The aim of this talk

Goal : To describe precisely the blow-up set, *for an arbitrary blow-up solution*.

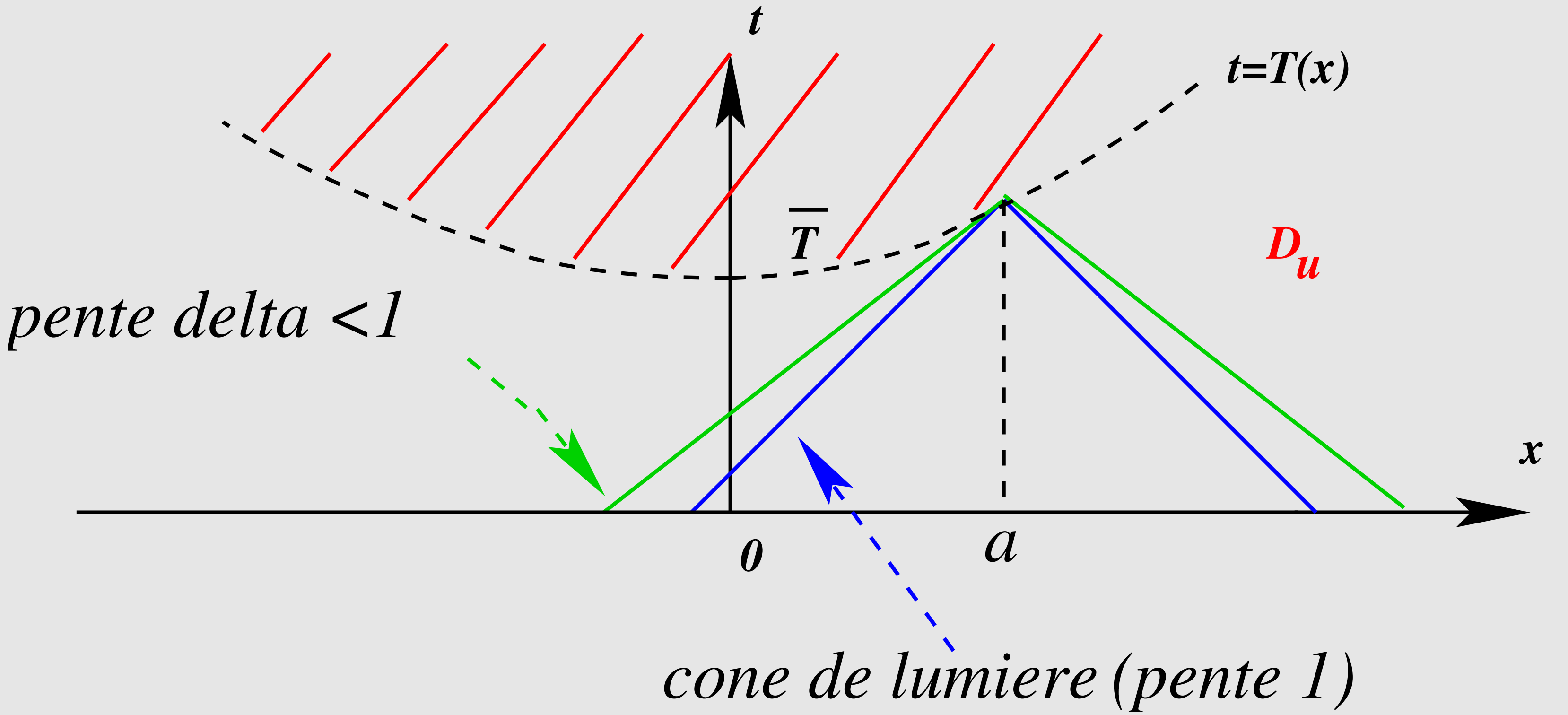
Tools : A whole program is needed.

Rk. We handle only the case $N = 1$. The only obstacle to go to higher dimensions $N \geq 2$: the solution of an elliptic problem.

Rk. We don't aim at describing some *particular* solution.

