Supercritical geometric optics for nonlinear Schrödinger equations

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Introduction

Semi-classical limit of the nonlinear Schrödinger equations

$$i\varepsilon\partial_t u^\varepsilon + \frac{\varepsilon^2}{2}\Delta u^\varepsilon = |u^\varepsilon|^{2\sigma} u^\varepsilon, \quad u^\varepsilon(0,x) = a_0^\varepsilon(x) e^{i\phi_0(x)/\varepsilon}. \tag{1}$$

$u^\varepsilon = u^\varepsilon(t,x) \in \mathbf{C}$, $t \in \mathbf{R}$, $x \in \mathbf{R}^n$, $\phi_0(x) \in \mathbf{R}$, $a_0^\varepsilon(x) \in \mathbf{C}$, $\varepsilon \in (0,1]$.

Question: behavior of the classical solutions when $\varepsilon \to 0$.

Assumptions: -$n \in \{1,2,3\}$, $\sigma \in \mathbf{N}$;
- $\phi_0 \in H^\infty(\mathbf{R}^n)$ does not depend on ε;
- $a_0^\varepsilon \in H^\infty(\mathbf{R}^n)$ has an asymptotic development of the form

$$a_0^\varepsilon(x) = a_0(x) + \varepsilon a_1(x) + \varepsilon^2 a_2^\varepsilon(x),$$

$a_0, a_1 \in H^\infty(\mathbf{R}^n)$, a_2^ε uniformly bounded in $H^\infty(\mathbf{R}^n)$.

Motivations: -supercritical geometrical optics;
-the Cauchy problem for H^1-supercritical nonlinearities.

Motivations : supercritical geometrical optics

The equation is supercritical as far as geometrical optics is concerned.

$\longrightarrow$ involved several interesting phenomena. In particular, small perturbations of the initial amplitude are amplified to order 1 in small time.

$\longrightarrow$ An interesting feature of NLS is that we can simplify the geometry (no creation of harmonics).

We seek an approximate solution of the form:

$$u^\varepsilon(t,x) \sim \left(A_0(t,x) + \varepsilon A_1(t,x) + \varepsilon^2 A_2(t,x) + \ldots\right) e^{i\phi(t,x)/\varepsilon} .$$

Instability: A classical fact in supercritical régimes is that the leading order amplitude A_0 depends on the initial first corrector a_1.
Small perturbations of size $O(\varepsilon^\alpha)$ of the initial amplitude induce in time $O(\varepsilon^{1-\alpha})$ perturbations of size $O(1)$ of the amplitude $A_0(t,x)$ (Carles for NLS).

Motivations : supercritical geometrical optics

$\rightarrow$ Nonlinear dispersive waves. Let

$$A \in \mathbf{C}, \quad \phi(t,x) = k \cdot x - \omega t, \quad u^\varepsilon(t,x) = A \exp(i\phi(t,x)/\varepsilon).$$

u^ε is solution provided that

$$\omega = \frac{1}{2}k^2 + |A|^{2\sigma}.$$

Simplest example of nonlinear dispersive waves.

$\rightarrow$ The limit system for the quadratic observables is the system of compressible Euler equations :

$$\begin{cases} \dfrac{\partial \phi}{\partial t} + \dfrac{1}{2}|\nabla \phi|^2 + |A|^{2\sigma} = 0, & \text{dispersion relation,} \\ \dfrac{\partial |A|^2}{\partial t} + \operatorname{div}(|A|^2 \nabla \phi) = 0, & \text{from the conservation of density for NLS.} \end{cases}$$

For NLS, we can simplify the geometry, yet the geometry is not simple!!

Remark : Similar formal analysis for the NLW (Luke; Lebeau).

The cascade

Seek

$$u^\varepsilon(t,x) \sim \left(A_0(t,x) + \varepsilon A_1(t,x) + \varepsilon^2 A_2(t,x) + \dots\right) e^{i\phi(t,x)/\varepsilon}.$$

The BKW Cascade

$$O\left(\varepsilon^0\right): \quad \partial_t \phi + \frac{1}{2}|\nabla\phi|^2 + |A_0|^{2\sigma} = 0,$$

$$O\left(\varepsilon^1\right): \quad \partial_t A_0 + \nabla\phi \cdot \nabla A_0 + \frac{1}{2} A_0 \Delta\phi = -2i\sigma |A_0|^{2\sigma-2} \operatorname{Re}\left(A_0 \overline{A_1}\right) A_0.$$

Typical facts in supercritical geometrical optics (cf Cheverry & Guès; Serre):
$\longrightarrow$ strong coupling between the phase and the main amplitude.
$\longrightarrow$ the system is not closed (no matter how many terms are computed).

$\longrightarrow$ However, we can determine ϕ (P. Gérard):

$$(\rho, v) := \left(|A_0|^2, \nabla\phi\right) \quad \text{solves} \quad \begin{cases} \partial_t \rho + \operatorname{div}(\rho v) = 0, \\ \partial_t v + v\cdot\nabla v + \nabla \rho^\sigma = 0. \end{cases}$$

References

Two cases where the mathematical analysis is well developped:
1) For analytic initial data and general nonlinearities (Patrick Gérard; Laurent Thomann).

