Bounds on Kolmogorov spectra for the Navier Stokes equations

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- A new estimate on Leray weak solutions
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Navier – Stokes equations

The equations of motion of an incompressible viscous fluid

$$\begin{aligned} &\partial_t u + (u \cdot \nabla) u = \nu \Delta u - \nabla p + f \\ &\nabla \cdot u = 0 \\ &u(x,0) = u_0(x) \ , \ \nabla \cdot u_0 = 0 \qquad \text{initial data} \end{aligned} \tag{1}$$

Forcing term $f : \nabla \cdot f = 0$ Take f to be zero at present

Space-time domain

$$D = \mathbb{R}^3 \qquad (x,t) \in D \times \mathbb{R}^+ := Q$$

Alternatively $D = \mathbb{T}^3$ and

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A bounded smooth domain $D \subseteq \mathbb{R}^3$; we leave this open.

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Weak solutions

The usual definition of a weak solution over $t \in [0,T]$ is that:

1. Integrability conditions

$$u \in L^\infty([0,T];L^2(D)) \cap L^2([0,T];\dot{H}^1(D)) \ ,$$
$$p \in L^{5/3}(Q) \qquad (2)$$

2. The pair (u,p) is a distributional solution of (1)

3. The energy inequality is satisfied

$$\frac{1}{2}\int_D |u(x,t)|^2\,dx + \nu\int_0^t\int_D |\nabla u(x,s)|^2\,dxds \le \frac{1}{2}\int_D |u_0(x)|^2\,dx \qquad (3)$$

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The existence of weak solutions

Theorem (Leray (1934))

Given $u_0 \in L^2(D)$ divergence free, then there exists at least one weak solution to (1) *globally in time. Weak solutions satisfy*

$$u \in L_t^\infty(L_x^2)$$

as well as

$$u \in C_t(L_x^2 : \text{weak} \quad \text{topology})$$

A lot is known about such solutions, for example that

$$u \in L_t^s(L_x^p) \,, \qquad \frac{3}{p} + \frac{2}{s} = \frac{3}{2}$$

Uniqueness and global regularity are unknown

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- The Fourier transform of $u(x,t)$ exists $a.e.\ t$

$$\hat{u}(k,t) = \frac{1}{2\pi^{3/2}} \int e^{-ik\cdot x'} u(x',t)\, dx'$$

and $\hat{u}(\cdot,t) \in L^\infty_t(L^2_x)$

- The Fourier transform is smooth in t

Theorem

The function $\hat{u}(k,t)$ is C^1 as a function of t for every k (when $D = \mathbb{T}^3$ at least).

- Define the energy spectrum as the spherical integrals

$$E(\kappa,t) := \frac{1}{V} \int_{|k|=\kappa} |\hat{u}(k,t)|^2\, dS(k)\ , \qquad 0 \le \kappa < +\infty \tag{4}$$

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$$\int_0^\infty E(\kappa,t)\,d\kappa = V\|u(\cdot,t)\|_{L^2}^2$$

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$$[u] = \frac{L}{T}\,, \quad [|\hat{u}|^2] = \frac{L^{2(d+1)}}{T^2} \quad [\nu] = \frac{L^2}{T} \quad [E(\cdot,t)] = \frac{L^3}{T^2}$$

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Reynold's number

There is a literature on the Reynold's number defined in terms of the energy spectrum

- $Re := \frac{UL}{\nu}$ dimensionless parameter
- Intrinsic Reynold's number (Gammond & Gage)

$$Re_1 := \frac{\Lambda}{\eta_K} , \qquad \Lambda := \frac{\int_0^\infty \kappa^{-1} E(\kappa)\, d\kappa}{\int_0^\infty E(\kappa)\, d\kappa} \qquad \eta_K := \Big(\frac{\nu^3}{\varepsilon}\Big)^{1/4}$$

  for $\varepsilon := 2\nu \int_0^\infty \kappa^2 E(\kappa)\, d\kappa$, the rate of energy dissipation
- Proposal for a mathematical Reynold's number

$$Re_2 := \frac{\|u\|_{\dot{H}^{1/2}}}{\nu}$$

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- The small scale turbulent motions of a fluid are statistically isotropic
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Kolmogorov's scaling law

- Prediction: For high Reynold's number flows which exhibit fully developed turbulence, the energy spectrum has universal behavior

$$E_K(\kappa) = C_0 \varepsilon^{2/3} \kappa^{-5/3} \tag{5}$$

These are the unique exponents for which the dimensions match
In fact the exponents are independent of space dimension