Vocabulary : Non characteristic points and characteristic points

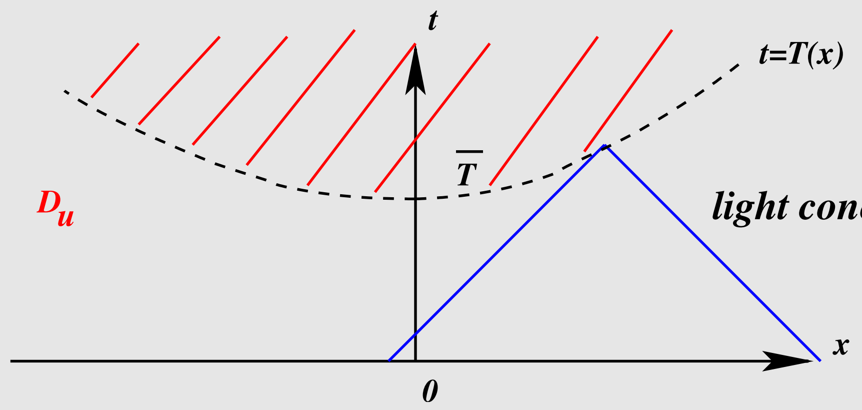
A point a is said *non characteristic* if the domain contains a cone with vertex $(a, T(a))$ and slope $\delta < 1$.



The point is said *characteristic* if not.

Known results, for an arbitrary solution

- The blow-up set $\Gamma = \{(x, T(x))\} \subset \mathbb{R}^2$.
- By defifnition, Γ is 1-Lipschitz. No more regularity was known.
- Notation: $I_0 \subset \mathbb{R}$ is the set of all *non* characteristic points.
- $I_0 \neq \emptyset$ (Indeed, $\bar{x}$ such that $T(\bar{x}) = \min_{x \in \mathbb{R}} T(x)$ is non characteristic).



Questions and new results

- **Existence**
  - Are there characteristic points? *yes*
- **Regularity**
  - Is I_0 open? *yes*
  - Is Γ (or Γ_{I_0}) of class C^1 ? *yes*
  - More regularity ? *yes from N. Nouaili,* $C^{1,\alpha}$
- **Asymptotic behavior (profile)**
  - How does the solution behave near a non characteristic point? *we have the profile*
  - and near a characteristic point? *we have a description*

Rk. Regularity and asymptotic behavior are linked.

The plan

- ▷ Part 1: Existence of characteristic points ($N = 1$).
- ▷ Part 2: A Liouville theorem and regularity of the blow-up set ($N = 1$).
- ▷ Part 3: A Lyapunov functional and blow-up rate ($N \geq 1$).
- ▷ Part 4: Blow-up profiles ($N = 1$).

Rk. Only Part 3 is done is dimension $N \geq 1$. The only obstacle in extending the other parts to the dimension $N \geq 2$ comes from the solution of some elliptic problem.

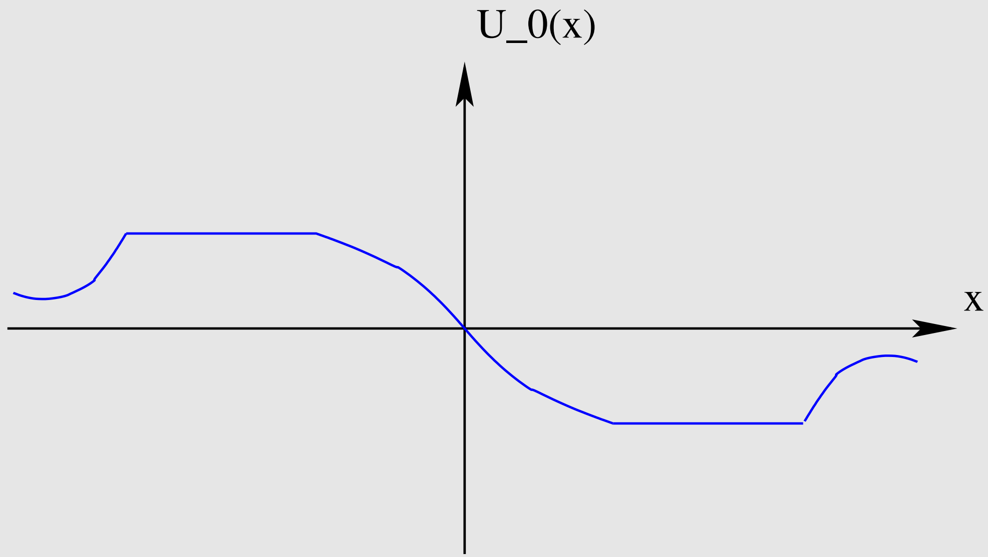
Rk. The order of this presentation goes from the easiest (to state) to the most complicated. The chronological order is actually 3, 4, 1, 2.

