Singular Solutions of Kinetic Equations

Existence of singular solutions
of non linear kinetic equations associated with
some singularity phenomena: two examples.

In Collaboration with J. J. L. Velazquez & S. Mischler.

Plan of the talk

1. Introduction:

   Uehling Uhlenbeck equation & singularity problem.

2. The linearized problem.
3. The non linear problem.
4. Another example:

   Smoluchowski equation and gelation.

The dilute gas of Bosons

Dilute gas of boson particles with interacting potential :

$$v(x - x') = 4\,\pi\,a\,\hbar\,\delta(x - x') \equiv g\delta(x - x'); \qquad a: \text{ scattering length.}$$

The particles P: mass $m = 1$, momentum p, energy $|p|^2/2$.
Only binary elastic collisions i.e. :
Two particles P_1, P_2 collide and give rise to two particles P_3, P_4:

$$p_1 + p_2 = p_3 + p_4 \quad \text{conservation of the momentum}$$

$$|p_1|^2 + |p_2|^2 = |p_3|^2 + |p_4|^2 \quad \text{conservation of the energy.}$$

The Uehling Uhlenbeck Equation

$f \equiv f(x,p,t)$:distribution of particles with momentum p at time t at point x. Satisfies the UEHLING UHLENBECK (UU) equation:

$$\begin{aligned}
\frac{\partial f}{\partial t} + p \cdot \nabla_x f &= Q(f) \\
Q(f) &= \frac{2\, g^2}{(2\,\pi)^5} \int\int\int_{\mathbb{R}^9} W\,(p_1,p_2,p_3,p_4)\, q(f) dp_2 dp_3 dp_4 \\
q\,(f) &= f_3 f_4 (1+f_1)(1+f_2) - f_1 f_2 (1+f_3)(1+f_4) \\
W\,(p_1,p_2,p_3,p_4) &= \omega(p_1,p_2,p_3,p_4)\, \delta(p_1+p_2-p_3-p_4) \times \\
&\quad \times \delta\left(|p_1|^2 + |p_2|^2 - |p_3|^2 - |p_4|^2\right)
\end{aligned}$$

• The function ω is determined by solving the quantum mechanical problem of collision particles:
The interaction of bosons is short ranged: $\omega = Constant.$

L. W. Nordheim: *Proc. Roy. Soc. London*, **A 119** (1928).

E. A. Uehling & G. E. Uhlenbeck: *Physical Review* **43** (1933).

E. Zaremba, T. Nikuni, A. Griffin J. Low Temp. Phys. **116** (1999).

R. Baier, T. Stockkamp: arXiv:hep-ph/0412310, (Jan. 2005).

Homogeneous gas

$$f(x,p,t) \equiv f(p,t)$$

•The equation becomes: $$\frac{\partial f}{\partial t} = Q(f)$$

By the symetries of W we have :

• Conservation of particles number, momentum and energy:

$$\frac{d}{dt}\int_{\mathbb{R}^3} f(p)\,dp = 0,\ \frac{d}{dt}\int_{\mathbb{R}^3} f(p)\,p\,dp = 0,\ \frac{d}{dt}\int_{\mathbb{R}^3} f(p)\,|p|^2\,dp = 0.$$

(at least formally...)

The entropy

The entropy is defined as

$$H(f)(t) = \int_{\mathbb{R}^3} h(f(t,p))\,dp$$
$$h(f) = (1+f)\,\ln(1+f) - f\,\ln(f)$$

It is increasing along the trajectories of the solutions:

$$\begin{aligned}\frac{\partial H(f)}{\partial t} &= \int_{\mathbb{R}^3} Q(f)\,h'(f)\,dp \\ &\equiv \frac{1}{4}D(f) \geq 0,\end{aligned}$$

Moreover:

Equilibria as Maxima of the entropy.

The maxima with zero momentum ($P = 0$) are:

$$F_{\beta,\mu}(p) = \frac{1}{e^{\beta|p|^2-\mu} - 1} \quad \beta > 0,\ \mu \leq 0$$

$$\beta = (k_B T)^{-1}, \quad (T: \text{ temperature of the gas.})$$

Remark. Given β (or T):

$$\frac{1}{e^{\beta|p|^2-\mu} - 1} \leq \frac{1}{e^{\beta|p|^2} - 1}, \qquad \text{for all } \mu < 0.$$

For a fixed temperature T: maximal particle number N_T.
Or, for a fixed particle number N: a MINIMAL temperature T_N.
If $T < T_N$?

Singular Equilibria

The answer was given by Bose & Einstein in 1924/1925:

$$F_{\beta,\mu}(p) \;=\; \frac{1}{e^{\beta|p|^2-\mu}-1}, \quad \text{for all } \mu \le 0,\ \beta > 0$$

$$G_{\beta,\rho}(p) \;=\; \frac{1}{e^{\beta|p|^2}-1} + \rho\,\delta_0, \quad \text{for all } \beta > 0,\ \rho > 0.$$

A consequence of the fact: *Let* $a \in \mathbb{R}^3$ *and* $\alpha \in \mathbb{R}$ *be fixed and* $(\varphi_n)_{n\in\mathbb{N}}$; $\quad \varphi_n \to \alpha\,\delta_a$. *Then, for any* $f \in L^1_2$*:*

$$H(f+\varphi_n) \underset{n\to\infty}{\longrightarrow} H(f) \text{ and } N(f+\varphi_n) \underset{n\to\infty}{\longrightarrow} \alpha + N(f).$$

Proof. SUPPOSE, for the sake of simplicity that $\varphi_n \equiv 0$ if $|p-a| \geq 2/n$. Then

$$H(f+\varphi_n) = \int_{|p-a|\geq 2/n} h(f(p,t),p)\,dp$$

$$+ \int_{|p-a|\leq 2/n} h((f(p,t)+\varphi_n(p),p)\,dp.$$

Using $|h(z)| \leq c\sqrt{z}$ we obtain:

$$\int_{|p-a|\leq 2/n} |h(f(p,t)+\varphi_n(p))|dp \;\leq\; c\frac{2}{\sqrt{n^3}}\left(\int_{|p-a|\leq 2/n}[f(p,t)+\varphi_n(p)]dp\right)^{1/2}$$

$$\longrightarrow 0 \text{ as } n \to +\infty.$$

REMARK. The entropy estimate $H(f) < \infty$ does not give any size estimate on f since it DOES NOT PREVENTS THE CONCENTRATION of f.

Consider now the Cauchy problem:

$$\frac{\partial f}{\partial t} = Q(f)$$

$$f(p,0) = f_0(p),$$

$f_0 :$ with number of particles N, energy E

and $T < T_N$

If $f_0(p) = f_0(|p|)$, X. Lu shows in JSP 2004:

• Existence of a GLOBAL solution in the WEAK sense (measures)

• Convergence in the WEAK sense to the corresponding equlibrium (with particle number N and energy E)

Since $T < T_N$ this equilibrium is singular (even if f_0 is regular):

Finite or infinite time formation of singularity?

Bose Einstein condensation

When the temperature is too low, or the initial particle number too large, the gas of bosons undergoes a phase transition: a condensate is formed **in finite time**.
A macroscopic part of the population of particles occupies the lowest possible energy level of the system (the fundamental state). This is the Bose Einstein CONDENSATE. After the condensation the gas+condensate is described by a system of two coupled equations ...

Isotropic case: $f = f(|p|, t)$

$$\text{Simplification: } Q(f) = \frac{1}{8} \int \int_{D(\varepsilon_1)} q(f) \widetilde{w}(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4) \, d\varepsilon_3 d\varepsilon_4$$

$$q(f) = f_3 f_4 (1 + f_1)(1 + f_2) - f_1 f_2 (1 + f_3)(1 + f_4)$$

$$\widetilde{w}(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4) = \frac{\min\{\sqrt{\varepsilon_1}, \sqrt{\varepsilon_2}, \sqrt{\varepsilon_3}, \sqrt{\varepsilon_4}\}}{\sqrt{\varepsilon_1}}$$

$$D(\varepsilon_1) = \{(\varepsilon_3, \varepsilon_4)) : \varepsilon_3 + \varepsilon_4 \geq \varepsilon_1\}, \quad \text{where } \varepsilon_i = |p_i|^2$$

$$\varepsilon_2 = \varepsilon_3 + \varepsilon_4 - \varepsilon_1$$

Singularity Formation. A description.

Following:

D. V. Semikov & I. I. Tkachev (Phys. Rev. Lett. 1995)

R. Lacaze, P. Lallemand, Y. Pomeau & S. Rica (Phys. D 2001).

Near the time singularity, $T > 0$ and the origin $\varepsilon = 0$, $f >> 1$.