- Considerable experimental and numerical evidence has been garnered to support this conjecture.
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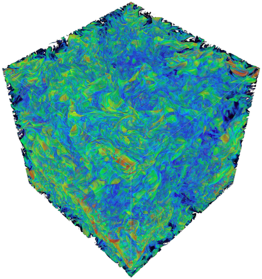
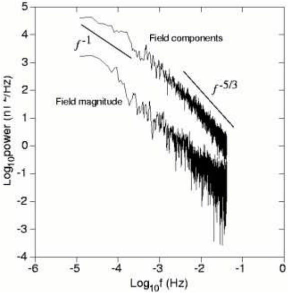
SDSC simulation
by Chowasia, Donzis and Yeung



Turbulence Spectrum



Estimates on weak solutions

- The energy inequality (3) can be viewed as the statement that the ball $B_R(0) \subseteq L^2_x$ is an invariant set for Navier – Stokes flow

$$u_0(\cdot) \in B_R(0) \quad \Longrightarrow \quad \forall t > 0 \, , u(\cdot, t) \in B_R(0)$$

- Another invariant set. Define $A := \{(\hat{u}(k))_{k \in \mathbb{R}^3} \; : \; |k||\hat{u}(k)| < R_1\} \cap B_R(0)$

Theorem (A. Biryuk (2003))

If $R^2 < \nu R_1$ then A is an invariant set for Navier – Stokes flow.

Proof given at end of talk if there is time

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Global estimates on weak solutions

- Bounds on $L^\infty_t(L^\infty_k(\mathcal{F}u))$, supposing that the initial data lies in the set A, then for all $k \in \mathbb{R}^3$,

$$\sup_{t\geq 0} |\hat{u}(k,t)| \leq \frac{R_1}{|k|} \tag{6}$$

- Time average quantities obey better estimates:

Corollary

For all $k \in \mathbb{R}^3$ and all $T \geq 0$, then $\nu \int_0^T |\hat{u}(k,s)|^2\, ds \leq \frac{R_1^2}{|k|^4}$

- The quantity $\sup_t \| |k|\hat{u}(\cdot,t)\|_{L^\infty}$ scales like the BV norm $\sup_t \|\partial_x u(\cdot,t)\|_{L^1}$, for which there are no known bounds. P. Constantin (1992) has a global bound on $\sup_t \|\nabla_x \times u(\cdot,t)\|_{L^1}$

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Estimates on spectra

Proposition (1)

The spectrum of a weak solution with initial data $u_0 \in A$ *satisfies a* *global upper bound*

$$E(\kappa, t) = \frac{1}{V} \int_{|k|=\kappa} |\hat{u}(k,t)|^2 \, dS(k) \leq \frac{R_1^2}{V\kappa^2} 4\pi\kappa^2 = \frac{4\pi R_1^2}{V}$$

Proposition (2)

Time averages of energy spectra have a uniform decay rate. Weak solutions with initial data $u_0 \in A$ *satisfy*

$$\frac{1}{T} \int_0^T E(\kappa, t) \, dt = \frac{1}{VT} \int_0^T \int_{|k|=\kappa} |\hat{u}(k,t)|^2 \, dS(k) dt \leq \frac{4\pi\kappa^2}{T} \frac{R_1^2}{\nu V \kappa^4} = \mathcal{O}(\kappa^{-2})$$

rate of decay

How does the energy spectrum of a solution compare to the Kolmogorov prediction.

Theorem

The exponent 2 is larger than 5/3.

Is this a problem with the theory?

- One resolution could be that Navier – Stokes flows which exhibit spectral behavior like the Kolmogorov law are in the support of a probability measure P on $L^2(D)$-divergence-free.
- And that for all R, R_1 then supp$P \cap A = \emptyset$.

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Better resolution of this dilemma

Bounds on the inertial range where this spectral behavior is manifest

Theorem

The upper and lower bounds for the inertial range $[\kappa_1, \kappa_2]$ *over which the Kolmogorov spectral function* E_K *does not violate our estimates*

$$\kappa_1 = \left(\frac{C_0 V}{4\pi R_1}\right)^{3/5} \varepsilon^{2/5} \tag{7}$$

$$\kappa_2 = \left(\frac{R_1^2}{\nu C_0 V T}\right)^3 \frac{1}{\varepsilon^2} \tag{8}$$

Maximum time for which this behavior persists is $T_0 : \kappa_1 = \kappa_2(T)$

$$T_0 = \frac{R_1^{11/5}}{\nu \varepsilon^{4/5}} \left(\frac{4\pi}{C_0^6 V^6}\right)^{1/5}$$