2) For general initial data in Sobolev spaces for the cubic ($\sigma = 1$) defocusing equation (Emmanuel Grenier).

Recall that one of the main difficulty of weakly nonlinear optics comes from interactions. (self-interaction; interaction of several waves) (cf Joly-Métivier-Rauch).

Here we simplify the geometry (no creation of harmonics), yet the stability analysis is more difficult.

At the linearized level, there is an exponential amplification factor in Gronwall's estimates, and hence small error terms of order $O(\varepsilon^\infty)$ are instantaneously amplified to order $O(1)$ (cf Cheverry; Cheverry-Guès; Cheverry-Guès-Métivier).

1) For analytic initial, one can define a very good BKW solution with a remainder of size $O(e^{-c/\varepsilon})$.

References

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1) For analytic initial data and general nonlinearities (Patrick Gérard; Laurent Thomann).

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One can use the specific struture of the equations to define a phase/amplitude representation of the solution.

The main result here is that we can extend the Grenier's results about Sobolev data to higher order nonlinearities.
This approach provides a local version of the modulated energy functional used by Fanghua Lin & Ping Zhang, following the approach initiated by Yann Brenier.

Motivations : the Cauchy problem for H^1-supercritical nonlineariti

Consider the Cauchy problem

$$i\frac{\partial u}{\partial t} + \Delta u = |u|^{2\sigma} u \quad ; \quad u_{|t=0} = u_0, \tag{NLS}$$

$(t,x) \in I \times \mathbf{R}^n$, $0 \in I$, $u(t,x) \in \mathbf{C}$.

The Cauchy problem is well posed, globally in time for $x \in \mathbf{R}^n$ with

$$n = 1,2 \text{ and } \sigma \in \mathbf{N} \quad ; \quad n = 3 \text{ and } \sigma = 1,2.$$

(Ginibre–Velo ; Cazenave-Weissler ; Kato ; Yajima ; Tsustumi)
(Colliander–Keel–Staffilani–Takaoka–Tao)

The question of whether blow up occurs for $\sigma = 3$ (H^1-supercritical defocusing NLS) is an open problem.
Yet, they are ill-posedness results.
Norm-inflation: Christ–Colliander–Tao ; Burq–Gérard–Tzvetkov ; Carles.
Loss of regularity: Lebeau (NLW) , Carles (cubic NLS).

Loss of regularity for H^1-supercritical NLS

Theorem (Alazard & Carles; Thomann). There exist $\varphi_n \in C_0^\infty(\mathbf{R}^3)$ such that

$$\|\varphi_n\|_{H^1(\mathbf{R}^3)} + \|\varphi_n\|_{L^8(\mathbf{R}^3)} \lesssim \|\varphi_n\|_{H^{9/8}(\mathbf{R}^3)} \xrightarrow[n\to+\infty]{} 0,$$

and a sequence $t_n > 0$ converging to 0, such that the Cauchy problem

$$i\frac{\partial \psi_n}{\partial t} + \Delta\psi_n = |\psi_n|^6\psi_n \quad ; \quad \psi_n(0,x) = \varphi_n,$$

has a unique classical solution ψ_n, defined on $[0,t_n]$, such that

$$\|\psi_n(t_n)\|_{H^k(\mathbf{R}^3)} \xrightarrow[n\to+\infty]{} +\infty, \quad \forall k > 1.$$

Remark. Let $s < 7/6 = d/2 - 1/\sigma (> 9/8)$. There exists classical data of arbitrary small H^s norm such that the solution of the focusing NLS blows up in arbitrary small times (virial argument of Glassey+ scale invariance).

Remark. Laurent Thomann’s thesis (2007).

The solution becomes ε-oscillatory

Thm (AC). Let $d \geqslant 1$, $\sigma \geqslant 1$ and $0 \neq a_0 \in \mathcal{S}(\mathbf{R}^n)$ such that the solution u^ε

$$i\varepsilon\partial_t u^\varepsilon + \frac{\varepsilon^2}{2}\Delta u^\varepsilon = |u^\varepsilon|^{2\sigma}u^\varepsilon \quad ; \quad u^\varepsilon(0,x) = a_0(x).$$

exists on a time interval independent of ε. Then, the solution becomes ε-oscillatory

$$\exists \tau > 0 / \quad \forall k \in]0,1], \quad \liminf_{\varepsilon \to 0} \left\| |\varepsilon D_x|^k u^\varepsilon(\tau) \right\|_{L^2} > 0.$$

One can extend this results to weak solutions.

Rk. : what is the analogue in classical mechanics ?
Between two adjacent air masses, the air flows instantenously from the region of high pressure to the region of low pressure (the pressure gradient force drive winds).