(mUU) $$\frac{\partial f}{\partial t} = \mathcal{Q}(f) \sim Q(f)$$ (modified UU equation)

$$\mathcal{Q}(f) = \frac{1}{8} \iint_{D(\varepsilon_1)} \widetilde{q}(f) \widetilde{w}(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4)\, d\varepsilon_3 d\varepsilon_4$$

$$\widetilde{q}(f) = f_3 f_4 (f_1 + f_2) - f_1 f_2 (f_3 + f_4)$$

There is a solution of mUU of the form:

$$f(\varepsilon, t) = A^{-1/2}(T-t)^{-\alpha}\Phi\left(\frac{\varepsilon}{(T-t)^A}\right)$$

$$-\left(\nu + x\frac{d}{dx}\right)\Phi = \mathcal{Q}(\Phi), \text{ and } \nu = \alpha/A.$$

where, Φ is bounded, and satisfies

$$\Phi(x) \sim \frac{1}{x^\nu} \quad \text{as } x \to +\infty.$$

Then, for all $\varepsilon > 0$: $f(\varepsilon, t) \sim A^{-1/2}(T-t)^{-\alpha}\left(\frac{\varepsilon}{(T-t)^A}\right)^{-\nu}$

$$\equiv A^{-1/2}\varepsilon^{-\nu}, \text{ as } t \to T^-.$$

- as $x \to +\infty$:

$$\Phi(x) \sim x^{-\nu} - \frac{C(\nu)}{2(\nu-1)} x^{-3\nu+2} + \mathcal{O}(x^{-5\nu+4})$$

with $C(7/6) = C(3/2) = 0.$

Therefore: $\nu \neq 7/6, \ \ \nu \neq 3/2.$

- Near the origin:

$$\Phi(x) = a(\nu)\, x^{-7/6} + \cdots, \quad \text{as } x \to 0$$

For the correct value of $\nu : a(\nu) = 0.$

Numerical value: $\nu = 1,234\cdots \ \in (7/6, 3/2)$

Equilibrium steady solutions of mUU

It is easy to check that: $\widetilde{q}(1) = \widetilde{q}(\varepsilon^{-1}) = 0$

and therefore:

$$\mathcal{Q}(1) = \mathcal{Q}(\varepsilon^{-1}) = 0.$$

They come from the regular solutions of $Q(f) = 0$: $\dfrac{1}{e^{\beta|p|^2-\mu} - 1}$

Non-Equilibrium steady solutions:

Another solution obtained by V. E. Zakharov et. al:

$$\mathcal{Q}(\varepsilon^{-7/6}) = 0.$$

- Although $\widetilde{q}(\varepsilon^{-7/6}) \neq 0.$

- In the original variables $p \in \mathbb{R}^3$:

for some constant $C > 0$: $\displaystyle\int_{|p|\leq K} \mathcal{Q}(|p|^{-7/3})\, dp = -C$ for all $K > 0$

So we have actually: $\mathcal{Q}(|p|^{-7/3}) = -C\delta_{p=0}.$

These two sets of results by:

• R. Lacaze, P. Lallemand, Y. Pomeau & S. Rica:

Near the origin: $f(\varepsilon, t) \sim a(\nu)\, g(t)\, \varepsilon^{-7/6} + \cdots,$ as $\varepsilon \to 0$.

• V. E, Zakharov et. al: $\mathcal{Q}(\varepsilon^{-7/6}) = 0$.

seem to indicate a particular role of the power $\varepsilon^{-7/6}$ as $\varepsilon \sim 0$.

Our main result (very partial): That behaviour is stable, at least locally in time.

Main Theorem

Suppose that:

$$|f_0(\varepsilon) - A\,\varepsilon^{-7/6}| \;\leq\; \frac{B}{\varepsilon^{7/6-\delta}}, \quad 0 \leq \varepsilon \leq 1,$$

$$|f_0'(\varepsilon) + \frac{7}{6}\,A\,\varepsilon^{-13/6}| \;\leq\; \frac{B}{\varepsilon^{13/6-\delta}}, \quad 0 \leq \varepsilon \leq 1$$

$$f_0(\varepsilon) \;\leq\; B\frac{e^{-D\varepsilon}}{\varepsilon^{7/6}}, \quad k \geq 1$$

for A, B, C, δ positive constants.

Then there are: a unique solution of UU, $f \in \mathbf{C}^{1,0}((0,T) \times (0,+\infty))$, a function $\lambda(t) \in \mathbf{C}[0,T] \cap \mathbf{C}^1(0,T)$, and constants $L > 0$, $T > 0$ such that:

$$0 \leq f(\varepsilon,t) \leq L\frac{e^{-D\varepsilon}}{\varepsilon^{7/6}}, \quad \text{if } \varepsilon > 0,\ t \in (0,T),$$

$$|f(\varepsilon,t) - \lambda(t)\,\varepsilon^{-7/6}| \leq L\,\varepsilon^{-7/6+\delta/2}, \quad \varepsilon \leq 1,\ t \in (0,T),$$

$$|\lambda(t)| \leq L, \quad \text{for } t \in (0,T).$$

Due to the precise behaviour $f(\varepsilon, t) \sim \varepsilon^{-7/6}$ at $\varepsilon = 0$,

this solution satisfies:

$$\frac{d}{dt}\left(\int_{|\varepsilon| \leq K} \sqrt{\varepsilon}\, f(\varepsilon, t)\, d\varepsilon\right) = -C\lambda^3(t) + \mathcal{O}(K^{1/10}),$$

as $K \to 0$:

$\Longrightarrow$ no conservation of the number of particles.

Plan of the proof

- Linearisation of the “modified” U-U equation:

$$\frac{\partial f}{\partial t} = \mathcal{Q}(f)$$

  around $\varepsilon^{-7/6}$. The fundamental solution. The linear semigroup. (Largely based on Zakharov work. Our main contribution: precise size estimates.)

- Treat the Ueling Uhlenbeck equation as a nonlinear perturbation.

The work by Zakharov et al.

- Systematic method for the deduction, under suitable hypothesis, of kinetic equations of this type from system of PDE's with a Hamiltonian formulation.

- Surface water waves, Langmuir waves etc...

- Sistematic method to find homogeneous non equilibrium steady states.

- General method to study the linear stability of these steady states.

A. M. Balk, V. E. Zakharov: A. M. S. Translations Series 2, Vol. 182, 1998, 1-81.

Linearisation

We linearise around $f(\varepsilon) = \varepsilon^{-7/6}$: $f(t,\varepsilon) = \varepsilon^{-7/6} + F(t,\varepsilon)$

$$\widetilde{q}\left(\varepsilon^{-7/6} + F\right) = \widetilde{q}\left(\varepsilon^{-7/6}\right) + \widetilde{\ell}\left(\varepsilon^{-7/6}, F\right) + \widetilde{n}\left(\varepsilon^{-7/6}, F\right)$$

$\widetilde{\ell}\left(\varepsilon^{-7/6}, F\right)$: linear with respect to F. Consider the equation:

$$\frac{\partial F}{\partial t} = \frac{1}{8}\int\int_{D(\varepsilon_1)} \widetilde{\ell}\left(\varepsilon^{-7/6} + F\right)\widetilde{w}(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4)\, d\varepsilon_3 d\varepsilon_4$$

and obtain the following equation for F: (a and K explicit)

$$\frac{\partial F}{\partial t} = \mathcal{L}(F) \equiv -\frac{a}{\varepsilon^{1/3}}F(\varepsilon) + \frac{1}{\varepsilon^{4/3}}\int_0^{\infty} K\left(\frac{r}{\varepsilon}\right) F(r)\, dr$$

The fundamental solution of $\mathcal{L}$

$$F_t(t,\varepsilon,\varepsilon_0) = -\frac{a}{\varepsilon^{1/3}} F(t,\varepsilon,\varepsilon_0) + \frac{1}{\varepsilon^{4/3}} \int_0^\infty K\left(\frac{r}{\varepsilon}\right) F(t,r,\varepsilon_0)\, dr$$

$$F(0,\varepsilon,\varepsilon_0) = \delta(\varepsilon - \varepsilon_0).$$

Theorem. *For all $\varepsilon_0 > 0$, there exists a unique solution:*

$$F(t,\varepsilon,\varepsilon_0) = \frac{1}{\varepsilon_0} F\left(\frac{t}{\varepsilon_0^{1/3}}, \frac{\varepsilon}{\varepsilon_0}, 1\right)$$

such that:

For $\varepsilon \in (0,2)$ *the function* $F(t,\varepsilon,1)$ *can be written as:*

$$F(t,\varepsilon,1) = e^{-a\,t}\delta(\varepsilon-1) + \sigma(t)\,\varepsilon^{-7/6} + \mathcal{R}_1(t,\varepsilon) + \mathcal{R}_2(t,\varepsilon),$$

where $\sigma \in \mathbf{C}[0,+\infty)$ *satisfies:*

$$\sigma(t) = \begin{cases} A\,t^4 + \mathcal{O}(t^{4+\varepsilon}) & \text{as } t \to 0^+, \\ \mathcal{O}(t^{-(3v_0-5/2)}) & \text{as } t \to +\infty \end{cases}$$

A *is an explicit numerical constant,* $\varepsilon > 0$ *is an arbitrarily small number,* $v_0 \sim 1.84020... > 11/6$.

$\mathcal{R}_1$ *and* $\mathcal{R}_2$ *satisfy:*

$$\mathcal{R}_1(t,\varepsilon) \equiv 0 \quad \text{for } |\varepsilon - 1| \geq \frac{1}{2},$$

$$|\mathcal{R}_1(t,\varepsilon)| \leq C\frac{e^{-(a-\varepsilon)t}}{|\varepsilon - 1|^{5/6}} \quad \text{for } |\varepsilon - 1| \leq \frac{1}{2},$$

$$\mathcal{R}_2(t,\varepsilon) \leq \begin{cases} \dfrac{C}{t^{5/2+\varepsilon}}\left(\dfrac{t^3}{\varepsilon}\right)^{\tilde{b}} & \text{for } 0 \leq t \leq 1 \\ \dfrac{C}{t^{3v_0-\varepsilon}}\left(\dfrac{t^3}{\varepsilon}\right)^{\tilde{b}} & \text{for } t > 1. \end{cases}$$

$\tilde{b}$ *is an arbitrary number in* $(1, 7/6)$. *On the other hand, for* $\varepsilon > 2$,

$$F(t, \varepsilon, 1) \leq \begin{cases} \dfrac{C}{t^{\frac{9}{2}+\varepsilon}} \left(\dfrac{t^3}{\varepsilon}\right)^{\frac{11}{6}} & \text{for } 0 \leq t \leq 1 \\ \dfrac{C}{t^{1+3v_0-\varepsilon}} \left(\dfrac{t^3}{\varepsilon}\right)^{\frac{11}{6}} & \text{for } t > 1. \end{cases}$$

Remarks.