Part 1 : Existence of characteristic points

We recall: Any *solution to the Cauchy problem has (at least) a non characteristic point (the minimum of the blow-up set).*

Th. There exist *initial data which give solutions with a characteristic point.*

Example : We take odd initial data, with two large plateaus of different signs. Then, the solutions blows up, and the origin is a characteristic point.



Part 2 : Regularity of the blow-up set

- Near a non characteristic point:

  Th. *The set of non characteristic points* I_0 **is open** *and* $T(x)$ **is of class** C^1 *on this set* ($C^{1,\alpha}$ *by N. Nouaili*).

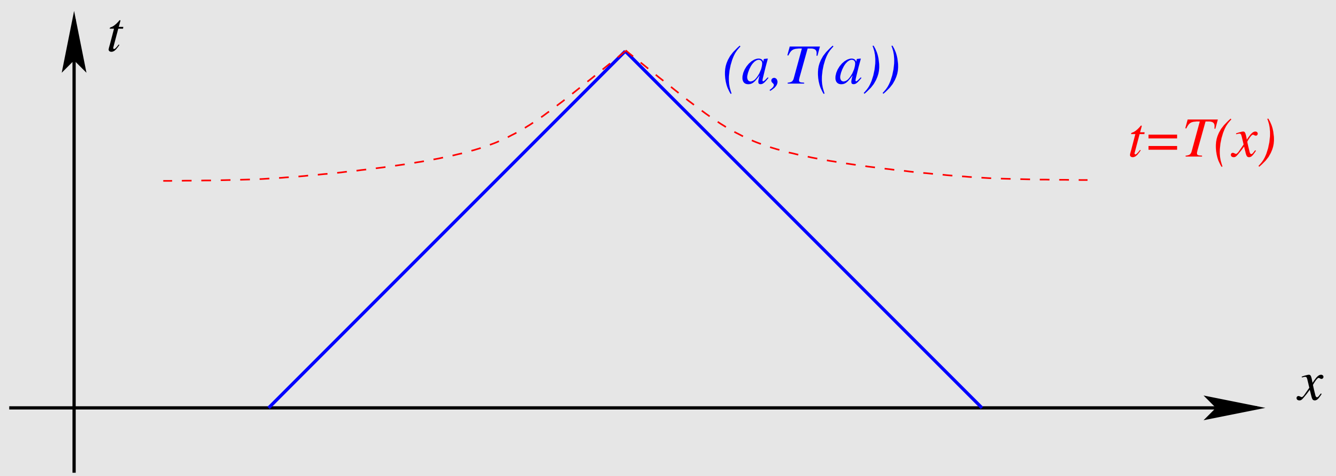
- Near a characteristic point:

  Th. *The interior of the set of characteristic points is empty (all the points belong to the boundary).*

  Conjecture : Is it made only of isolated points?