The Grenier's approach

$\rightarrow$ We cannot use the classical hydrodynamic form when ρ vanishes:

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho v) = 0, \\ \partial_t v + v \cdot \nabla v + \nabla \rho^\sigma = \dfrac{\varepsilon^2}{2} \nabla \left(\dfrac{1}{\sqrt{\rho}} \Delta \sqrt{\rho} \right). \end{cases}$$

$\rightarrow$ Grenier: seek u^ε under the form $u^\varepsilon = a^\varepsilon e^{i\phi^\varepsilon/\varepsilon}$ with a^ε complex-valued, and

$$\begin{cases} \partial_t \phi^\varepsilon + \dfrac{1}{2} |\nabla \phi^\varepsilon|^2 + |a^\varepsilon|^2 = 0 \quad ; \quad \phi^\varepsilon_{|t=0} = \phi_0, \\ \partial_t a^\varepsilon + \nabla \phi^\varepsilon \cdot \nabla a^\varepsilon + \dfrac{1}{2} a^\varepsilon \Delta \phi^\varepsilon = i \dfrac{\varepsilon}{2} \Delta a^\varepsilon \quad ; \quad a^\varepsilon_{|t=0} = a^\varepsilon_0, \end{cases}$$

Make a BKW asymptotic for the amplitude and for the phase: solve the BKW cascade for this system.
This yields closed systems which allow to determine

$$\phi^\varepsilon \sim \sum_{k=0}^{+\infty} \varepsilon^k \phi_k(t,x), \qquad a^\varepsilon \sim \sum_{k=0}^{\infty} \varepsilon^k a_k(t,x)$$

Study of the limit system

The first step is to solve the limit system

$$\begin{cases} \partial_t v + v\cdot\nabla v + \nabla\left(|a|^{2\sigma}\right) = 0 \quad ; \quad v_{|t=0} = \phi_0, \\ \partial_t a + v\cdot\nabla a + \frac{1}{2} a \,\mathrm{div}\, v = 0 \quad ; \quad a_{|t=0} = a_0. \end{cases} \tag{E}$$

Proposition. There is a unique maximal solution (ϕ, a) in $C^\infty([0, T^*[; H^\infty(\mathbf{R}^n))$.

$\longrightarrow$ (sound speed) The proof is based on a nonlinear change of unknown introduced by Makino-Ukai-Kawashima (see also Chemin; Serre; Grassin):

$$(v, u) := (\nabla\phi, a^\sigma)$$

solves a quasi-linear symmetric hyperbolic system.
(dichotomy between $\sigma = 1$ and $\sigma \geqslant 2$.)
(does not seem to be well adapted to NLS equations.)

$\longrightarrow$ (vacuum) Possible loss of one derivative : We prove that, for all $(\phi_0, a_0) \in H^{s+1}\times H^s$ with $s > n/2+1$, there exists $T^* > 0$ such that the Cauchy problem has a unique maximal solution (ϕ, a) in $C^0([0, T^*[; H^{s+1}\times H^{s-1})$.

Study of the limit system

Proposition. The lifespan T^* is finite for some initial data.
Proof: if $a = 0$ ($\Leftrightarrow a_0 = 0$), the limit system reduces to Burgers equation for v.

$a_0 = 0$ is not interesting... More seriously:

Proposition. The lifespan T^* is finite for all compactly support initial data (a_0, v_0).

Proof: follows from the pseudo-conformal identity:

$$\frac{d}{dt}\int \left(\frac{1}{2}\left|(x - tv(t,x))\right|^2 \rho(t,x) + \frac{t^2}{\sigma+1}\rho(t,x)^{\sigma+1}\right) dx$$
$$= \frac{t}{\sigma+1}(2 - n\sigma)\int \rho(t,x)^{\sigma+1}dx.$$

We verify that the geometry is not simple.
Open question: behavior of the solutions for large times (Jin-Levermore-Mc Laughlin 1D).

Study of the limit system

Consider the analogous system for a focusing nonlinearity

$$\begin{cases} \partial_t \phi + \dfrac{1}{2}|\partial_x \phi|^2 - |a|^{2\sigma} = 0 & ; \quad \phi_{|t=0} = \phi_0, \\ \partial_t a + \partial_x \phi \partial_x a + \dfrac{1}{2} a \partial_x^2 \phi = 0 & ; \quad a_{|t=0} = a_0. \end{cases}$$

(Euler elliptic)

Proposition. There are initial data for which the Cauchy problem has no solution.

Yet, one can justify the semi-classical limit for analytic initial data (Gérard; Thomann). See also, Clarke–Miller, DiFranco–Miller .

Main result

Theorem. There exists $T \in]0, T^*[$ such that, for all $\varepsilon \in]0,1]$ the Cauchy problem has a unique solution $u^\varepsilon \in C([0,T]; H^\infty(\mathbf{R}^n))$. Moreover,

$$\sup_{\varepsilon \in]0,1]} \sup_{t \in [0,T]} \left\{ \left\| u^\varepsilon(t) e^{-i\phi(t)/\varepsilon} \right\|_{H^k}^2 + \varepsilon^{-2} \left\| |u^\varepsilon(t)|^2 - |a(t)|^2 \right\|_{L^{\sigma+1}}^{\sigma+1} \right\} < +\infty,$$

where the index k is as follows:

- If $\sigma = 1$, then $k \in \mathbf{N}$ is arbitrary.
- If $\sigma = 2$ and $n = 1$, then we can take $k = 2$.
- If $\sigma = 2$ and $2 \leqslant n \leqslant 3$, then we can take $k = 1$.
- If $\sigma \geqslant 3$, then we can take $k = \sigma$.