• The initial Dirac measure at $\varepsilon = \varepsilon_0$ PERSISTS for all time $t > 0$ and is NOT REGULARISED: hyperbolic behaviour.

• The total mass of the Dirac measure DECAYS exponentially fast in time: it is "ASYMPTOTICALLY" regularised.

• The behaviour $\varepsilon^{-7/6}$ as $\varepsilon \to 0$ PERSISTS for all time.

Sketch of the proof.

$$F_t(t,\varepsilon) = -\frac{a}{\varepsilon^{1/3}} F(t,\varepsilon) + \frac{1}{\varepsilon^{4/3}} \int_0^\infty K\left(\frac{r}{\varepsilon}\right) F(t,r)\,dr$$

$$F(0,\varepsilon) = \delta(\varepsilon - 1)$$

Properties of the kernel K. $K \in \mathbf{C}^\infty((0,1) \cup (1,+\infty))$ satisfies:

$$K(r) \sim a_1 r^{1/2} \quad \text{as} \quad r \to 0, \quad K(r) \sim a_2 r^{-7/6} \quad \text{as} \quad r \to +\infty$$

$$K(r) \sim a_3 (1-r)^{-5/6} + a_4 + \mathcal{O}((1-r)^{1/6}) \quad \text{as} \quad r \to 1^-,$$

$$K(r) \sim a_5 (r-1)^{-5/6} + a_6 + \mathcal{O}((1-r)^{1/6}) \quad \text{as} \quad r \to 1^+,$$

Change of variables: $\varepsilon = e^x$,

$$F(t,\varepsilon) = \mathcal{G}(t,x), \quad K(r/\varepsilon) = K(e^{-(x-y)}) = e^{x-y}\mathcal{K}(x-y)$$

with $\mathcal{K}(x) = e^{-x}K(e^{-x})$. We arrive to the Cauchy problem:

$$\begin{cases} \dfrac{\partial}{\partial t}\mathcal{G}(t,x) = e^{-x/3}\left(-a\mathcal{G}(t,x) + \displaystyle\int_{-\infty}^{\infty} \mathcal{K}(x-y)\,\mathcal{G}(t,y)\,dy\right), \\ \mathcal{G}(0,x) = \delta(x), \end{cases}$$

In what space do we look for a solution $\mathcal{G}$?

Due to the behaviour of K at 0 and $+\infty$, THE BEHAVIOUR of $\mathcal{K}$ is

$$|\mathcal{K}(x)| \sim C_1 e^{\frac{x}{6}} \quad \text{for} \quad x < 0$$
$$|\mathcal{K}(x)| \sim C_2 e^{-\frac{3}{2}x} \quad \text{for} \quad x > 0.$$

Therefore, IF WE WANT

$$\int_{-\infty}^{\infty} \mathcal{K}(x-y)\mathcal{G}(t,y)\,dy < +\infty$$

WE NEED

$$|\mathcal{G}(t,x)| \le Ce^{-Mx} \quad \text{for} \quad x < 0, \qquad |\mathcal{G}(t,x)| \le Ce^{-mx} \quad \text{for} \quad x > 0$$

for some $m > -1/6$ and $M < 3/2$. Now we BOOTSTRAP for $x > 0$:

$$
\begin{aligned}
|\int_{-\infty}^{\infty} \mathcal{K}(x-y)\mathcal{G}(y)dy| &\leq |\int_{-\infty}^{0} \mathcal{K}(x-y)\mathcal{G}(y)dy| + |\int_{-\infty}^{x} \mathcal{K}(z)\mathcal{G}(x-z)dz| \\
&\leq \int_{-\infty}^{0} e^{-\frac{3}{2}(x-y)} e^{-My} dy + \int_{-\infty}^{x} e^{\frac{z}{6}} e^{-m(x-z)} dz \\
&\leq C\left(e^{-\frac{3}{2}x} + e^{-mx}\right).
\end{aligned}
$$

We deduce that, for $x > 0$ the right hand term of the equation satisfies:

$$
e^{-x/3}\left|-a\mathcal{G}(x) + \int_{-\infty}^{\infty} K(x-y)\,\mathcal{G}(y)\,dy\right| \leq C\left(e^{-(m+\frac{1}{3})x} + e^{-\frac{11}{6}x}\right),
$$

Therefore, $|\mathcal{G}(t,x)| \leq Ce^{-\frac{11}{6}x}$ for $x > 0$. This does not work for $x < 0$.

LAPLACE transform in t and FOURIER transform in x: $G(z,\xi)$.

If $\mathcal{G}(x) \leq Ce^{-\frac{11}{6}x}$ for $x > 0$, then considering $\xi = u + i\,v$, $u \in \mathbb{R}$, $v \in \mathbb{R}$ we have:

$$\left|e^{-i\,\xi\,x}\mathcal{G}(x)\right| \leq Ce^{(v-\frac{11}{6})\,x} \quad \text{for } \ x > 0$$

and, if $\mathcal{G}(x) \leq Ce^{-Mx}$ for $x < 0$:

$$\left|e^{-i\,\xi\,x}\mathcal{G}(x)\right| \leq Ce^{(v-M)\,x} \quad \text{for } \ x < 0.$$

Therefore: $G(z,\cdot)$ is ANALYTIC in the strip $M < v < 11/6$ $(M < 3/2)$.

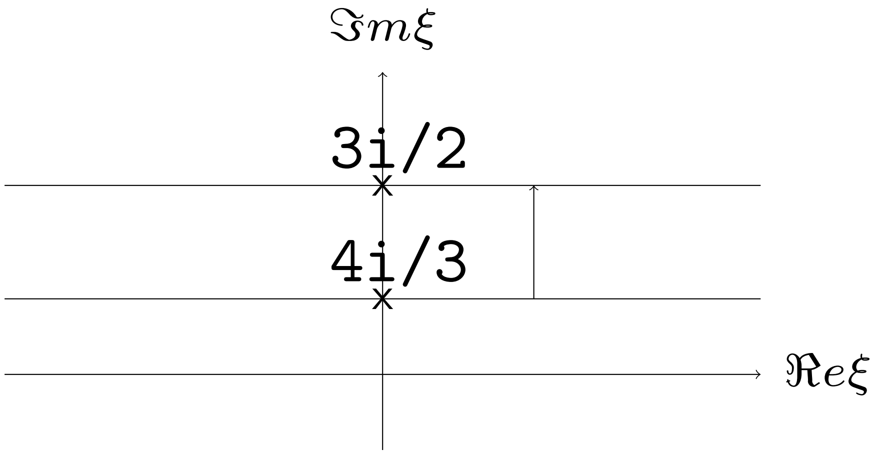
The Carleman equation.

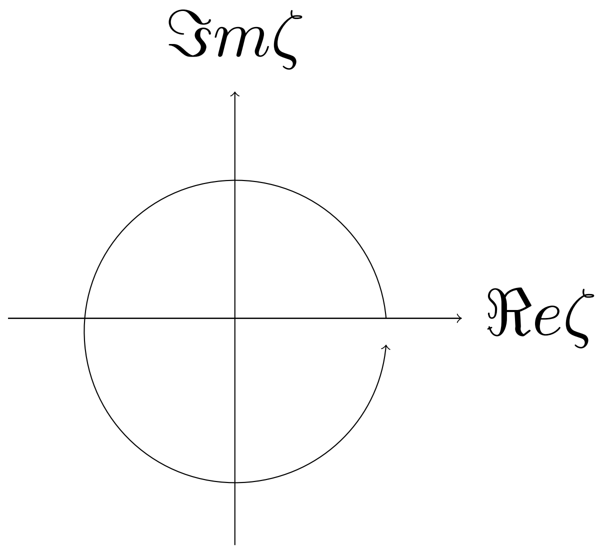
$$zG(z,\xi) = G(z,\xi - \frac{i}{3})\Phi(\xi - \frac{i}{3}) + \frac{1}{\sqrt{2\pi}}, \qquad (1)$$

where $\Phi(\xi) = -a + \widehat{\mathcal{K}}(\xi)$ and $\widehat{\mathcal{K}}$ is the Fourier transform of $\mathcal{K}$. The problem is then transformed in the following:

For any $z \in \mathbb{C}$, $\mathcal{R}ez > 0$, find a function $G(z,\cdot)$ analytic in the strip $S = \{\xi;\ \xi = u + iv,\ 4/3 < v < 11/6,\ u \in \mathbb{R}\}$ satisfying (1) on S.