  Th. *If a is a characteristic point, then*

$$\text{for } x \text{ near } a, \quad T(x) \le T(a) - |x-a| + C|x-a||\log|x-a||^{-\gamma} \text{ with } \gamma > 0.$$



Comments

Rk. Caffarelli and Friedman proved the C^1 regularity of I_0, under restrictive assumptions on the nonlinear term and initial data (which imply in particular that $I_0 = \mathbb{R}$).
Their proof relies on the **positivity of the fundamental solution for** $N \leq 3$ (impossible to extend to the case $N \geq 4$).

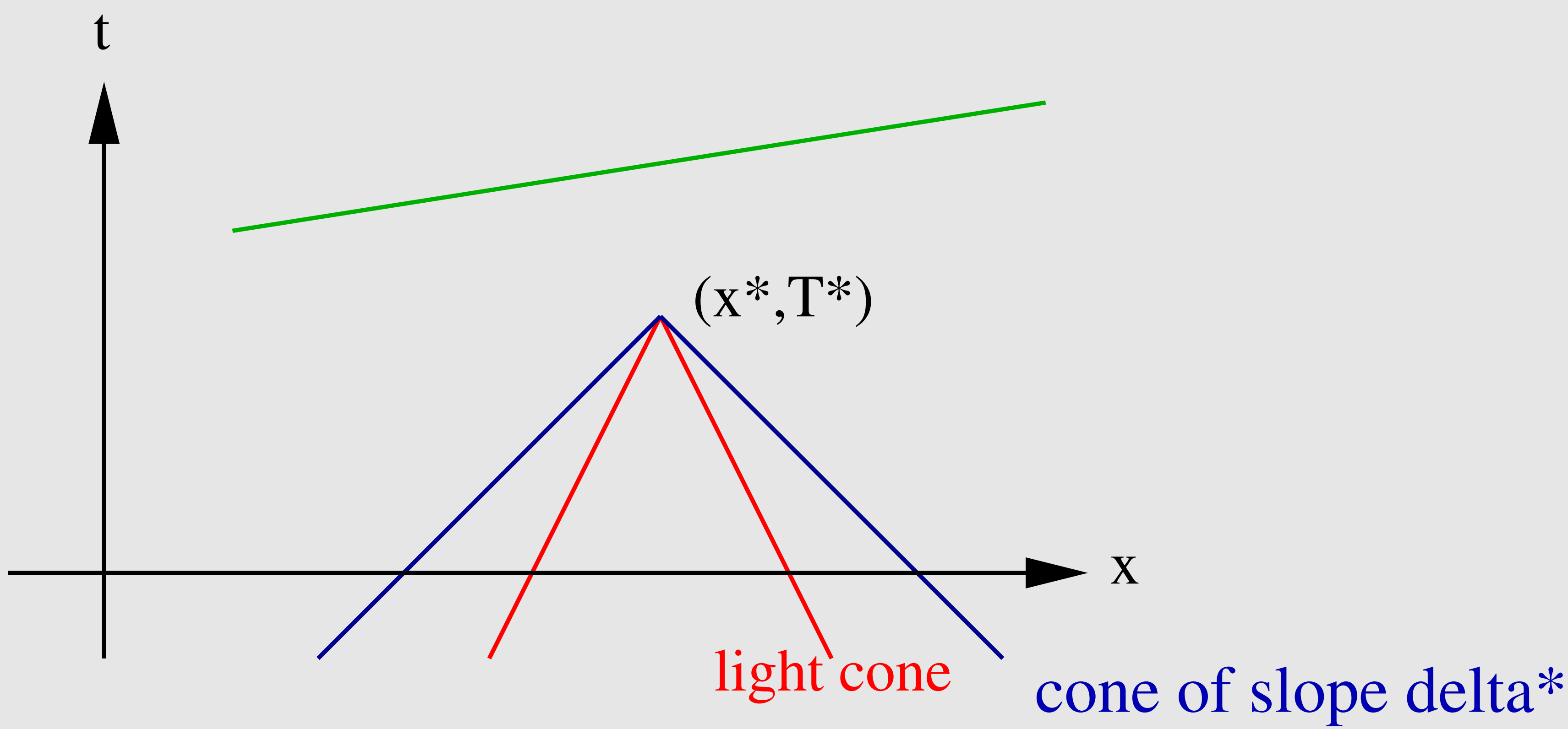
Idea of the proof of the regularity:
The techniques are based on

- - a very good understanding of the **behavior of the solution in selfsimilar variables in the energy space** related to the selfsimilar variable, together with a **Liouville Theorem** (see section 4 of this talk).
- - a **Liouville Theorem** (see next slide).