Remarks concerning the main result

- For $\sigma = 1$: consequence of Grenier's analysis.
- For $\sigma \geqslant 3$ and $n = 3$, the equation is H^1-supercritical. The existence on a time interval independent of $\varepsilon \in]0, 1]$ is new.

One can consider:
- initial data in $H^s(\mathbf{R}^n)$ with $s < +\infty$ large enough.
- some nonlinearities which are not homogeneous.
- external potential (previous result of Carles).
- exterior domains for $k = 1$ (Lin–Zhang when $\sigma = 1$).
- higher dimensions $n < 2\sigma - 2$ for sufficiently large σ.

Recall that:
- we cannot expect global in time results.
- If we assume only $a_0^\varepsilon = a_0 + o(1)$, then the conclusion fails.

Convergence of position and current densities

Ideally, we would like to prove that

$$\forall k \in \mathbf{N}, \quad \sup_{\varepsilon\in]0,1]} \sup_{t\in[0,T]} \left\{ \left\| u^\varepsilon(t) e^{-i\phi(t)/\varepsilon} \right\|_{H^k}^2 + \varepsilon^{-2} \left\| |u^\varepsilon(t)|^2 - |a(t)|^2 \right\|_{L^{\sigma+1}}^{\sigma+1} \right\} < +\infty.$$

Yet, as observed by Lin–Zhang, the case $k=1$ is enough to prove that:

Corollary. There exists $T \in]0, T^*[$ such that

$$|u^\varepsilon|^2 \mathop{\longrightarrow}_{\varepsilon\to 0} |a|^2 \quad \text{in } C\left([0,T]; L^{\sigma+1}(\mathbf{R}^n)\right).$$

$$\operatorname{Im}\left(\varepsilon \overline{u}^\varepsilon \nabla u^\varepsilon\right) \mathop{\longrightarrow}_{\varepsilon\to 0} |a|^2 \nabla\phi \quad \text{in } C\left([0,T]; L^{\sigma+1}(\mathbf{R}^n) + L^1(\mathbf{R}^n)\right).$$

In particular, there is only one Wigner measure associated to $(u^\varepsilon)_\varepsilon$, given by

$$\mu(t, dx, d\xi) = |a(t,x)|^2 dx \otimes \delta\left(\xi - \nabla\phi(t,x)\right).$$

Leading order behavior of the wave function

Theorem. For any $T \in]0, T^*[$, there exists $\varepsilon(T) > 0$ such that $u^\varepsilon \in C([0,T]; H^\infty)$ for $\varepsilon \in]0, \varepsilon(T)]$, and

$$\left\| u^\varepsilon e^{-i\phi/\varepsilon} - a e^{i\phi^{(1)}} \right\|_{L^\infty([0,T];H^k)} = O(\varepsilon),$$

where k is as above and $\phi^{(1)}$ given by (cf Grenier's BKW approach)

$$\begin{cases} \partial_t \phi^{(1)} + \nabla\phi \cdot \nabla\phi^{(1)} + 2\sigma \,\mathrm{Re}\left(\overline{a} a^{(1)}\right) |a|^{2\sigma-2} = 0, \\ \partial_t a^{(1)} + \nabla\phi \cdot \nabla a^{(1)} + \nabla\phi^{(1)} \cdot \nabla a + \frac{1}{2} a^{(1)} \Delta\phi + \frac{1}{2} a \Delta\phi^{(1)} = \frac{i}{2}\Delta a, \\ \phi^{(1)}\big|_{t=0} = 0 \quad ; \quad a^{(1)}\big|_{t=0} = a_1. \end{cases}$$

The phase shift $\phi^{(1)}$ is a function of a, ϕ, and a_1, where recall that

$$a_0^\varepsilon(x) = a_0(x) + \varepsilon a_1(x) + O(\varepsilon^2).$$

Rk. Ghost effect: $\phi^{(1)} \neq 0$ in general. Implies instabilities for the semi-classical equations (Carles).

Filtering

We filter out the oscillations by the change of unknown:

$$a^\varepsilon(t,x) := u^\varepsilon(t,x)e^{-i\phi(t,x)/\varepsilon}.$$

The key point is that

- To prove that u^ε exist for a time independent of ε, it is enough to prove uniform L^∞ estimates (semi-linear equation).
- $\|u^\varepsilon(t)\|_{L^\infty} = \|a^\varepsilon(t)\|_{L^\infty} \lesssim \|a^\varepsilon(t)\|_{H^s}$ for $s > n/2$.
- we expect uniform estimates in Sobolev spaces for a^ε.
- Obviously, uniform estimates in Sobolev spaces for u^ε are not expected to hold, due to the rapid oscillations described by ϕ.