We introduce the NEW SET OF VARIABLES:





$$\zeta = T(\xi) \equiv e^{6\pi(\xi - \frac{4}{3}i)}, \qquad g(z,\zeta) = G(z,\xi), \qquad \widetilde{\varphi}(\zeta) = \Phi(\xi)$$

Then g SOLVES:

$$zg(z, x - i0) = \varphi(x)\, g(z, x + i0) + \frac{1}{\sqrt{2\pi}} \quad \text{for all } x \in \mathbb{R}^+$$

g is analytic and bounded in D,

where,

$$D = \{\zeta \in T(\mathbb{C});\ \zeta = re^{i\theta},\ r > 0,\ 0 < \theta < 2\pi\},$$

and, for any $x \in \mathbb{R}^+$:

$$g(z, x + i0) = \lim_{\varepsilon \to 0} g(z, xe^{i\varepsilon}),\ \ g(z, x - i0) = \lim_{\varepsilon \to 0} g(z, xe^{i(2\pi - \varepsilon)})$$

$$\varphi(x) = \lim_{\varepsilon \to 0} \widetilde{\varphi}(xe^{i\varepsilon}).$$

The Wiener Hopf method

The key of the argument is:

- To write the function $\varphi(\zeta)/z$ for $\zeta \in \mathbb{R}^+$ as

$$\frac{\varphi(\zeta)}{z} = \frac{M(z, \zeta + i0)}{M(z, \zeta - i0)}, \quad \text{for} \quad \zeta \in \mathbb{R}^+,$$

where $M(z, \xi)$ is an analytic function of ξ on $\mathbb{C} \setminus \mathbb{R}^+$.

- To write the function $M(z, x - i0)$ for $x \in \mathbb{R}^+$ as

$$\frac{M(z, x - i0)}{\sqrt{2\pi}\, z} = W(z, x + i0) - W(z, x - i0) \quad \text{for} \quad x \in \mathbb{R}^+,$$

where $W(z, \xi)$ is an analytic function of ξ on $\mathbb{C} \setminus \mathbb{R}^+$.

• This makes that the equation on g may be written:

$$M(z, x - i0)g(z, x - i0) + W(z, x - i0) =$$
$$M(z, x + i0)g(z, x + i0) + W(z, x + i0), \quad \text{for all} \quad x \in \mathbb{R}^+$$

with $M(z, \cdot)g(z, \cdot) + W(z, \cdot)$ analytic in $\mathbb{C} \setminus \mathbb{R}^+$.

• The function $C(z, \cdot)$ defined by means of:

$$C(z, \cdot) \equiv M(z, \cdot)g(z, \cdot) + W(z, \cdot)$$

is then analytic in $\mathbb{C} \setminus \{0\}$.

• Finally to identify this function $C(z, \cdot)$ showing that it is analytic also at $\xi = 0$ and then in all $\mathbb{C}$.

The decomposition of φ/z

If the following integral is convergent:

$$H(z,\zeta) = \frac{1}{2\pi i}\int_0^\infty \ln\left(\frac{\varphi(\lambda)}{z}\right)\frac{d\lambda}{\lambda-\zeta}.$$

then, the Plemej Sojoltski formulas give, for $\zeta \in \mathbb{R}^+$:

$$\begin{aligned} H(\zeta+i0) &= \frac{1}{2}\ln\left(\frac{\varphi(\zeta)}{z}\right) + \frac{1}{2\pi i}\, pv\int_0^\infty \ln\left(\frac{\varphi(\lambda)}{z}\right)\frac{d\lambda}{\lambda-\zeta} \\ H(\zeta-i0) &= -\frac{1}{2}\ln\left(\frac{\varphi(\zeta)}{z}\right) + \frac{1}{2\pi i}\, pv\int_0^\infty \ln\left(\frac{\varphi(\lambda)}{z}\right)\frac{d\lambda}{\lambda-\zeta} \end{aligned}$$

Therefore : $$\frac{\varphi(\lambda)}{z} = \frac{e^{H(z,\zeta+i0)}}{e^{H(z,\zeta-i0)}} \equiv \frac{M(z,\zeta+i0)}{M(z,\zeta-i0)}.$$

$M(z,\zeta)$ ANALYTIC in $\zeta \in \mathbb{C} \setminus \mathbb{R}^+$: follows from Integrability properties of $\ln(\varphi)$ (To check later)

Moreover, if M has suitable bounds as $x \to 0$ and $x \to +\infty$, we may define:

$$W(z,\zeta) = \frac{1}{2\pi i}\int_0^\infty \frac{M(z,\lambda-i0)}{z}\frac{d\lambda}{\lambda-\zeta}$$

and, by the same argument:

$$\frac{M(z, x - i0)}{\sqrt{2\pi}\, z} = W(z, x + i0) - W(z, x - i0), \quad \text{for any } x > 0$$

The function

$$C(z, \cdot) \equiv M(z, \cdot) g(z, \cdot) + W(z, \cdot)$$

is then analytic in $\mathbb{C} \setminus \{0\}$. The size estimates on W and M allow to show:

$$|C(z, \zeta)| \leq |\zeta|^{-1+\rho} \quad \text{as} \quad |\zeta| \to 0$$
$$|C(z, \zeta)| \leq |\zeta|^{1-\delta} \quad \text{as} \quad |\zeta| \to +\infty$$

for some $\rho > 0$ and $\delta > 0$.

$C(z,\zeta)$ is then analytic also at 0 and does not depend on ζ i. e.

$$\forall z \in \mathbb{C} \setminus \mathbb{R}^- : \quad C(z,\zeta) = C(z),$$

whence, IF A SOLUTION g EXISTS:

$$g(z,\zeta) = \frac{C(z) - W(z,\zeta)}{M(z,\zeta)},$$

where, $$C(z) = \lim_{\zeta \to 0} W(z,\zeta) = \frac{1}{2\pi i} \int_0^\infty \frac{M(z,\lambda - i0)}{z} \frac{d\lambda}{\lambda}$$

Due to the behaviour of $\ln(\varphi(\zeta))$ and $M(z,\zeta)$ as $\Re e\zeta \to \pm\infty$, the INTEGRALS which define H and M above do NOT CONVERGE. They have to be slightly MODIFIED.

Theorem. *For any* $z \in \mathbb{C} \setminus \mathbb{R}^-$*, there exists a unique bounded solution* g*, given by:*

$$g(z,\zeta) = \frac{1}{2\pi i}\frac{\zeta}{z}\int_0^\infty \frac{M(z,\lambda - i0)}{M(z,\zeta)}\frac{d\lambda}{\lambda\,(\lambda-\zeta)}$$

where,

$$M(z,\zeta) = \exp\left[\frac{1}{2\pi i}\int_0^\infty \ln\left(\frac{\varphi(\lambda)}{z}\right)\left(\frac{1}{\lambda-\zeta} - \frac{1}{\lambda-\lambda_0}\right)\,d\lambda\right],$$

$\lambda_0 \in \mathbb{C} \setminus \mathbb{R}^+$ *is arbitrary and* $\alpha(z) = \dfrac{1}{2\pi\, i}\ln\left(-\dfrac{z}{a}\right)$.

Example of technical lemma

Lemma 1. *Suppose that, for some* $\varepsilon > 0$ f *is analytic in the cone*

$$C(2\varepsilon_0) \equiv \left\{\zeta \in \mathbb{C};\, \zeta = |\zeta| e^{i\theta},\ \theta \in (-2\varepsilon_0, 2\varepsilon_0)\right\}.$$

Let us also assume that:

$$\int_0^\infty \frac{|f(re^{i\theta})|}{1+r^2}\, dr < +\infty, \text{ for any } \theta \in (-2\varepsilon_0, 2\varepsilon_0)$$

$$\lim_{\lambda \to 0 \lambda \in C(2\varepsilon_0)} f(\lambda) = L_1, \quad \lim_{\lambda \to \infty,\, \lambda \in C(2\varepsilon_0)} f(\lambda) = L_2,$$

$$|f'(\lambda)| = o(1/\lambda), \quad \text{as } \lambda \to 0,\ \lambda \to +\infty,\ \lambda \in C(2\varepsilon_0).$$

Then, for any $\lambda_0 \in \mathbb{C} \setminus C(2\varepsilon_0)$, *the function*

$$F(\zeta) = \frac{1}{2\pi i} \int_0^{\infty} f(\lambda) \left(\frac{1}{\lambda - \zeta} - \frac{1}{\lambda - \lambda_0} \right) \, d\lambda$$

is analytic in the domain

$$D(\varepsilon_0) = \left\{ \zeta \in \mathcal{S};\ \zeta = |\zeta| e^{i\theta},\ \theta \in (-\varepsilon_0, 2\pi + \varepsilon_0) \right\}$$

Moreover:

$$F(\zeta) = -\frac{L_1}{2\pi i} \ln \zeta + o\left(\ln |\zeta|\right), \quad \text{as } \zeta \to 0,\ \zeta \in D(\varepsilon_0)$$

$$F(\zeta) = -\frac{L_2}{2\pi i} \ln \zeta + o\left(\ln |\zeta|\right), \quad \text{as } \zeta \to +\infty,\ \zeta \in D(\varepsilon_0).$$

Theorem. *For any $z \in \mathbb{C} \setminus \mathbb{R}^-$, there exists a unique bounded solution G, given by:*

$$G(z,\xi) = \frac{3i}{2\pi\, z} \int_{\mathcal{I}m\, y = \frac{5}{3}} e^{6\pi\alpha(z)\,(y-\xi)} \frac{\mathcal{V}(y)}{\mathcal{V}(\xi)} \frac{dy}{\left(e^{6\pi(y-\xi)} - 1\right)}$$

where, $$\mathcal{V}(\xi) = \exp[-3i \int_{\mathcal{I}m\, y = \frac{4}{3}} \ln\left(\frac{\Phi(y+i0)}{-a}\right) \times e^{6\pi y} \left(\frac{1}{e^{6\pi y} - e^{6\pi\xi}} - \frac{1}{e^{6\pi y} - a e^{6\pi\delta i}}\right) dy].$$

and $\delta \in \mathbb{C}$ is arbitrary such that $\Im m\delta \neq 4\, i/3 + 2\, k\, \pi$.

