Good reasons to study Euler equation.C’est un bon sujet pour retraité cela correspond à des phénomènes macroscopiques étudiés depuis 250 ans et cela contient presque toutes les difficultés du non linéaire. Applications often correspond to very large Reynolds number =ratio between *the strenght of the non linear effects* and *the strenght of the linear viscous effects.*

$$\Re = \frac{UL}{\nu}. \Rightarrow \Re \sim 2 \times 10^7 \text{airplanes}$$

A theorem valid for any finite Reynolds number should be in **some cases,** in particular in the absence of boundary, compatible with results concerning infinite Reynolds number. In fact it is the case Reynolds$= \infty$ which drive other results. The parabolic structure and the scalings does not carry enough information to deal with the $3d$ Navier-Stokes equations. *Simple examples with the same scalings but with no conservation of energy may (concerning regularity) exhibit very different behaviour.*

1. Hamilton Jacobi type equation

$$\partial_t \phi - \nu \Delta \phi + \frac{1}{2} |\nabla \phi|^2 = 0 \text{ in } \Omega \times \mathbb{R}_t^+ ,$$
$$\partial_t \nabla \phi - \nu \Delta \nabla \phi + \nabla \phi \cdot \nabla (\nabla \phi) = 0 \text{ in } \Omega \times \mathbb{R}_t^+$$
$$\phi(x,t) = 0 \text{ for } x \in \partial \Omega \, , \phi(.,0) = \phi_0(.) \in L^\infty(\Omega) \, ,$$

For $\nu > 0$ global smooth solution. May become singular (with shocks) for $\nu = 0$.

2. Montgomery-Smith example :

$$\partial_t u - \nu \Delta_D u + \frac{1}{2}(-\Delta)^{\frac{1}{2}} u^2 = 0 \text{ in } \Omega \times \mathbb{R}_t^+ \,,$$

$$u(x,t) = 0 \text{ for } x \in \partial\Omega \,, u(.,0) = u_0(.) \in L^\infty(\Omega) \,,$$

$$\int_\Omega u_0(x,t)\phi_1(x)dx = -m(t) < 0 - \Delta\phi_1 = \lambda_1\phi_1 \,, \phi_1(x) \geq 0$$

$$\frac{d}{dt}\int_\Omega u(x,t)\phi_1(x)dx + \nu\lambda_1 \int_\Omega u(x,t)\phi_1(x)dx = -\frac{\sqrt{\lambda_1}}{2}\int_\Omega u(x,t)^2\phi_1(x)dx \,.$$

$$m(t) = -\int_\Omega u(x,t)\phi_1(x)dx$$

$$(\int u(x,t)\phi_1(x)dx)^2 \leq \int_\Omega u(x,t)^2\phi_1(x)dx \int_\Omega \phi_1(x)dx$$

$$\frac{dm}{dt} + \nu\lambda_1 m \geq \frac{\sqrt{\lambda_1}\int_\Omega \phi_1(x)dx}{2} m^2 \,, m(0) > \frac{2\nu\sqrt{\lambda_1}}{\int_\Omega \phi_1(x)dx} \Rightarrow \text{Blow up}$$

The above example has been introduced with $\Omega = \mathbb{R}^3$ by Montgomery-Smith under the name of “cheap Navier-Stokes equations" with the purpose of underlying the role of the conservation of energy (which is not present in the above examples) in the Navier-Stokes dynamic. His proof shows that the same blow up property may appear in any space dimension for the solution of the “cheap hyper viscosity equations"

$$\partial_t u + \nu(-\Delta)^m u + \frac{1}{2}|\nabla|u^2 = 0$$

On the other hand one should observe that the simple proof given above do not applies to the "cheap hyperviscosity" in a bounded domain with convenient boundary conditions. The reason is that there may be no eigenvector of $(-\Delta)^m$ with a constant sign.

$$\partial_t u(x,t) + u(x,t)\cdot\nabla_x u(x,t) - \nu\Delta u(x,t) = \nabla_x p(x,t)\ \nabla_x \cdot u(x,t) = 0\,.$$
$$\partial_t \omega(x,t) + u(x,t)\cdot\nabla_x \omega(x,t) = \omega(x,t)\cdot\nabla_x u(x,t)$$
$$u\cdot n = 0 \text{ and } \nu u(,t) = 0\,,\ \text{on } \partial\Omega$$
$$\frac{1}{2}\frac{d}{dt}\int |u(x,t)|^2 dx + \nu\int_\Omega |\nabla_x u(x,t)|^2 dx \le 0 \text{ energy estimate}$$
$$\omega = \nabla\wedge u \mapsto \nabla_x u \text{ Zero order pdo}$$

For $\nu = 0$ existence of a smooth solution for finite time with initial data in $C^{1,\alpha}$ or $H^s\,, s \ge \frac{n}{2}+1$. This goes back to Lichtenstein based on the comparison with the Riccati equation and with no use of energy estimate.