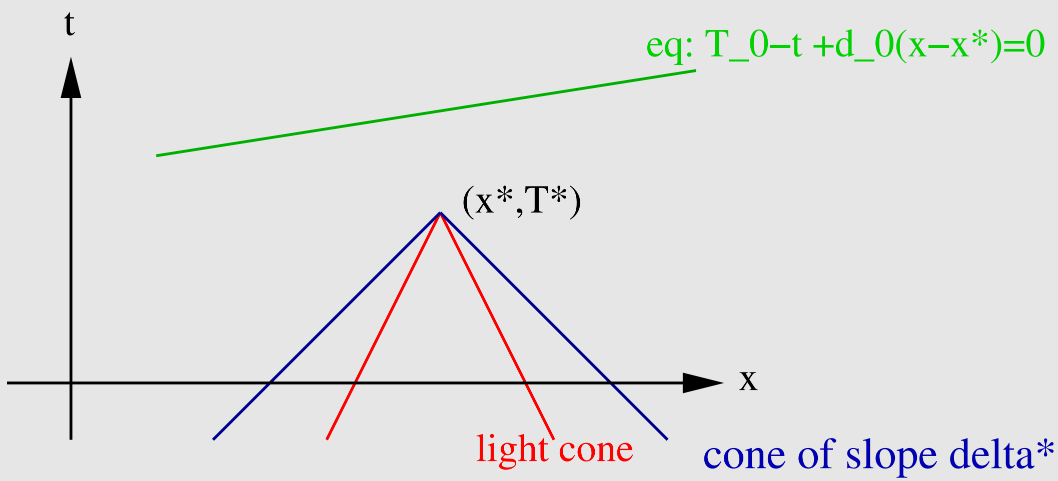
A Liouville theorem (N=1)

Th. Consider $u(x,t)$ a solution of $u_{tt} = u_{xx} + |u|^{p-1}u$ such that:

- u is defined in the *infinite* blue cone,
- u is less than $(T^* - t)^{-\frac{2}{p-1}}$ (in L^2 average).



A Liouville Theorem



Then,
- either $u \equiv 0$,
- or there exists $T_0 \geq T^*$, $d_0 \in [-\delta_*, \delta_*]$ and $\theta_0 = \pm 1$ such that u is actually defined below the green line by

$$u(x,t) = \theta_0 \kappa_0(p) \frac{(1 - d_0^2)^{\frac{1}{p-1}}}{(T_0 - t + d_0(x - x^*))^{\frac{2}{p-1}}}.$$

Remark: u blows up on the green line.

Comments

- The limiting case $\delta^* = 1$ is still open.
- $N \geq 2$: we expect the result to be valid. The only obstruction comes from the classification of stationary solutions.

The proof:

- The proof has a completely different structure from the proof for the heat equation.
- The proof is based on various energy arguments and on a dynamical result.

Part 3 : A Lyapunov functional and the blow-up rate ($N \geq 1$)

Selfsimilar transformation for all $x_0 \in \mathbb{R}^N$

$$w_{x_0}(y,s) = (T(x_0)-t)^{\frac{2}{p-1}} u(x,t),\ y = \frac{x-x_0}{T(x_0)-t},\ s = -\log(T(x_0)-t).$$

(x,t) in the light cone of vertex $(x_0, T(x_0)) \Longleftrightarrow (y,s) \in B(0,1) \times [-\log T(x_0), \infty)$.