•The convergence of the integrals rely on the behaviour both local and as $\Re e\, \lambda \to \pm\infty$ of the function $\ln(\Phi)$.

The function $\Phi(\xi) := -a + \widehat{\mathcal{K}}(\xi)$:

$$\Phi(\xi) = -a + \sum_{j=0}^{\infty} \frac{A_1(j)}{(1 - 6i\xi + 12j)} + \sum_{j=0}^{\infty} \frac{A_2(j)}{(1 - 3i\xi + 3j)} +$$

$$+ \sum_{j=0}^{\infty} \frac{A_3(j)}{(3 + 2i\xi + 2j)} + \sum_{j=0}^{\infty} \frac{A_4(j)}{(10 + 3i\xi + 6j)}; \quad A_i(j), \text{ explicit.}$$

Poles: $\xi = (\frac{3}{2} + j)\, i;\ (\frac{10}{3} + 2j)\, i;\ -(\frac{1}{3} + j)\, i;\ -(\frac{1}{6} + 2j)\, i;\quad j = 0, 1, \cdots$

and: $\Phi(\xi) \sim -a + \frac{b_1}{\xi^{1/6}} + \frac{b_2}{\xi}$ as $|\xi| \to +\infty$ and $\Im m \xi$ bounded.

The zeros of Φ.

The only exact results on the zeros of Φ are:

• The function Φ has a simple zero at the point $\xi = 7i/6$. It corresponds to the fact that $k^{-7/6}$ is a solution of the linearised equation.

• Moreover, it also has a simple zero at $\xi = 13i/6$. This corresponds to the fact that k^{-1} is also a solution of the linearised equation.

• NO OTHER ZERO of Φ is known in general. But OTHER ZEROS of Φ determine the behaviour of the term $\sigma(t)$ and the lower order terms $\mathcal{R}_1$ and $\mathcal{R}_2$ in the expansion of the fundamental solution.

We assume and have numerically checked:

• The point $\xi = 7i/6$ is the only zero of Φ in the strip $\mathcal{I}m\xi \in (-1/6, 5/3)$.

• The zeros of Φ nearest to $13i/6$ are two simple zeros at $\xi = \pm u_0 + iv_0$ with: $u_0 = 0.331...,\quad v_0 = 1.84020...$
These are the only zeros of Φ in the strip $\mathcal{I}m\xi \in (-1/3, 5/2)$.

• The graph of the function $\Phi(\xi)$ does not make any complete turn around the origin when ξ moves along any curve connecting the two extremes of the strip $7/6 < \Im m\xi < 3/2$..

We draw part of the curves: $\Phi(\xi)$ $\xi = b + i\,r$, $-\infty < r < +\infty$.