$$y' \le C y^{\frac{3}{2}}\,; y = \|u\|^2_{H^s}$$

Importance of Energy estimate :

Blow up for some solutions of infinite energy in $(\mathbf{R}^2/L)^2 \times R$ Constantin

$$u = (u_1(x_1,x_2,t), u_2(x_1,x_2,t), x_3\gamma(x_1,x_2,t)) = (\tilde{u}, x_3\gamma)$$

$$\nabla_x \cdot u = 0 \Rightarrow \nabla_{x_1,x_2}\tilde{u} + \gamma = 0$$

$$\Rightarrow \partial_t \nabla \wedge \tilde{u} + \tilde{u}\nabla \wedge \tilde{u} = \gamma \nabla_x \wedge \tilde{u}$$

vertical component $\Rightarrow \partial_t\gamma + \tilde{u}\nabla\gamma = -\gamma^2 + I(t)$

x_1, x_2 peridodicity $\Rightarrow I(t) = \frac{2}{L^2}\int_{(\mathbf{R}^2/L)^2} (\gamma(x_1,x_2,t))^2 dx_1 dx_2$

A nice Ricatti equation :

$$\Rightarrow \partial_t\gamma + \tilde{u}\nabla\gamma = -\gamma^2 + \frac{2}{L^2}\int_{(\mathbf{R}^2/L)^2} (\gamma(x_1,x_2,t))^2 dx_1 dx_2$$

Instability above the H^s , $s > n/2 + 1$ threshold : In $W^{1,p}$ for all $1 < p \leq \infty$
Exemple on the Torus with pressure less fluid :

$$
\begin{aligned}
&(u_1(x_2), 0, u_3(x_1 - tu_1(x_2), x_2))\\
&\partial_{x_2} u_3(x_1, x_2, 0) = \partial_{X_2} u_3(x_1 - tu_1(x_2), x_2))\\
&-t\partial_{X_1} u_3(x_1 - tu_1(x_2), x_2))\partial_{x_2} u_1(x_2)
\end{aligned}
$$

This give examples of unstable solutions but in the mean time No proof of existence of solution for the Cauchy problem with initial data below H^s No existence proof concerning weak solutions. In $2d$

$$\omega \cdot \nabla_x u = 0 \Rightarrow \partial_t \omega + u \cdot \nabla_x \omega = 0 \qquad (1)$$

$\Rightarrow$ Existence of weak solutions with initial data
$\omega(x, 0) = \omega_0(x) \in L^p 1 < p \leq \infty$ or for ω_0 a signed measure (Delors).
Uniqueness for $\omega_0 \in L^\infty$ (Youdovitch)

Examples in $2d$ and $3d$ of very bad solutions $u \in L^\infty(\mathbb{R}_t, L^2(\mathbb{R}^n)$ with space time compact support ! ! Constructed by accumulating oscillations (explicit construction by Scheffer and Shnirelman) less explicit uses first plane waves solution (as Luc Tartar) then sum of plane waves and finally Baire theorem.

Theorem For a any bounded open domain $O \subset \mathbb{R}^n_x \times \mathbb{R}_t$ there exists a solution $(u,p) \in L^\infty(\mathbb{R}^n_x \times \mathbb{R}_t)$ with

$$x \in O \Rightarrow |u(x,t)| = 1\,, x \notin O \Rightarrow |u(x,t)| = 0$$

This solution is "strong" there exists smooth $u_k, \nabla_x u_k = 0$ converging in L^p to (u,p) with

$$\partial_t u_k + P(u_k \cdot \nabla_x u_k) \rightarrow 0 \text{ in } H^{-1}, P \text{ Leray projector}$$

To avoid the above examples and the lack of existence of smooth solution one may use (PL Lions) dissipative solutions :

Let $u(x,t)$ be a smooth solution of the Euler equation in Ω and $w(x,t)$ any smooth, tangent to the boundary, divergence free, vector field then with $E(v) = \partial_t w + P(w\cdot\nabla_x w)$ one has

$$\partial_t u + \nabla_x(u\otimes u) + \nabla_x p = 0$$
$$\partial_t w + \nabla_x(w\otimes w) + \nabla_x q = E(w)$$
$$\frac{d|u-w|^2}{dt} + 2(D(w)(u-w),(u-w)) = 2(E(w),u-w)$$

Or by integration :

$$|u(t)-w(t)|^2 \le e^{\int_0^t 2\|D(w)\|_\infty(s)ds}|u(0)-w(0)|^2$$
$$+2\int_0^t e^{\int_s^t 2\|D(w)\|_\infty(\tau)d\tau}(E(w),u-w)(s)ds$$

- Any classical solution is a dissipative solution.