Symmetrize

The amplitude a^ε solves

$$\begin{cases} \partial_t a^\varepsilon + \nabla\phi \cdot \nabla a^\varepsilon + \frac{1}{2} a^\varepsilon \Delta\phi - i\frac{\varepsilon}{2}\Delta a^\varepsilon = -\frac{i}{\varepsilon}\left(|a^\varepsilon|^{2\sigma} - |a|^{2\sigma} \right) a^\varepsilon. \\ a^\varepsilon_{|t=0} = a^\varepsilon_0. \end{cases}$$

One has

$$\frac{1}{2}\frac{d}{dt}\|a^\varepsilon\|^2_{L^2} = \frac{1}{2}\frac{d}{dt}\|u^\varepsilon\|^2_{L^2} = 0,$$

$$\frac{1}{2}\frac{d}{dt}\|\nabla a^\varepsilon\|^2_{L^2} - \frac{1}{\varepsilon}\int \operatorname{div}(\operatorname{Im}(\overline{a}^\varepsilon \nabla a^\varepsilon))\left(|a^\varepsilon|^{2\sigma} - |a|^{2\sigma} \right) \\ = -\operatorname{Re}\int_{\mathbf{R}^n} \left(\nabla a^\varepsilon \cdot \nabla\nabla\phi + \frac{1}{2} a^\varepsilon \nabla\Delta\phi\right)\nabla \overline{a}^\varepsilon \, dx.$$

Hence

$$\frac{1}{2}\frac{d}{dt}\|a^\varepsilon\|^2_{H^1} - \frac{1}{\varepsilon}\int \operatorname{div}(\operatorname{Im}(\overline{a}^\varepsilon \nabla a^\varepsilon))\left(|a^\varepsilon|^{2\sigma} - |a|^{2\sigma} \right) \leqslant C_\phi \|a^\varepsilon\|^2_{H^1}.$$

Symmetrize the estimates

We have

$$\frac{1}{2}\frac{d}{dt}\|a^\varepsilon\|_{H^1}^2 - \frac{1}{\varepsilon}\int \mathrm{div}(\mathrm{Im}(\overline{a}^\varepsilon \nabla a^\varepsilon))\left(|a^\varepsilon|^{2\sigma} - |a|^{2\sigma}\right) \leqslant C_\phi \|a^\varepsilon\|_{H^1}^2 .$$

The idea is then to find a second energy functional $\mathcal{E}^\varepsilon$ such that

$$\frac{1}{2}\frac{d\mathcal{E}^\varepsilon}{dt} + \frac{1}{\varepsilon}\int \mathrm{div}(\mathrm{Im}(\overline{a}^\varepsilon \nabla a^\varepsilon))\left(|a^\varepsilon|^{2\sigma} - |a|^{2\sigma}\right) \leqslant C_{a,\phi}(\|a^\varepsilon\|_{H^1}^2 + \mathcal{E}^\varepsilon).$$

$\longrightarrow$ uniform in ε energy estimate

$$\|a^\varepsilon(t)\|_{H^1}^2 + \mathcal{E}^\varepsilon(t) \leqslant e^{C_{a,\phi}(t)}(E^\varepsilon(0) + \mathcal{E}^\varepsilon(0)).$$

For the semi-classical limit, this strategy goes back to the work of Y. Brenier; P. Zhang; F. Lin and P. Zhang (modulated energy estimate).

Symmetrize the equations

We seek $\mathcal{E}^\varepsilon$ such that

$$\frac{1}{2}\frac{d\mathcal{E}^\varepsilon}{dt} + \frac{1}{\varepsilon}\int \operatorname{div}(\operatorname{Im}(\overline{a}^\varepsilon \nabla a^\varepsilon))\left(|a^\varepsilon|^{2\sigma} - |a|^{2\sigma}\right) \leqslant C_{a,\phi}(E^\varepsilon + \mathcal{E}^\varepsilon).$$

To find $\mathcal{E}^\varepsilon$, we seek a nonlinear change of unknown to symmetrize the equations. We seek g^ε and q^ε such that

$$(1) \quad \partial_t q^\varepsilon + g^\varepsilon \operatorname{div}(\operatorname{Im}(\overline{a}^\varepsilon \nabla a^\varepsilon)) + \nabla\phi \cdot \nabla q^\varepsilon + \frac{\sigma+1}{2} q^\varepsilon \Delta\phi = 0,$$

$$(2) \quad q^\varepsilon g^\varepsilon = \frac{1}{\varepsilon}\left(|a^\varepsilon|^{2\sigma} - |a|^{2\sigma}\right), \qquad (3) \quad g^\varepsilon = O(1).$$

Symmetrize the equations

We seek $\mathcal{E}^\varepsilon$ such that

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$$(2) \quad q^\varepsilon g^\varepsilon = \frac{1}{\varepsilon}\left(|a^\varepsilon|^{2\sigma} - |a|^{2\sigma}\right), \qquad (3) \quad g^\varepsilon = O(1).$$

Example. If $\sigma = 1$ then

$$q^\varepsilon := \frac{1}{\varepsilon}\left(|a^\varepsilon|^2 - |a|^2\right).$$

satisfies

$$\partial_t q^\varepsilon + \operatorname{div}\left(\operatorname{Im}\left(\overline{a}^\varepsilon \nabla a^\varepsilon\right)\right) + \operatorname{div}(q^\varepsilon \nabla\phi) = 0,$$

and hence one has the desired splitting with $g^\varepsilon = 1$.