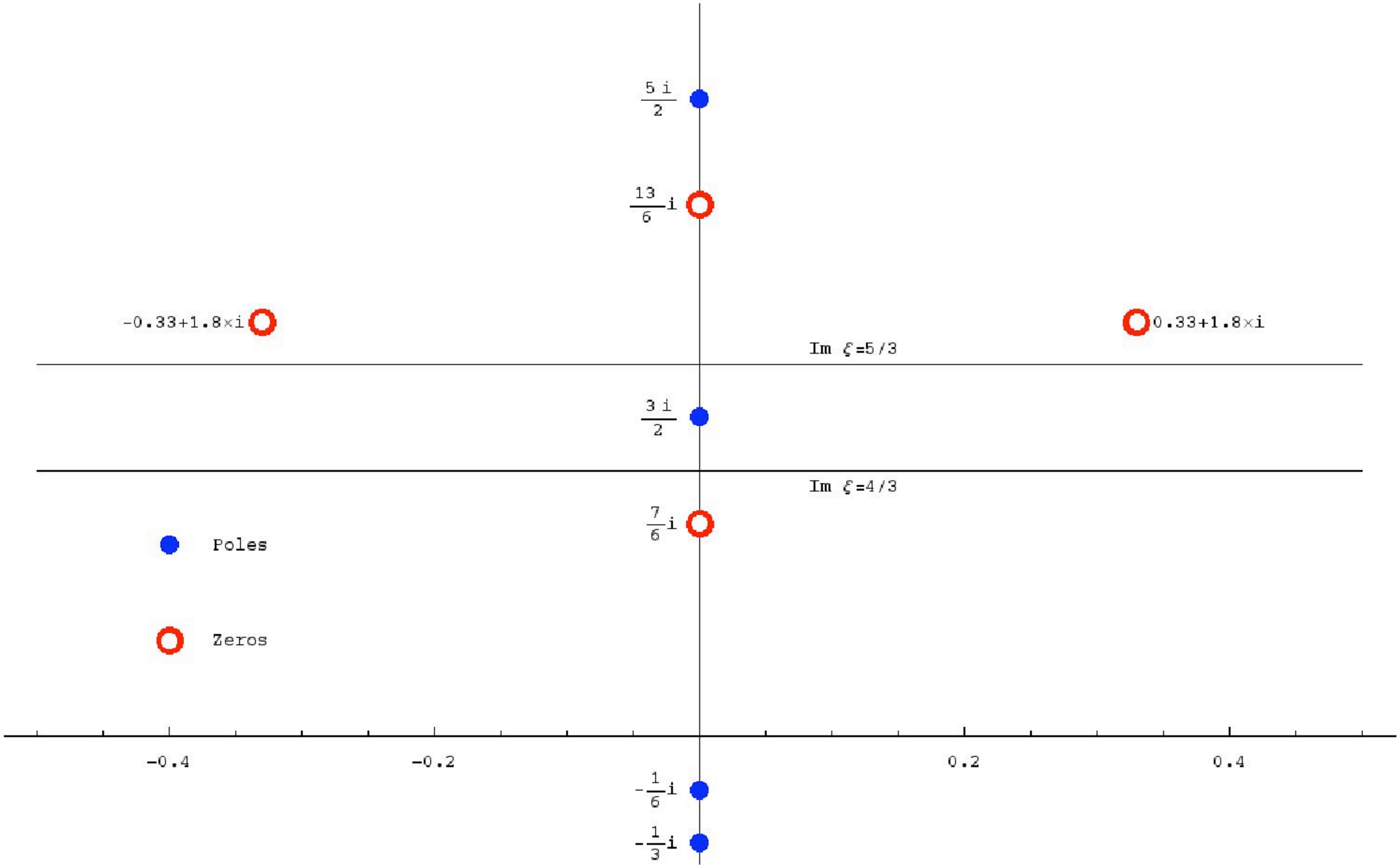


Figure 1: Some zeros and poles of Φ

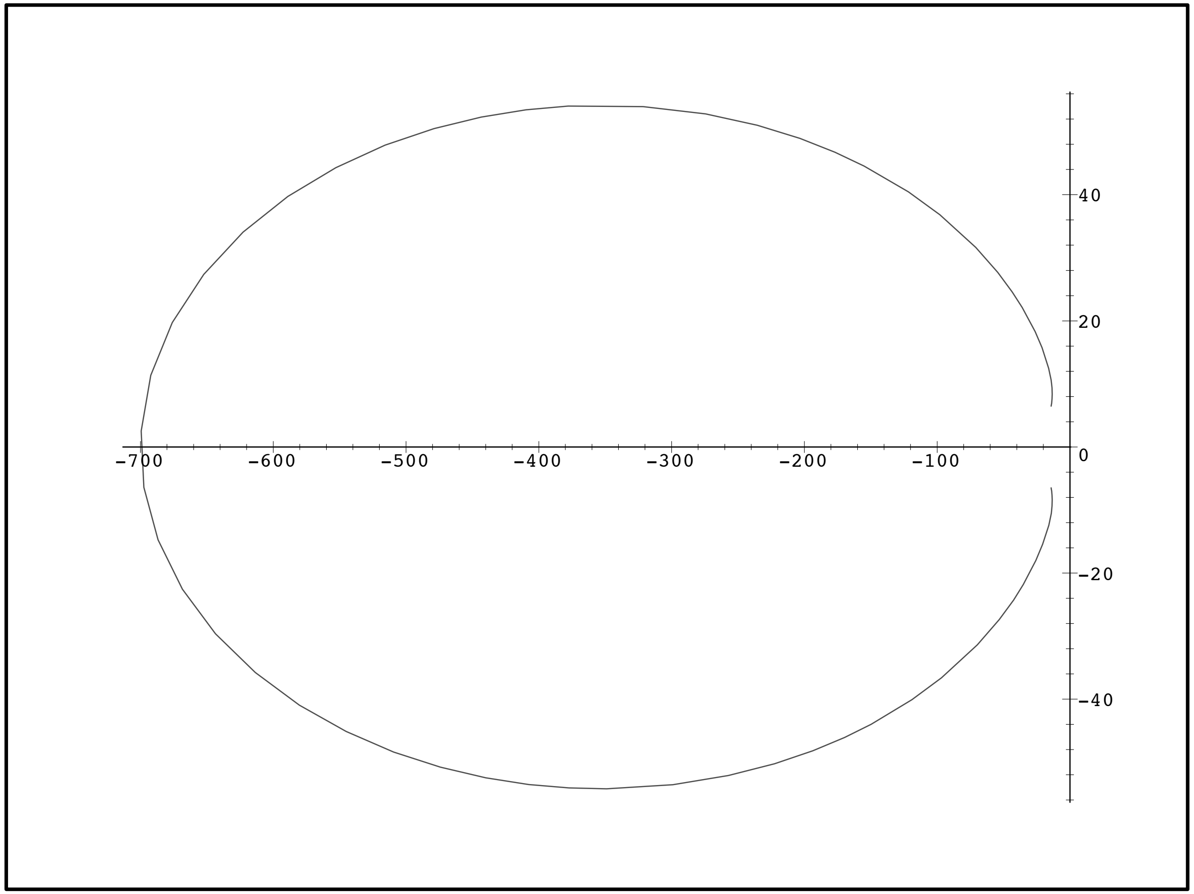


Figure 2: $b = -1/4$

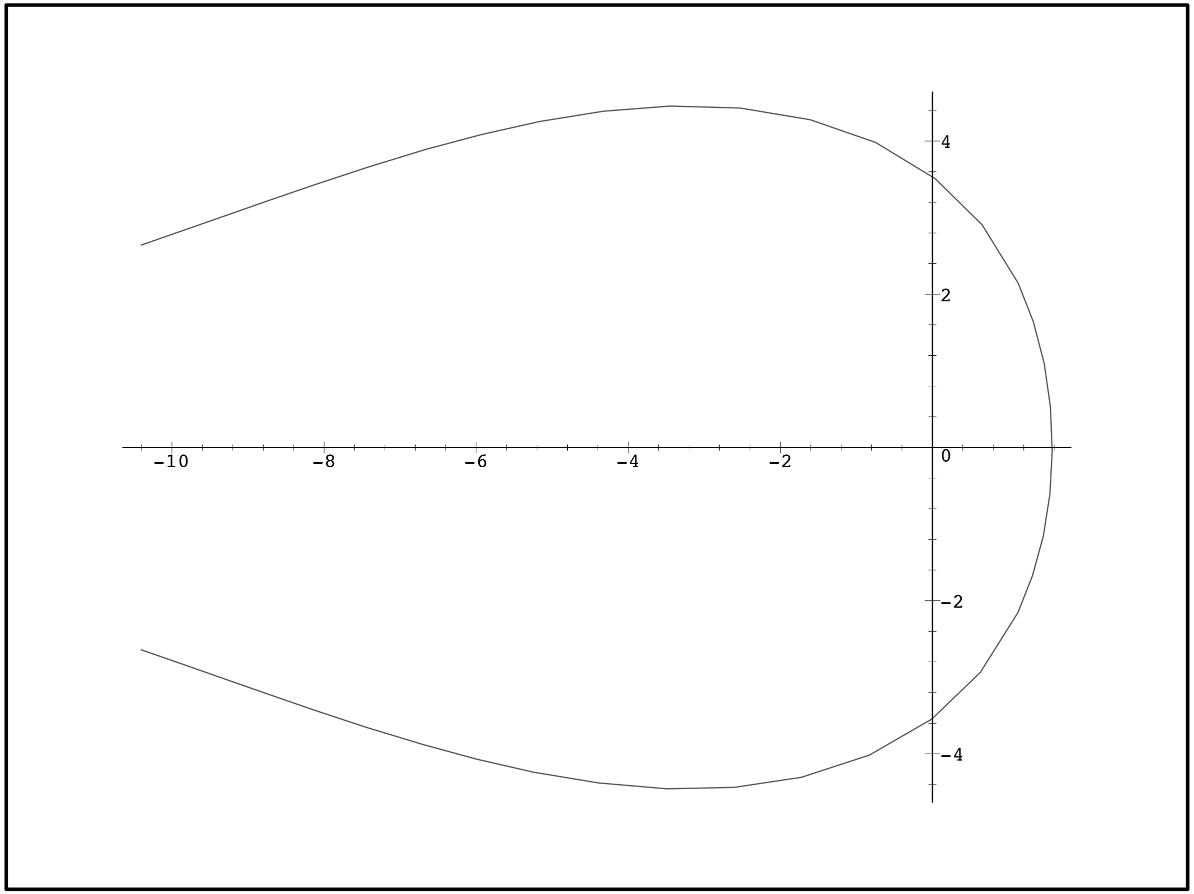


Figure 3: $b = 1$

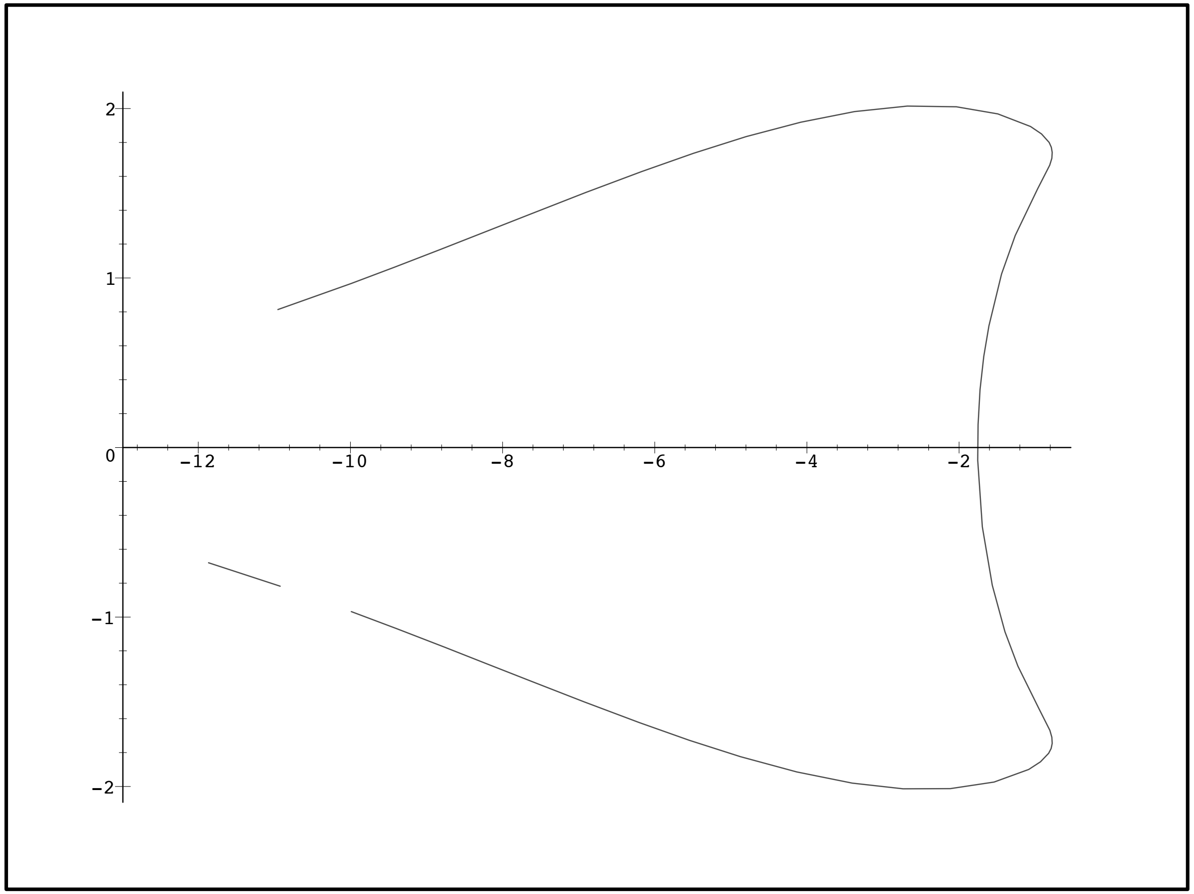


Figure 4: $b = 4/3$

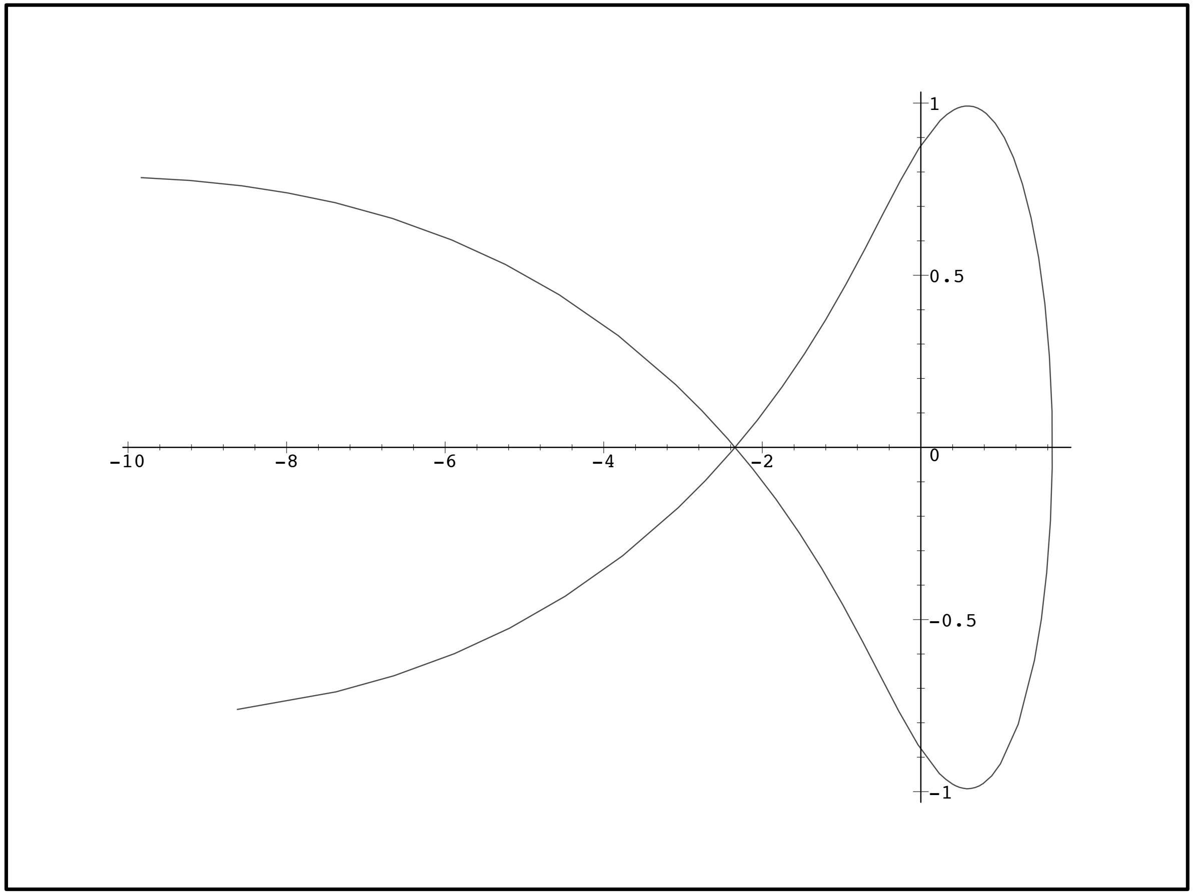


Figure 5: $b = 5/3$

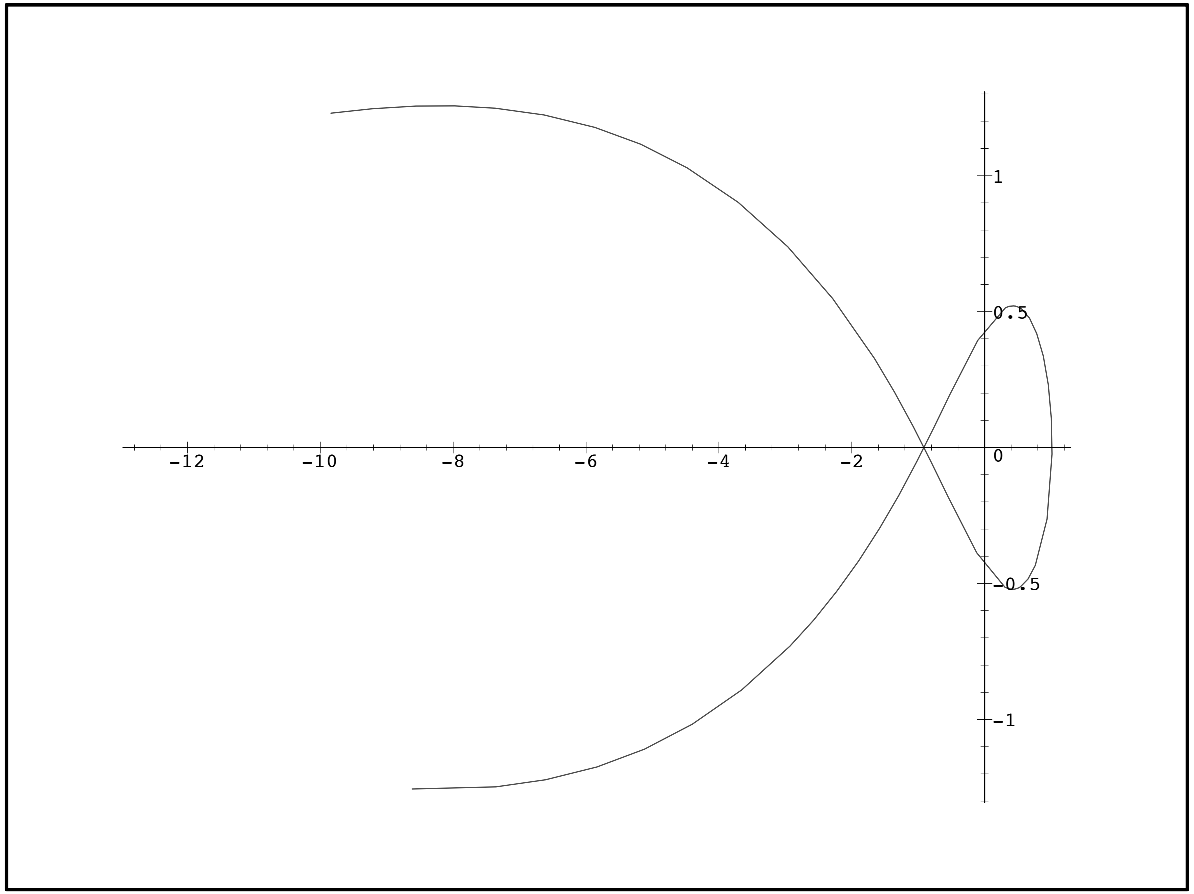


Figure 6: $b = 21/12$

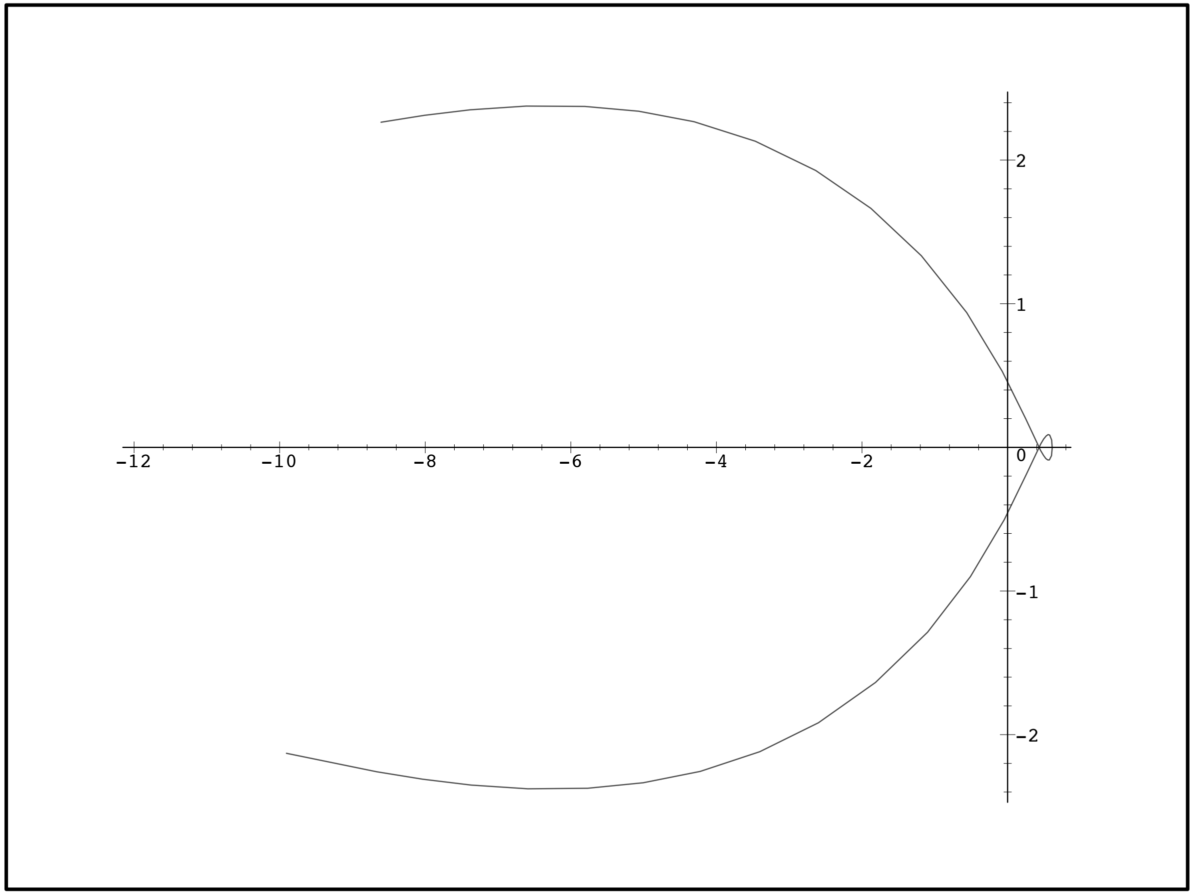


Figure 7: $b = 23/12$

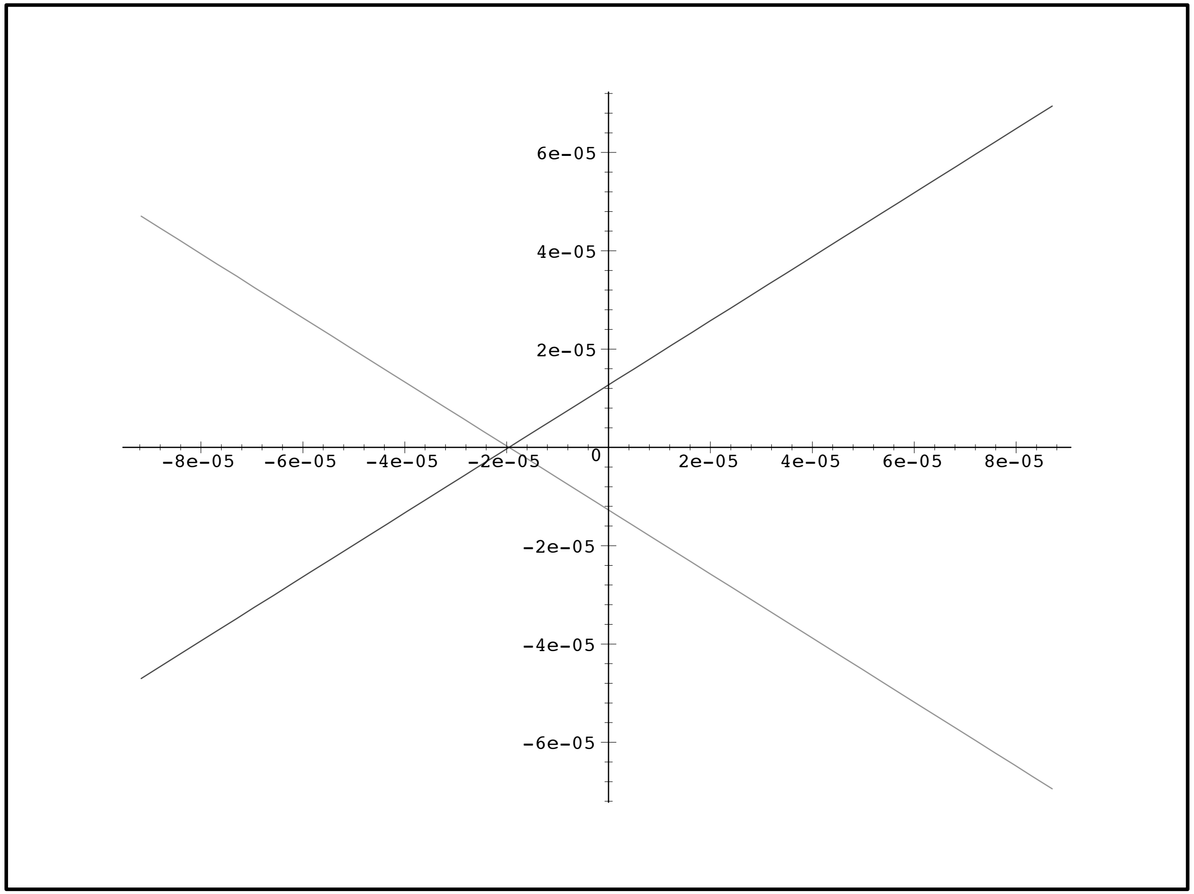


Figure 8: $b = 1.840205625$

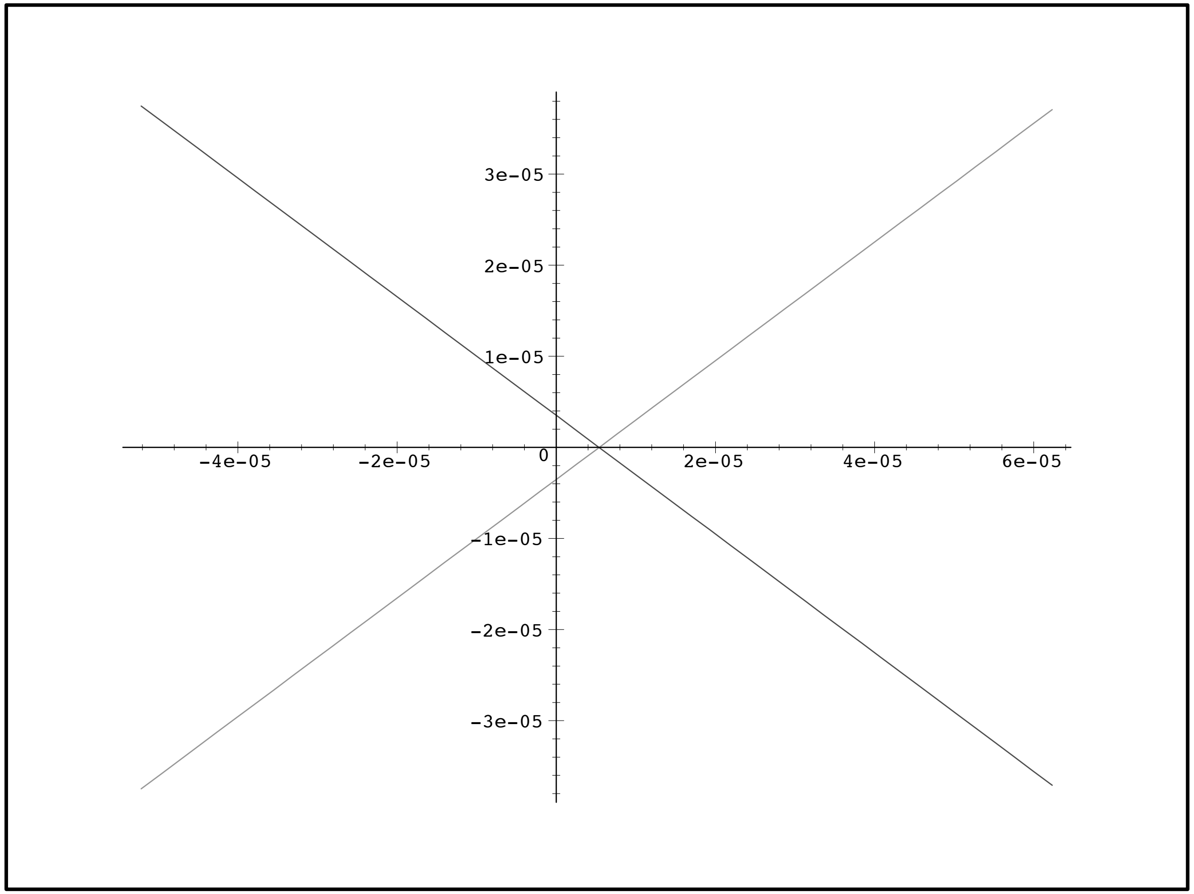


Figure 9: $b = 1.8402088125$

The solution g in the x, t variables

$$G(z,\xi) = \frac{3i}{2\pi\, z} \int_{\mathcal{I}m\, y=\frac{5}{3}} e^{6\pi\alpha(z)\,(y-\xi)}\, \frac{\mathcal{V}(y)}{\mathcal{V}(\xi)} \frac{dy}{\left(e^{6\pi(y-\xi)} - 1\right)}$$