- With $w = 0$ one obtains for the dissipative solution the relation $|u(t)|^2 \leq |u(0)|^2$ this is not in contradiction with some non conservation of energy coming from lack of regularity and justifies the name dissipative. Scheffer Shnirelman De Lellis and Szekelyhidi are not dissipative solutions.

- Assume that w is a classical solution and u a dissipative solution then one has

$$|u(t) - w(t)|^2 \leq e^{\int_0^t 2\|D(w)\|_\infty(s)ds}|u(0) - w(0)|^2$$

in particular if there exists a classical solution any dissipative solution with the same initial data coincide with it.

• **In the absence of boundary** Any family of Leray solutions of $3d$ Navier-Stokes equations converge to a dissipative solution. In particular if there exists for $0 < t < T$ a smooth solution of Euler equation this is the limit of any sequence and the energy dissipation goes to zero with ν :

$$\lim_{\nu \to 0} \nu \int_0^T \int |\nabla_x u(x,t)|^2 dx dt = 0$$

Boundary value problem for Navier Stokes

The limit $\nu \to 0$ for solution of the Navier Stokes equation in a domain $\Omega \subset \mathbf{R}^d, \quad d = 2, 3$ with homogenous boundary condition. The Dirichlet boundary condition $u_\nu = 0$ is not the easiest one however it is the one to consider. It can be deduced in the smooth regime from the Boltzmann equation when the interaction with the boundary is described by a scattering kernel. It generates the pathology which is observed in physical experiments. Problem come from the boundary layer generated by the jump in tangential component of the vorticity. The non linearity may imply the propagation of this jump inside the media. The problem si basically open even in $2d$ when one has with smooth initial data smooth solution both for Euler and Navier Stokes.

Theorem Kato Consider the two problems for divergence free flows ($\nabla \cdot u = 0$

$$\partial_t u_\nu - \nu \Delta u_\nu + \nabla(u_\nu \otimes u_\nu) + \nabla p_\nu = 0\,, u_\nu(x,0) = u_0(x)\,, u_\nu(x,t) = 0 \text{ on } \partial\Omega$$
$$\partial_t u + \nabla(u \otimes u) + \nabla p_\nu = 0\,, u(x,0) = u_0(x)\,, u(x,t)\cdot n = 0 \text{ on } \partial\Omega$$

With smooth solution for $0 < t < T$ for Euler equation are equivalent :

$$(i)\ u_\nu(t) \to \overline{u}(t) \text{ in } L^2(\Omega) \text{ uniformly in } t \in [0,T]$$
$$(ii)\ u_\nu(t) \to u(t) \text{ weakly in } L^2(\Omega) \text{ for each } t \in [0,T]$$
$$(iii)\ \nu \int_0^T \int_\Omega |\nabla u_\nu(x,t)|^2 dxdt \to 0$$
$$(iv)\ \nu \int_0^T \int_{\Omega \cap \{d(x,\partial\Omega) < \nu\}} |\nabla u_\nu(x,t)|^2 dx \to 0$$
$$(v)\ \nu \frac{\partial u_\nu^{tang}}{\partial n} \to 0 \text{ in } \mathcal{D}'(\partial\Omega \times [0,T])$$

Proof energy estimates and for (v) boundary layer correction.

Il y a moyen de donner des conditions suffisantes de convergeance mais ceci en construisant la couche limite de Prandlt. Ce qui exige l'analyticité des données initiales. Les solutions ainsi construites satisfont bien sur au critère de Kato. Mais la construction n'est valable en général que pour un temps fini, appartion de singulairtés

Basic open problems and tools.

- Appearance of singularity for smooth solutions of $3d$ Euler equation.

- Existence of weak solutions in $3d$ for initial data less regular than $H^{\frac{5}{2}}$ This is an important issue for at least two reasons.

1 Several problems like the appearance of coherent structures are generated by turbulent initial data.

2 If singularities appears for smooth solutions at a time T then extension of the solution after this time implies weak solutions ! !

• Uniqueness of weak solutions. With the exemples of Scheffer, Shnirleman and De Lellis the only good candidate seems to be the dissipative solutions then uniqueness follows from the uniqueness for Navier-Stokes OK in $2d$ in $3d$? ? ? Très différent de ce qui se passe pour Euler compressible ou pour Hamilton Jacobi

• Criteria for conservation or decay of energy for the Euler equation (conservation of energy for Leray solution is still an open problem)

1 Exhibit genuine dissipative solutions that would be weak solutions.

2 Give criteria for conservation of energy. Le problème de la conservation de l'énergie est toujours ouvert pour les solutions de Leray

This is related the introduction of a filter $\widehat{G}(k)$

$$G_l = l^3 G(\frac{x}{l})\,, u_l = G \star u, \tau_l = G_l \star (u \times u) - (G_l \star u) \otimes (G_l \star u)\,,$$
$$\Pi_l = -\nabla_x u_l : \tau_l$$
$$\text{with } u \in L^3(\mathbb{R}^3 \times \mathbb{R}_t)\,,\ \partial_t \frac{|u|^2}{2} + \nabla_x \cdot (\frac{|u|^2}{2} + p)u = \lim_{l \to 0} \Pi_l(u)$$