Symmetrize the equations

We seek $\mathcal{E}^\varepsilon$ such that

$$\frac{1}{2}\frac{d\mathcal{E}^\varepsilon}{dt} + \frac{1}{\varepsilon}\int \operatorname{div}(\operatorname{Im}(\overline{a}^\varepsilon \nabla a^\varepsilon))\left(|a^\varepsilon|^{2\sigma} - |a|^{2\sigma}\right) \leqslant C_{a,\phi}(E^\varepsilon + \mathcal{E}^\varepsilon).$$

To find $\mathcal{E}^\varepsilon$, we seek a nonlinear change of unknown to symmetrize the equations. We seek g^ε and q^ε such that

$$(1) \quad \partial_t q^\varepsilon + g^\varepsilon \operatorname{div}(\operatorname{Im}(\overline{a}^\varepsilon \nabla a^\varepsilon)) + \nabla\phi \cdot \nabla q^\varepsilon + \frac{\sigma+1}{2} q^\varepsilon \Delta\phi = 0,$$

$$(2) \quad q^\varepsilon g^\varepsilon = \frac{1}{\varepsilon}\left(|a^\varepsilon|^{2\sigma} - |a|^{2\sigma}\right), \qquad (3) \quad g^\varepsilon = O(1).$$

This allows to find

$$\mathcal{E}^\varepsilon := \|q^\varepsilon\|_{L^2}^2.$$

Symmetrize the equations

This strategy allows to find

$$\mathcal{E}^\varepsilon := \|q^\varepsilon\|_{L^2}^2,$$

and also to derive a local version of the modulated energy functional.

Proposition. Set $\psi^\varepsilon := \nabla a^\varepsilon$. The modulated energy $e^\varepsilon := |\psi^\varepsilon|^2 + (q^\varepsilon)^2$, solves

$$\partial_t e^\varepsilon + \operatorname{div}(e^\varepsilon \nabla \phi) + \operatorname{div}\big(2\,\mathrm{Im}(q^\varepsilon \overline{a}^\varepsilon \psi^\varepsilon)\big) + \operatorname{div}\left(\varepsilon\,\mathrm{Im}\left(\overline{\psi}^\varepsilon \cdot \nabla \psi^\varepsilon\right)\right)$$
$$= -(q^\varepsilon)^2 \Delta \phi - \mathrm{Re}\left(\left(2\psi^\varepsilon \cdot \nabla\nabla\phi + a^\varepsilon \nabla \Delta \phi\right) \cdot \overline{\psi}^\varepsilon\right).$$

This yields the desired modulated energy estimate ($k=1$) by integration and Gronwall's lemma.

Furthermore, the system satisfied by $(a^\varepsilon, \nabla a^\varepsilon, q^\varepsilon)$ is a hyperbolic symmetric system plus some skew-symmetric terms. Therefore, we can derive energy estimates in Sobolev norms.

Recall that a^ε solves

$$\begin{cases} \partial_t a^\varepsilon + \nabla\phi \cdot \nabla a^\varepsilon + \frac{1}{2} a^\varepsilon \Delta\phi - i\frac{\varepsilon}{2}\Delta a^\varepsilon = -\frac{i}{\varepsilon}\left(|a^\varepsilon|^{2\sigma} - |a|^{2\sigma}\right) a^\varepsilon. \\ a^\varepsilon_{|t=0} = a_0^\varepsilon. \end{cases}$$

To symmetrize the equations, split the term $|a^\varepsilon|^{2\sigma} - |a|^{2\sigma}$ as a product

$$|a^\varepsilon|^{2\sigma} - |a|^{2\sigma} = g^\varepsilon \beta^\varepsilon = (G\,B)(|a^\varepsilon|^2, |a|^2) = G(r_1,r_2)B(r_1,r_2)\Big|_{(r_1,r_2)=(|a^\varepsilon|^2,|a|^2)},$$

where 1) the good term is seen as a coefficient; 2) we form an evolution equation for the bad term. We want to chose β^ε such that

$$\partial_t \beta^\varepsilon + L(a,\phi,\partial_x)\beta^\varepsilon + g^\varepsilon \operatorname{div}\left(\varepsilon \operatorname{Im}(\overline{a}^\varepsilon \nabla a^\varepsilon)\right) = 0,$$

and L is a first order differential operator.

We split

$$|a^\varepsilon|^{2\sigma} - |a|^{2\sigma} = g^\varepsilon \beta^\varepsilon = G(|a^\varepsilon|^2, |a|^2)B(|a^\varepsilon|^2, |a|^2).$$

We want

$$\partial_t \beta^\varepsilon + L(a, \phi, \partial_x)\beta^\varepsilon + g^\varepsilon \operatorname{div}(\varepsilon \operatorname{Im}(\overline{a}^\varepsilon \nabla a^\varepsilon)) = 0.$$

Introduce

$$\rho := |a|^2, \quad v = \nabla\phi, \quad \rho^\varepsilon := |a^\varepsilon|^2 = |u^\varepsilon|^2.$$

By using the conservation laws for the densities, we compute

$$\partial_t \beta^\varepsilon + (\partial_{r_1} B)(\rho^\varepsilon, \rho) \operatorname{div}(\operatorname{Im}(\varepsilon \overline{a}^\varepsilon \nabla a^\varepsilon) + \rho^\varepsilon v) + (\partial_{r_2} B)(\rho^\varepsilon, \rho) \operatorname{div}(\rho v) = 0.$$