$$\mathcal{V}(\xi) = \exp[-3i \int_{\mathcal{I}m\, y=\frac{4}{3}} \ln\left(\frac{\Phi(y+i0)}{-a}\right) \times$$

$$e^{6\pi y}\left(\frac{1}{e^{6\pi y} - e^{6\pi\xi}} - \frac{1}{e^{6\pi y} - a e^{6\pi\delta i}}\right) dy].$$

In the (t, x) variables: invert Fourier and Laplace transform:

$$g(t,x) = \frac{1}{(2\,\pi)^{3/2}\ i} \int_{c-\infty\,i}^{c+\infty\,i} e^{zt} \left[\int_{-\infty+b\,i}^{\infty+b\,i} e^{ix\xi}\ G(z,\xi)\,d\xi \right] dz,$$

for some suitable choosed $b \in \mathbb{R}$ and $c \in \mathbb{R}$.

In particular we have to choose $\Im m b \in (7/6, 11/6)$ to have good decay estimates on $e^{ix\xi}\ G(z,\xi)$ along the integration path.

Asymptotic behaviour for $x \to -\infty$.

Using the Theorem of residues: deform the integration contour downward until the first pole of $G(z,\xi)$ is reached.
This pole is $\xi = 7i/6$. It follows:

$$
\begin{aligned}
\mathcal{F}^{-1}(G)(z,x) &= e^{-\frac{7x}{6}} h(z) + \frac{1}{\sqrt{2\pi}} \int_{\mathcal{I}m\,\xi=\tilde{b}} e^{ix\xi} G(z,\xi) d\xi \\
h(z) &= \sqrt{2\,\pi}\, i\, \mathcal{R}es\left(G(z,\cdot), \xi = 7i/6\right).
\end{aligned}
$$

The inverse Laplace transform gives then:

$$
g(t,x) \sim \sigma(t)\, e^{-7x/6}, \quad \text{as } x \to -\infty.
$$

Same method for $x \to +\infty$.

More Remarks.

Everything is encoded in the function $\Phi(\xi)$:

• The uniqueness of the solution: from the argument property of Φ along horizantal lines contained in the strip $7/6 < \Im m\xi < 3/2$.

• The persistency of the Dirac measure: comes from the fact that $\Phi(\xi) \to a$ as $|\xi| \to \pm\infty$.

• The decay of the total mass of the Dirac measure: $a > 0$.

• The asymptotic behavior as $x \to \pm\infty$: come from the zeros and poles of Φ.