• Convergence of the solution of Navier-Stokes to the solution of Euler equation in the presence of boundary and of a smooth solution of the Euelr equation

Absence de convergeance ⇔ Décroissance d'énergie

Tools

Conservation of energy

$$\partial_t \int \frac{|u_\nu|^2}{2} dx + \nu \int \|\nabla_x u_\nu\|^2 dx \leq 0$$

With this estimate one may have :

$$\lim_{\nu \to 0} (u_\nu \otimes u_\nu) \neq (\lim_{\nu \to 0} u_\nu) \otimes (\lim_{\nu \to 0} u_\nu)$$

Lien avec la théorie statistique de la turbulence.

On représente les vitesses du fluide sous la forme $u(x,t) = U + \tilde{u}$ avec $\tilde{u}$ une variable aléatoire de moyenne nulle. On introduit des moments $\langle . \rangle$.

Dans la théorie il s'agit de moyennes d'ensemble mais dans la pratique on prend aussi la moyenne par rapport à un nombre fini de réalisations (numériques ou expérimentales), la moyenne spatiale ou temporelle sur une réalisation ce qui conduit à supposer que l'on dispose d'un théorème ergodique. Enfin on suppose que ces constructions sont uniformes par rapport à ν. Ce qui me semble le point le plus délicat à prouver

- Kolmogorov On suppose que les corrélations d'ordre 2 sont homogènes et isotropes :

$$\langle (u(x+r) - u(x)) \otimes (u(x+r) - u(x)) \rangle$$

ne dépendent que du module de r et sont invariant par transformation Galiléenne alors si dans une région dite zone inertielle elles suivent une loi de puissance on a :

$$\langle |u(x+r) - u(x))|^p \rangle^{\frac{1}{p}} \simeq C\epsilon r^{\frac{1}{3}} \;, \epsilon = \nu \langle \|\nabla u\|^2 \rangle$$
$$\nu^{\frac{3}{4}} \epsilon^{-\frac{1}{4}} \leq r \leq L$$

Cela veut dire (avec les hypothèses d'uniformité vérifées et ϵ ne tendant pas vers 0 avec ν) qu'avec probabilité 1 les solutions limites pour $\nu \to 0$ sont dans $B_p^{\alpha,\infty}$ pour $\alpha < 1/3$ et $\alpha = 1/3$ est le cas limite. Conséquences Cohérences

En l'absence de frontière cela indique que l'on obtient à la limite des solutions faibles dans $B_p^{\frac{1}{3},\infty}$ si et seulement si la dissipation d'énergie ne tend pas trop vite vers zéro ou mieux reste bornée inférieurement.

1 La relation ci dessus ne doit pas être valable pour une suite u_ν bornée dans $B_p^{\alpha,\infty}$ pour $\alpha > 1/3$ et en particulier la dissipation d'énergie d'énergie dans l'espace entier doit tendre vers zéro. Prouvé par Constantin Edriss Titi with the hypothesis

$$u_\nu \text{ borné dans } L^3(0,T;B_3^{\alpha,\infty})/,,\alpha > \frac{1}{3}$$

2 Une donnée initiale $u \in C^{\frac{1}{3}}$ telle que (Eyink)

$$G_l = l^3 G(\frac{x}{l})\,, u_l = G \star u, \tau_l = G_l \star (u \times u) - (G_l \star u) \otimes (G_l \star u)\,,$$
$$\lim_{l \to \infty} \Pi_l = -\nabla_x u_l : \tau_l \neq 0$$

mais sans construction de solution correspondante d'Euler.

3 Dans le cas de conditions aux limites (no slip) l'hypothèse (la loi) de Kolmogorov n'exclut pas que u_ν converge (fortement $L^2(\Omega \times [0,T))$ vers une solution dans $L^2(0,T; B_3^{\alpha,\infty})/,,\alpha < \frac{1}{3}$ avec

$$\lim_{\nu\to 0} \nu \int_0^T \int_\Omega |\nabla_x u_\nu(x,t)|^2 dxdt > 0 \tag{2}$$

Meme en présence d'une solution régulière d'Euler (est celle ci la bonne ?) Ceci n'est pas contradictoire avec l'énoncé de Kato.

On considère les dfférentes réalisation du fluide comme des variables aléatoires et on prend des moyennes $\langle .,. \rangle$ En pratique :

- Un nombre N d'observation par

For proofs Use energy estimate and a for (v) a boundary layer correction.