Equation on $w = w_{x_0}$: For all $(y,s) \in B(0,1) \times [-\log T(x_0), \infty)$:

$$\partial_s^2 w - \tfrac{1}{\rho}\mathrm{div}\,[\rho \nabla w - \rho(y.\nabla w)y] + \tfrac{2(p+1)}{(p-1)^2} w - |w|^{p-1}w$$

$$= -\tfrac{p+3}{p-1}\partial_s w - 2y.\nabla \partial_s w$$

where $\rho(y) = (1-|y|^2)^\alpha$ and $\alpha \equiv \dfrac{2}{p-1} - \dfrac{N-1}{2} \geq 0.$

A Lyapunov functional (Antonini-Merle)

$$E(w) = \int_B \left(\frac{1}{2}(\partial_s w)^2 + \frac{1}{2}\left(|\nabla w|^2 - (y.\nabla w)^2 \right) + \frac{(p+1)}{(p-1)^2} w^2 - \frac{1}{p+1}|w|^{p+1} \right) \rho dy,$$

where $B = B(0,1)$.

Thanks to a Hardy-Sobolev inequality, $E = E(w, \partial_s w)$ is well defined in the energy space

$$\mathcal{H} = \left\{ q \in H^1_{loc} \times L^2_{loc}(B) \ \mid \ \|q\|^2_{\mathcal{H}} \equiv \int_B \left(q_1^2 + |\nabla q_1|^2 (1-y^2) + q_2^2 \right) \rho dy < +\infty \right\}.$$

Properties of the Lyapunov functional E

Lemme 1 (Monotonicity) *For all* s_1 *and* s_2*:*
($p < p_c$*, Antonini-Merle),*

$$E(w(s_2)) - E(w(s_1)) = -2\alpha \int_{s_1}^{s_2} \int_B (\partial_s w)^2 (1-|y|^2)^{\alpha-1} dy ds.$$

Lemme 2 (A blow-up criterion) *Consider a solution* W *such that* $E(W(s_0)) < 0$ *for some* $s_0 \in \mathbb{R}$*, then* W *blows up in finite time* $S > s_0$*.*

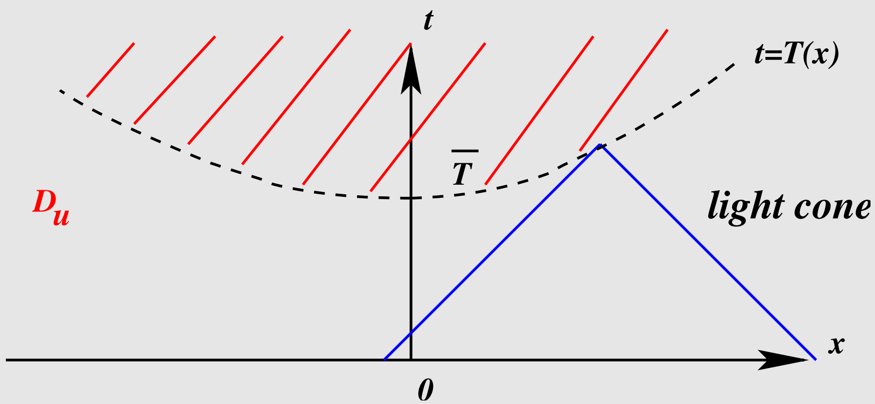
The blow-up rate ($N \geq 1$ and $p \leq p_c$)

The heat equation (Giga and Kohn 87, Giga, Matsui and Sasayama 2004)

$$0 < \kappa(p)(\bar{T}-t)^{-\frac{1}{p-1}} \leq \|u(t)\|_{L^\infty} \leq C(\bar{T}-t)^{-\frac{1}{p-1}}.$$

Remark : the blow-up rate is given by the solution of the associated ODE $v' = v^p$, $v(\bar{T}) = +\infty$.

The wave equation: We look for a *local* **blow-up rate** near the singular surface (i.e. near every local blow-up time, $t \to T(x_0)$), in $H^1 \times L^2$ **of the section of the light cone.**



Hint : Is the rate given by the associated ODE $v'' = v^p$?