Comparison with the classical quantities

- Kolmogorov lengthscale $\eta_K := \left(\nu^3/\varepsilon\right)^{1/4}$

$$\frac{2\pi}{\eta_K} = 2\pi\left(\frac{\varepsilon}{\nu^3}\right)^{1/4} < \kappa_2 = \left(\frac{R_1^2}{\nu C_0 VT}\right)^3 \frac{1}{\varepsilon^2}$$

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Comparison with a Navier – Stokes velocity field

there can be various definitions of proximity to $\hat{u}_K(k) \simeq \varepsilon^{1/3}|k|^{11/6}$

- **Definition 1:** $\|u_K - u(\cdot,t)\|_{L^2_x} \leq C_1$.
  Since $\hat{u}_K \notin L^2$ this is not a satisfactory criterion.
- Dyadic decomposition $u = \sum_j \Delta_j u$ with support

$$\mathrm{supp}(\widehat{\Delta_j u}(k)) \subseteq A_j$$

where $A_j := \{2^{j-1} < |k| < 2^{j+1}\}$.

Definition 2: $\|\Delta_j(u - u_K)\|_{L^1} 2^{11j/3} \leq C_2$ for all j in the range $\log_2(\kappa_1) \leq j \leq \log_2(\kappa_2)$

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Definition 2: $\|\Delta_j(u - u_K)\|_{L^1} 2^{11j/3} \leq C_2$ for all j in the range $\log_2(\kappa_1) \leq j \leq \log_2(\kappa_2)$

continued comparison

- There is the question as to whether $E(\kappa, t)$ has spectral behavior for individual solutions, or does it hold in an average sense, over a statistical ensemble of solutions with probability measure P.
- Therefore study the ensemble averages

$$\langle E(\kappa, t)\rangle := \int_{|k|=\kappa} \langle |\hat{u}(k,t)|^2\rangle \, dS(k)$$

Definition 3: Use Definition 2 for ensemble averages of solutions.

- In fact P should be ergodic with regard to NS flow, so that asymptotically the P average should approximate the time average

$$\langle E(\kappa, t)\rangle \simeq \frac{1}{T}\int_0^T E(\kappa, t)\, dt$$

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Theorem (Bounds on Kolmogorov spectra)

In order that $u(x,t)$ exhibit Kolmogorov-like behavior of its spectral energy function, in either of the senses of Definition 2 or Definition 3 over an inertial range $[\kappa_1, \kappa_2]$, then

$$\kappa_1 \ , \quad \kappa_2 \ , \quad T_0$$

must satisfy the above three relations, up to a constant.

proof of the $L^\infty_t(L^\infty_k(\mathcal{F}u))$ estimate

- For fixed k the field $\hat{u}(k) \in \mathbb{C}^2_k \subseteq \mathbb{C}^3$
  Because of incompressibility $k \cdot \hat{u}(k) = 0$
  Suppose that $\|u(\cdot)\|_{L^2} \leq R$
- The Fourier transform satisfies

$$\begin{aligned} \partial_t \hat{u}(k) &= -\nu|k|^2\hat{u}(k) - ik\Pi_k \int \hat{u}(k-k_1)\cdot\hat{u}(k_1)\,dk_1 + \hat{f}(k,t) \\ &:= X(u)_k \end{aligned}$$

- Consider the vector field $X(u)$ when $|\hat{u}(k)| = R_1/|k|$. Then

$$\mathrm{re}(\hat{u}(k)\cdot X(u)_k) < -\nu|k|^2(R_1/|k|)^2 + (R_1/|k|)|k|R^2 + |\hat{f}|(R_1/|k|)$$

which is negative when $R^2 + |\hat{f}(k)|/|k| < \nu R_1$

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proof of corollary

- A fact about the vector field $X(\hat{u})$ is that solutions obey

$$|\hat{u}(k,T)|^2 - |\hat{u}_0(k)|^2 + 2\nu \int_0^T |k|^2 |\hat{u}(k,t)|^2 \, dt$$
$$= 2\mathrm{im}[\int_0^T \bar{\hat{u}}(k) \cdot \int \hat{u}(k-k_1) \cdot k_1 \hat{u}(k_1)) \, dk_1 dt]$$

(setting $f = 0$ for simplicity)

- Writing $I^2(k) = (2\nu)^3 \int_0^T |k|^4 |\hat{u}(k,t)|^2 \, dt$
this gives an inequality

$$I^2(k) - 2R^2 I(k) - (2\nu R_1)^2 \leq 0$$

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Thank you