It does not seems that anything more can be said in full generality. This is related to the lack of well posedness for the Prandlt problem. Consider in $2d$

With no boundary (periodic or the whole space) the existence on an intervall $[0, T]$ of a smooth solution implies both the convergence of any sequence of Navier Stokes solution

With the energy equality (inequality) the solution of the NS converges weakly to a function $\overline{u}$

$$\partial_t u_\nu - \nu \Delta u_\nu + \nabla (u_\nu \otimes u_\nu) = -\nabla p_\nu \,, \tag{3}$$

$$\nabla \cdot u_\nu = 0 \,, u_\nu(x,0) = u_0(x) \text{ given} \tag{4}$$

$$\frac{1}{2}|u(.,t)|^2 + \int_0^t \int_\Omega |\nabla u(x,t)|^2 dx = \frac{1}{2}|u(.,0)|^2 \tag{5}$$

the obstruction of to $\overline{u}$ being a weak solution of the Euler equation lies in the expression :

$$\overline{u_\nu \otimes u_\nu} \text{ may be } \neq \overline{u_\nu} \otimes \overline{u_\nu}$$

With no boundary convergence to a strong solution as long as such solution exists.

Defect of convergence related to a deterministic version of the Kolmogorov spectra :

$$\overline{u_\nu \otimes u_\nu} = \overline{u_\nu} \otimes \overline{u_\nu} + \overline{(u_\nu - \overline{u_\nu}) \otimes (u_\nu - \overline{u_\nu}))} \tag{6}$$

$$RT(\overline{u}) = \overline{(u_\nu - \overline{u_\nu}) \otimes (u_\nu - \overline{u_\nu}))}; \tag{7}$$

$$\partial_t \overline{u} + \nabla \overline{u} \otimes \overline{u} + \nabla RT(\overline{u}) + \nabla p = 0 \tag{8}$$

Now $RT(\overline{u})(x,t)$ is a symmetric positive measure valued matrix.

Extend u_ν by zero outside Ω and introduce the deterministic correlation spectra or Wigner transform at the scale $\sqrt{\nu}$:

$$R\widehat{T(}u_\nu)(x,t,k) = \frac{1}{2\pi^d} \int_{\mathbf{R}_y^d} e^{iky} \nu(x - \frac{\sqrt{\nu}}{2} y) \otimes u_\nu(x + \frac{\sqrt{\nu}}{2} y) dy \tag{9}$$

and its weak limit $RT(\overline{u})(x,t,k)$ one has

$$\int_{\mathbf{R}^d_k} RT(\overline{u})(x,t,k)dk = RT(\overline{u})(x,t)$$

$$\phi(x)^2 RT(\overline{u})(x,t) = \overline{\frac{1}{2\pi^d}\int_{\mathbf{R}^d_y} e^{iky}(\phi u_\nu)(x-\frac{\sqrt{\nu}}{2}y)\otimes(\phi u_\nu)(x+\frac{\sqrt{\nu}}{2}y)dy}$$

RT is a local object with **local isotropy : Galilean invariance** one has

$$\widehat{RT}(\overline{u})(x,t,k) = \frac{E(x,t,|k|)}{|k|^{d-1}}(Id - \frac{k\otimes k}{|k|^2}) \tag{10}$$

Furthermore under the assumption (traditional in turbulence) that RT is invariant under rotation and depends only on the tensor $(\nabla\overline{u}+\nabla^\perp\overline{u})$ one has in $3d$ the formulas :

$$\widehat{RT}(\overline{u})(x,t,k) = a(|(\nabla\overline{u}+\nabla^\perp\overline{u})|)Id + \mu(|(\nabla\overline{u}+\nabla^\perp\overline{u})|)(\nabla\overline{u}+\nabla^\perp\overline{u}) + \lambda(|(\nabla\overline{u}+\nabla$$

With in $2d$ $\lambda \equiv 0$

Remark and open problems

• Observe that the term $a(|(\nabla \overline{u} + \nabla^{\perp} \overline{u})|) Id$ can be absorbed in the pressure gradient.

• When ever there is a smooth solution for the Euler equation (with no boundary) $RT(\overline{u}) \equiv 0$

• Taking in account all the hypothesis made giving a formula that would relate μ and λ and the spectra $E(|k|, t, x)$ should be a first step in a mathematical treatment of turbulent models say like $\epsilon - k$

- A second step would be to prove that in any case and for smooth initial data the defect measure is invariant under rotation and therefore the formula (9) is valid.