An upper bound on the blow-up rate in selfsimilar variables

Th. For all $x_0 \in \mathbb{R}^N$ and $s \geq -\log T(x_0) + 1$,

$$\int_B \left(\frac{1}{2}(\partial_s w)^2 + \frac{1}{2}|\nabla w|^2(1-|y|^2) + \frac{(p+1)}{(p-1)^2}w^2 + \frac{1}{p+1}|w|^{p+1} \right) \rho dy \leq K$$

where the constant K depends only on N, p, and an upper bound on $T(x_0)$, $1/T(x_0)$ and $\|(u_0, u_1)\|$.

Getting rid of the weights

Reducing $B = B(0,1)$ to $B_{1/2} = B(0, \frac{1}{2})$, we get:

Cor. For all $x_0 \in \mathbb{R}^N$ and $s \geq -\log T(x_0) + 1$,

$$\int_{B_{1/2}} \left((\partial_s w)^2 + |\nabla w|^2 + w^2 + |w|^{p+1} \right) dy \leq K.$$

Upper bound in the original $u(x,t)$ variables

Th. sup. *For all* $x_0 \in \mathbb{R}^N$ *and* $t \in [\frac{3}{4}T(x_0), T(x_0))$:

$$(T(x_0)-t)^{\frac{2}{p-1}} \frac{\|u(t)\|_{L^2(B(x_0,\frac{T(x_0)-t}{2}))}}{(T(x_0)-t)^{N/2}}$$

$$+(T(x_0)-t)^{\frac{2}{p-1}+1} \left(\frac{\|u_t(t)\|_{L^2(B(x_0,\frac{T(x_0)-t}{2}))}}{(T(x_0)-t)^{N/2}} + \frac{\|\nabla u(t)\|_{L^2(B(x_0,\frac{T(x_0)-t}{2}))}}{(T(x_0)-t)^{N/2}} \right) \le K.$$

Rk. We have a lower bound of the same size when x_0 is non characteristic (see section 4 on profiles).

Idea of the proof of the upper bound

- Selfsimilar transformation and existence of a Lyapunov functional
- Interpolation to gain regularity
- Gagliardo-Nirenberg estimates.

Remark: The critical case $p = p_c$ is degenerate, therefore, we need more ideas.

Part 4: Asymptotic behavior ($N = 1$)

Take $x_0 \in \mathbb{R}$ **non characteristic**. Using a covering argument for x near x_0, we obtain that $\|w_{x_0}(s)\|_{H^1 \times L^2(B)}$ is bounded.

Question: Does $w_{x_0}(y, s)$ have a limit or not, as $s \to \infty$ (that is as $t \to T(x_0)$).

Remark: In the context of Hamiltonian systems, **this question is delicate**, and there is no natural reason for such a convergence, since **the wave equation is time reversible**.

See for similar difficulty and approach, results for

- the **critical KdV** (Martel and Merle),
- **NLS** (Merle and Raphaël).

Stationary solutions when $N = 1$.

We look for solutions of

$$\frac{1}{\rho}\left(\rho(1-y^2)w'\right)' - \frac{2(p+1)}{(p-1)^2}w + |w|^{p-1}w = 0.$$

We work in $\mathcal{H}_0$, the (**stationary energy space**) defined by

$$\mathcal{H}_0 = \{r \in H^1_{loc}(-1,1) \mid \|r\|^2_{\mathcal{H}_0} \equiv \int_{-1}^{1}\left(r'^2(1-y^2)+r^2\right)\rho dy < +\infty\}.$$

Prop. Take $N = 1$. Consider $w \in \mathcal{H}_0$ a stationary solution. Then, either $w \equiv 0$ or there exist $d \in (-1,1)$ and $\omega = \pm 1$ such that $w(y) = \omega\kappa(d,y)$ where

$$\forall (d,y) \in (-1,1)^2,\ \ \kappa(d,y) = \kappa_0 \frac{(1-d^2)^{\frac{1}{p-1}}}{(1+dy)^{\frac{2}{p-1}}} \text{ and } \kappa_0 = \left(\frac{2(p+1)}{(p-1)^2}\right)^{\frac{1}{p-1}}.$$

Remark: We have 3 connected components.