To have an equation of the desired form, we impose

$$\partial_{r_1} B(r_1, r_2) = G(r_1, r_2).$$

Since $G(r_1, r_2)B(r_1, r_2) = r_1^\sigma - r_2^\sigma$, this suggests to choose

$$(\beta^\varepsilon)^2 = \frac{2}{\sigma+1}(\rho^\varepsilon)^{\sigma+1} - 2\rho^\sigma \rho^\varepsilon + f(\rho).$$

To obtain an operator L which is linear with respect to β^ε we choose

$$(\beta^\varepsilon)^2 = \frac{2}{\sigma+1}(\rho^\varepsilon)^{\sigma+1} - \frac{2}{\sigma+1}\rho^{\sigma+1} - 2\rho^\sigma(\rho^\varepsilon - \rho).$$

We formally compute:

$$\partial_t \beta^\varepsilon + \varepsilon g^\varepsilon \operatorname{div}(\operatorname{Im}(\overline{a}^\varepsilon \nabla a^\varepsilon)) + v\cdot\nabla\beta^\varepsilon + \frac{\sigma+1}{2}\beta^\varepsilon \operatorname{div} v = 0.$$

Taylor's formula yields

$$\frac{2}{\sigma+1}(\rho^\varepsilon)^{\sigma+1} - \frac{2}{\sigma+1}\rho^{\sigma+1} - 2\rho^\sigma(\rho^\varepsilon - \rho) = (\rho^\varepsilon - \rho)^2 Q_\sigma(\rho^\varepsilon, \rho),$$

with

$$Q_\sigma(r_1, r_2) := 2\sigma \int_0^1 (1-s)\left(r_2 + s(r_1 - r_2)\right)^{\sigma-1} ds \geqslant C_\sigma\left(r_1^{\sigma-1} + r_2^{\sigma-1}\right).$$

Let $\sigma \in \mathbf{N}$. Introduce

$$G_\sigma(r_1, r_2) = \frac{P_\sigma(r_1, r_2)}{\sqrt{Q_\sigma(r_1, r_2)}} \quad ; \quad B_\sigma(r_1, r_2) := (r_1 - r_2)\sqrt{Q_\sigma(r_1, r_2)},$$

where

$$P_\sigma(r_1, r_2) = \frac{r_1^\sigma - r_2^\sigma}{r_1 - r_2} = \sum_{\ell=0}^{\sigma-1} r_1^{\sigma-1-\ell} r_2^\ell.$$

Example: For $\sigma = 1, 2, 3$, we compute

$$G_1 = 1, \qquad B_1 = r_1 - r_2.$$

$$G_2 = \sqrt{\frac{3}{2}} \frac{r_1 + r_2}{\sqrt{r_1 + 2r_2}}, \qquad B_2 = \sqrt{\frac{2}{3}}(r_1 - r_2)\sqrt{r_1 + 2r_2}.$$

$$G_3 = \sqrt{2} \frac{r_1^2 + r_1 r_2 + r_2^2}{\sqrt{(r_1 - r_2)^2 + 2r_2^2}}, \qquad B_3 = \frac{1}{\sqrt{2}}(r_1 - r_2)\sqrt{(r_1 - r_2)^2 + 2r_2^2}.$$

Let $\sigma \in \mathbf{N}$. Introduce

$$G_\sigma(r_1,r_2) = \frac{P_\sigma(r_1,r_2)}{\sqrt{Q_\sigma(r_1,r_2)}} \quad ; \quad B_\sigma(r_1,r_2) := (r_1 - r_2)\sqrt{Q_\sigma(r_1,r_2)},$$

where

$$P_\sigma(r_1,r_2) = \frac{r_1^\sigma - r_2^\sigma}{r_1 - r_2} = \sum_{\ell=0}^{\sigma-1} r_1^{\sigma-1-\ell} r_2^\ell.$$

We can divide by β^ε.

Proposition. $\beta^\varepsilon \in C^1_{t,x}$ and $g^\varepsilon \in C^0_{t,x}$. Moreover,

$$\partial_t \beta^\varepsilon + \varepsilon g^\varepsilon \operatorname{div}(\operatorname{Im}(\overline{a}^\varepsilon \nabla a^\varepsilon)) + v \cdot \nabla \beta^\varepsilon + \frac{\sigma+1}{2}\beta^\varepsilon \operatorname{div} v = 0.$$

We will prove $|a^\varepsilon|^{2\sigma} - |a|^{2\sigma} = O(\varepsilon)$. Set

$$\psi^\varepsilon := \nabla a^\varepsilon \quad ; \quad q^\varepsilon := \varepsilon^{-1}\beta^\varepsilon.$$