- Denote by

$$\epsilon(t) = \epsilon(T,\nu)(t) = \int_\Omega |\nabla u_\nu(x,t)|^2 dx\,, \quad \int_0^\infty \epsilon(t)dt \le \frac{1}{2}|u(.,0)|^2$$

then the determinist counter part of the statistical Kolmogorov law would be the relation

$$RT\widehat{(}u_\nu)(x,t,k) \simeq 1.5\epsilon(t)^{\frac{2}{3}}|k|^{-\frac{5}{3}}\frac{(Id - \frac{k\otimes k}{|k|^2})}{|k|^2} \text{ for } \nu \to 0$$

The existence of an estimate of the type () will imply the convergence to a weak solution (even if no regular solution do exist). The existence of the smooth solution will give

$$\lim_{\nu \to 0} \int_0^T \int_\Omega |\nabla u_\nu(x,t)|^2 dxdt = 0$$

Boundary value problem for Navier Stokes

To goal is to describe the very few results that do exist for the limit $\nu \to 0$ for solution of the Navier Stokes equation in a domain $\Omega \subset \mathbf{R}^d\,, \quad d = 2, 3$ with homogenous boundary condition The Dirichlet boundary condition $u_\nu = 0$ is not the easiest one (for instance in $2d\; u \cdot n = 0$ and $\nabla \wedge u = 0$ on the boundary are much easier for our purpose). However it is the one to consider.

1 It can be deduced in the smooth regime from the Boltzmann equation when the interaction with the boundary is described by a scattering kernel.

2 It generates the pathology which observed in physical experiments. Problem come from the boundary layer generate by the jump in tangential component of the vorticity. The non linearity may imply the propagation of this jump inside the media.

Proposition In any case (modulo extraction of subsequence) u_ν converges weakly to an interior dissipative solution $\overline{u}$.

This is not enough because for an interior dissipative solution w the test functions are of compact support or at least should belong to $H^1_0(\Omega)$ and and the solution u is not zero on the boundary.

Theorem Kato Consider the two problems for divergence free flows ($\nabla \cdot u = 0$

$$\partial_t u_\nu - \nu \Delta u_\nu + \nabla (u_\nu \otimes u_\nu) + \nabla p_\nu = 0 \,, u_\nu(x,0) = u_0(x) \,, u_\nu(x,t) = 0 \text{ on } \partial\Omega \quad (12)$$

$$\partial_t u + \nabla (u \otimes u) + \nabla p_\nu = 0 \,, u(x,0) = u_0(x) \,, u(x,t) \cdot n = 0 \text{ on } \partial\Omega \quad (13)$$

Assume that the equation (12) has a smooth solution for $0 < t < T$ then the following fact are equivalent :

(i) $u_\nu(t) \to \overline{u}(t)$ in $L^2(\Omega)$ uniformly in $t \in [0, T]$

(ii) $u_\nu(t) \to u(t)$ weakly in $L^2(\Omega)$ for each $t \in [0, T]$

(iii) $\nu \int_0^T \int_\Omega |\nabla u_\nu(x,t)|^2 dx \to 0$

(iv) $\nu \int_0^T \int_{\Omega \cap \{d(x,\partial\Omega) < \nu\} |\nabla u_\nu(x,t)|^2 dx \to 0}$

(v) $\nu \frac{\partial u_\nu^{tang}}{\partial n} \to 0$ in $\mathcal{D}'(\partial\Omega \times [0, T])$

(vi) $u_\nu(t) \to \overline{u}(t)$ weakly in $L^2(\Omega)$ for each $t \in [0, T]$ $\overline{u}(t)$ is smooth and $RT(\overline{u})(x, t, k) \equiv 0$

For proofs Use energy estimate and a for (v) a boundary layer correction.

It does not seems that anything more can be said in full generality. This is related to the lack of well posedness for the Prandlt problem. Consider in $2d$

The Prandlt Boundary Layer $\epsilon = \sqrt{\nu}$

$$\partial_t u_1^\epsilon - \epsilon^2 \Delta u_1^\epsilon + u_1^\epsilon \partial_{x_1} u_1^\epsilon + u_2^\epsilon \partial_{x_2} u_1^\epsilon + \partial_{x_1} p^\epsilon = 0$$

$$\partial_t u_2^\epsilon - \epsilon^2 \Delta u_2^\epsilon + u_1^\epsilon \partial_{x_1} u_2^\epsilon + u_2^\epsilon \partial_{x_2} u_2^\epsilon + \partial_{x_2} p^\epsilon = 0$$

$$\partial_{x_1} u_1^\epsilon + \partial_{x_2} u_2^\epsilon = 0 \quad u_1^\epsilon(x_1, 0) = u_2^\epsilon(x_1, 0) = 0 \text{ on } x_1 \in \mathbf{R}$$

$$X_1 = x_1, X_2 = \frac{x_2}{\epsilon},$$

$$\tilde{u}_1(x_1, X_2) = u_1(x_1, X_2), \tilde{u}_2(x_1, X_2) = \epsilon u_2(x_1, X_2)$$

$$\partial_t \tilde{u}_1 - \nu \partial_{x_2}^2 \tilde{u}_1 + \tilde{u} \partial_{x_1} u_1 + \tilde{u}_2 \partial_{x_2} \tilde{u}_1 + \partial_{x_1} \tilde{p} = 0$$