Stationary solutions if $N \geq 2$

If $N \geq 2$, we have no classification, unfortunately.
This is the only obstruction in generalizing our results to $N \geq 2$.

Of course, we already know that $\pm\kappa(d, \omega.y)$ is an $\mathcal{H}_0$ stationary solution for any $|d| < 1$ and $\omega \in \mathbb{R}^N$ with $|\omega| = 1$.

Now, back to $N = 1$.

Part 4A: Blow-up profile near a non characteristic point

Th. *There exist* $C_0 > 0$ *and* $\mu_0 > 0$ *such that*
if x_0 *is* **non characteristic**, *then there exist* $d(x_0) \in (-1,1)$, $\omega^*(x_0) = \pm 1$ *and* $s^*(x_0) \geq -\log T(x_0)$ *such that :*
(i) For all $s \geq s^*(x_0)$,

$$\left\| \begin{pmatrix} w_{x_0}(s) \\ \partial_s w_{x_0}(s) \end{pmatrix} - \omega^* \begin{pmatrix} \kappa(d(x_0),\cdot) \\ 0 \end{pmatrix} \right\|_{\mathcal{H}} \leq C_0 e^{-\mu_0(s-s^*)}$$

where the energy space

$$\mathcal{H} = \left\{ q \in H^1_{loc} \times L^2_{loc}(-1,1) \;\middle|\; \|q\|^2_{\mathcal{H}} \equiv \int_{-1}^{1} \left(q_1^2 + (q_1')^2 (1-y^2) + q_2^2 \right) \rho dy < +\infty \right\}.$$

(ii) $d(x_0) = T'(x_0)$.

Rk. We have exponentially fast convergence (hence, C^{1,μ_0} regularity of I_0).

Rk. $\|w_{x_0}(y,s) - \kappa(d(x_0),y)\|_{L^\infty(-1,1)} \to 0$.

Rk. The parameter of the profile $d(x_0)$ has a geometrical interpretation $(T'(x_0))$.

Difficulties of the proof of convergence

- - The set of non zero stationary solutions is made up of non isolated solutions (one parameter family):
  $\longrightarrow$ we need **modulation theory**.
- - The linearized operator around a non zero stationary solution **is non self-adjoint**:
  $\longrightarrow$ we need to use dispersive properties coming from the Lyapunov functional to control the negative part of the spectrum.

Partie 4B: Convergence for a *characteristic* point

Th. If $x_0 \in \mathbb{R}$ is **characteristic**, then, there exist $k(x_0) \geq 2$, $\omega_i^* = \pm 1$ and continuous $d_i(s) \in (-1,1)$ (ordonnés) for $i = 1, ..., k$ such that:
(i)

$$\|w_{x_0}(s) - \sum_{i=1}^{k(x_0)} \omega_i^* \kappa(d_i(s), \cdot)\|_{\mathcal{H}} \to 0 \text{ as } s \to \infty,$$

(ii) $d_1(s) \to 1$, $d_k(s) \to -1$ and $\omega_i^* \omega_{i+1}^* = -1$.
(iii)

$$\left| \frac{1}{2} \log\left(\frac{1+d_i(s)}{1-d_i(s)}\right) - \frac{1}{2} \log\left(\frac{1+d_j(s)}{1-d_j(s)}\right) \right| \to \infty \text{ for } i \neq j$$

as $s \to \infty$,
(iv)

$$E(w_{x_0}(s)) \to k(x_0) E(\kappa_0) \text{ as } s \to \infty.$$

Remark: As $s \to \infty$, w_{x_0} becomes like a **decoupled** sum of stationary solutions (“solitons”).

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Open questions

- Characteristic points are isolated (under preparation).
- The elliptic problem in dimension $N \geq 2$.
- At least, the radial case for $N \geq 2$.