$$\partial_{x_2} \tilde{p} = 0, \tilde{p}(x_1, x_2, t) = \tilde{p}(x_1, t) \quad \partial_{x_1} \tilde{u}_1 + \partial_{x_2} \tilde{u}_2 = 0$$

$$\tilde{u}_1(x_1, 0) = \tilde{u}_2(x_1, 0) = 0 \text{ for } x_1 \in \mathbf{R}$$

$$\lim_{x_2 \to \infty} \tilde{u}_2(x_1, x_2) = 0, \lim_{x_2 \to \infty} \tilde{u}_1(x_1, x_2) = U(x_1, t)$$

$$\frac{|U(x_1, t)|^2}{2} + \tilde{p}(x_1, t) = \text{Constant}$$

- The validity of the Prandlt approximation is consistant with the Kato theorem

$$\nu \int_0^T \int_{\Omega \cap d(x,\partial\Omega) \leq c\nu} |\nabla \wedge u_\nu(x,s)|^2 dx ds \leq C\sqrt{\nu}$$

- As long as the solution of of the Prandlt equations is smooth the solution of Navier Stokes converges to the solution of the Euler equation.

- The possibility of detachment and recirculation appear in the fact that the Prandlt problem is highly unstable.

- If initially the velocity profile is monotone then (Oleinik) global solution do exist for the Prandlt equation.

- The loss of regularity in finite time is proven for some class of non monotone initial profile (E W. B. Enquist)

- The existence of solution which is not described by the Prandlt equation has been constructed by Grenier using unstable solutions of the $2d$ Euler equation.

- With analytic initial data (in fact analytic with respect to the tangential variable is enough) one can prove abstract version of the Cauchy Kowalewsky theorem the existence of a smooth solution of the Prandlt equation for a fintie time and the convergence to the solution of the Euler equation during this same time (Asano, Caflisch-Sammartino and Cannone-Lombardo-Sammartino)

Relation between Prandlt and Kelvin Helmholtz

Comparison beetwen Prandlt and Kelvin Helmholtz problems in $2d$

The Kelvin Helmholtz problem concerns the evolution of a solution of the $2d$ Euler equation

$$\partial_t u + \nabla(u \otimes u) + \nabla p = 0\,, \quad \nabla \cdot u = 0 \tag{14}$$

with initial vorticity Ω being a measure concentrated on a curve

- Existence of a weak solution when $\Omega(x,0)$ is a Radon measure with distinguished sign (Delort) or with controlled change of sign (Lopes Filho, Nussenzveig Lopes and Xin). This is impaired by the non uniqueness result of Shnirelman

- For density located on a smooth curve the problem is equivalent to the Birkhoff Rott equation in the complex plane.

$$\partial_t \overline{z} = \frac{1}{2\pi i} p.v. \int \frac{d\lambda'}{z(t,\lambda) - z(t,\lambda')} \tag{15}$$

- As for the Prandlt one has for (14) a local in time existence uniqueness result in the class of analytic initial data (B. Frisch Sulem and Sulem) with a version of the Cauchy Kowalewsky theorem.

- As for the Prandlt equation one can construct a solution with blows up in finite. Use reversibility write (Caflish Orellana)

$$z(t,\lambda) = \lambda + \epsilon s_0 + r(\lambda,t) \tag{16}$$

with s_0 given by :

$$s_0(\lambda,t) = (1-i)\{(1-e^{-\frac{t}{2}-i\lambda})^{1+\nu} - (1-e^{-\frac{t}{2}+i\lambda})^{1+\nu}\} \quad (17)$$

$\epsilon > 0$ small enough , $r(\lambda,t)$ proven to be analytic for $t > 0$ and much smaller $(O(\epsilon^2))$ in $C^2(\lambda)$ Since $s_0(\lambda,0) \sim \lambda^{1+\nu}$:

$$z(\lambda,0) \in C^{1+\nu}\,, z(\lambda,0) \notin C^{1+\nu'} \text{ for } \nu' > \nu \quad (18)$$

- Different with some regularity ($C_t^{\alpha}(C_{\lambda}^{1+\nu}$) the problem can be “locally " reduced to the non linear perturbation of an elliptic equation :

$$\Omega(0,0)=1\,, z(\lambda,t)=(\alpha t+\beta(\lambda+\epsilon f(t,\lambda)) \text{ for } \sup\{|\lambda|,|t|\} \le M, f(0,0)=\nabla f(0,0)=0\,.$$

$$\epsilon|\beta|^2\partial_t\overline{f} = \frac{1}{2\pi i}p.v.\int_{z(t,\lambda')\in} \frac{\Omega(t,\lambda')d\lambda'}{(\lambda-\lambda')(1-\epsilon\frac{f(t,\lambda)-f(t,\lambda')}{\lambda-\lambda'})} + E(t,r(t\lambda)) \quad (19)$$

$r \mapsto E(t, r)$ denoting analytic functions in .

Next one use the expansion

$$\frac{1}{2\pi}pv\int \frac{d\lambda'}{(\lambda-\lambda')(1+\epsilon\frac{f(\lambda,t)-f(\lambda',t)}{\lambda-\lambda'})}d\lambda'$$

$$= \frac{\epsilon}{2\pi}\int \frac{f(\lambda,t)-f(\lambda',t)}{(\lambda-\lambda')^2}d\lambda' + \sum_{n\geq 2}\frac{\epsilon^n}{2\pi}\int \frac{(f(\lambda,t)-f(\lambda',t))^n}{(\lambda-\lambda')^{(n+1)}}d\lambda' .$$

and the formulas (Hilbert transform)

$$\frac{1}{2\pi}\int \frac{f(\lambda,t)-f(\lambda',t)}{(\lambda-\lambda')^2}d\lambda' = -\frac{i}{2}sign(D)f\ , \ \frac{1}{2\pi}vp\int \frac{f(\lambda,t)-f(\lambda',t)}{(\lambda-\lambda')^2}d\lambda' = |D|f$$

and deduce from (**??**) and (**??**) that the real and imaginary part of

$f(t,\lambda) = X(t,\lambda) + iY(t,\lambda)$ are in $(' \subset\subset)$ solutions of the system :

$$\partial_t X = \frac{\Omega_0^2}{2\beta^2}|D_\lambda| Y + \epsilon R_1(X,Y) + E_1(t,\lambda) \quad (20)$$

$$\partial_t Y = \frac{\Omega_0^2}{2\beta^2}|D_\lambda| X + \epsilon R_2(X,Y) + E_2(t,\lambda) \quad (21)$$

The consequence is that a non analytic solution as the one given by Caflsich and Orellana cannot remain in $C_t^{\alpha}(C_\lambda^{1+\nu})$ after the apparition of an Holder singularity it has to be more singular. In this case the challenge is threshold of regularity that will imply analyticity.

Improving the regularity threshold

Experiments and numerical simulation, done mostly for the Kelvin Helmholtz problem show the existence and the stability of vortex sheet after the singularity.

Furthermore these vortex sheet roll up and seem to lead to rectifiable curves but of infinite length. Therefore the “threshold of regularity" should be above and may include spirals with finite length. In fact the best (to the best of my knowledge) result is due to Sijue Wu. The hypothesis $C^{\alpha}_{loc}(\mathbb{R}_t; C^{1+\beta}_{loc}(\mathbb{R}_\lambda)$ is replaced by $H^1_{loc}(\mathbb{R}_t \times \mathbb{R}_\lambda)$. Estimates are done explicitly using theorems of G. David saying that for all chord arc curves $\Gamma : s \mapsto \xi(s), s$ the arc length the Cauchy integral operator

$$C_\Gamma(f) = pv \int \frac{f(s')}{\xi(s) - \xi(s')} d\xi(s')$$

is bounded in $L^2(ds)$. It is interesting to notice that these results will apply to logarithmic spirals $r = e^\theta$ but not to infinite length algebraic spirals.

Conclusion

- I tried to show many crucial the issues lie in the analysis of the Euler equation.

• In $3d$ space variables with no boundary the solution of incompressible Navier Stokes equation converges with $\nu \to 0$ to a dissipative solution of the Euler equation $\overline{u}$. Therefore to the classical solution whenever such classical solution exist. It will not be in the non uniqueness With the formula :

$$(u_\nu(x,t) - \overline{u}) \otimes (u_\nu(x,t) - \overline{u}) = \int R\widehat{T(}u_\nu)(x,t,k)dk$$

one observes that if a Onzager Kolmogorof type hypothesis holds

$$|R\widehat{T(}u_\nu)(x,t,k)| = \frac{E(|k|)}{4\pi|k|^2} \leq \epsilon^{\frac{2}{3}}|k|^{-\frac{5}{3}} \qquad (22)$$

then u_ν converges to a weak solution of the Euler equation. As conclusion one can observe that with no boundary (21) implies :

$$\lim_{\nu\to 0} \nu \int_0^T \int_\Omega |\nabla u_\nu|^2 dxdx \neq 0 \Rightarrow \text{ no regular solution of the Euler equation} \qquad (23)$$

- As observed above the situation is completely different when boundary effects are present and this is already true in $2d$.

- In this setting turbulent effect may appear and some results (invariance under rotation) of the Wigner -Measure -Rayleigh Tensor may be available. On the other hand the full description of what happen is surely out of reach even in $2d$.

- Existence of Kelvin Helmholtz type solution appear in the wake of turbulence generated by boundary effect. In agreement with the previous remark the support of the vorticity density is a very singular curve. On the other hand such phenomena is very stable leading to the need of stability theorem for singular solution. Surely much more difficult but may be more natural than result based on the Arnold Criteria.