Since

$$g^\varepsilon \operatorname{div}(\operatorname{Im}(\overline{a}^\varepsilon \psi^\varepsilon)) = \operatorname{Im}(g^\varepsilon \overline{a}^\varepsilon \operatorname{div}\psi^\varepsilon),$$

we find

$$\begin{cases} \partial_t a^\varepsilon + v\cdot\nabla a^\varepsilon - i\dfrac{\varepsilon}{2}\Delta a^\varepsilon = -\dfrac{1}{2}a^\varepsilon \operatorname{div} v - ig^\varepsilon q^\varepsilon a^\varepsilon, \\ \partial_t \psi^\varepsilon + v\cdot\nabla\psi^\varepsilon + ia^\varepsilon g^\varepsilon \nabla q^\varepsilon - i\dfrac{\varepsilon}{2}\Delta\psi^\varepsilon \\ \quad = -\dfrac{1}{2}\psi^\varepsilon \operatorname{div} v - \psi^\varepsilon\cdot\nabla v - \dfrac{1}{2}a^\varepsilon \nabla\operatorname{div} v - iq^\varepsilon\nabla\left(a^\varepsilon g^\varepsilon\right), \\ \partial_t q^\varepsilon + v\cdot\nabla q^\varepsilon + \operatorname{Im}(g^\varepsilon\overline{a}^\varepsilon \operatorname{div}\psi^\varepsilon) = -\dfrac{\sigma+1}{2}q^\varepsilon \operatorname{div} v. \end{cases}$$

$$\begin{cases} \partial_t a^\varepsilon + v \cdot \nabla a^\varepsilon - i\frac{\varepsilon}{2}\Delta a^\varepsilon = -\frac{1}{2}a^\varepsilon \operatorname{div} v - ig^\varepsilon q^\varepsilon a^\varepsilon, \\ \partial_t \psi^\varepsilon + v \cdot \nabla \psi^\varepsilon + ia^\varepsilon g^\varepsilon \nabla q^\varepsilon - i\frac{\varepsilon}{2}\Delta \psi^\varepsilon \\ \qquad = -\frac{1}{2}\psi^\varepsilon \operatorname{div} v - \psi^\varepsilon \cdot \nabla v - \frac{1}{2}a^\varepsilon \nabla \operatorname{div} v - iq^\varepsilon \nabla(a^\varepsilon g^\varepsilon), \\ \partial_t q^\varepsilon + v \cdot \nabla q^\varepsilon + \operatorname{Im}(g^\varepsilon \overline{a}^\varepsilon \operatorname{div} \psi^\varepsilon) = -\frac{\sigma+1}{2} q^\varepsilon \operatorname{div} v. \end{cases}$$

Corollary. $U^\varepsilon := (2q^\varepsilon, a^\varepsilon, \overline{a}^\varepsilon, \psi^\varepsilon, \overline{\psi}^\varepsilon)$ satisfies an equation of the form

$$\partial_t U^\varepsilon + \sum_{1 \leqslant j \leqslant n} A_j(v, a^\varepsilon g^\varepsilon, \overline{a}^\varepsilon g^\varepsilon)\partial_j U^\varepsilon + \varepsilon \mathcal{L}(\partial_x) U^\varepsilon = E(\Phi, U^\varepsilon, a^\varepsilon g^\varepsilon, \nabla(a^\varepsilon g^\varepsilon)),$$

with $\Phi = (\nabla\phi, \nabla^2\phi, \nabla^3\phi)$, A_j hermitian and lineair in their arguments, $\mathcal{L}(\partial_x) = \sum L_{jk}\partial_j\partial_k$ second-order skew-symmetric operator with constant coefficients, $E \in C^\infty$ vanishes at the origin.

$$\begin{cases} \partial_t a^\varepsilon + v\cdot\nabla a^\varepsilon - i\frac{\varepsilon}{2}\Delta a^\varepsilon = -\frac{1}{2}a^\varepsilon \operatorname{div} v - ig^\varepsilon q^\varepsilon a^\varepsilon, \\ \partial_t \psi^\varepsilon + v\cdot\nabla\psi^\varepsilon + ia^\varepsilon g^\varepsilon \nabla q^\varepsilon - i\frac{\varepsilon}{2}\Delta\psi^\varepsilon \\ \qquad = -\frac{1}{2}\psi^\varepsilon \operatorname{div} v - \psi^\varepsilon\cdot\nabla v - \frac{1}{2}a^\varepsilon\nabla \operatorname{div} v - iq^\varepsilon\nabla(a^\varepsilon g^\varepsilon), \\ \partial_t q^\varepsilon + v\cdot\nabla q^\varepsilon + \operatorname{Im}(g^\varepsilon \overline{a}^\varepsilon \operatorname{div}\psi^\varepsilon) = -\frac{\sigma+1}{2}q^\varepsilon \operatorname{div} v. \end{cases}$$

Corollary. $U^\varepsilon := (2q^\varepsilon, a^\varepsilon, \overline{a}^\varepsilon, \psi^\varepsilon, \overline{\psi}^\varepsilon)$ satisfies an equation of the form

$$\partial_t U^\varepsilon + \sum_{1\leqslant j\leqslant n} A_j(v, a^\varepsilon g^\varepsilon, \overline{a}^\varepsilon g^\varepsilon)\partial_j U^\varepsilon + \varepsilon\mathcal{L}(\partial_x)U^\varepsilon = E(\Phi, U^\varepsilon, a^\varepsilon g^\varepsilon, \nabla(a^\varepsilon g^\varepsilon)),$$

with $\Phi = (\nabla\phi, \nabla^2\phi, \nabla^3\phi)$, A_j hermitian and lineair in their arguments, $\mathcal{L}(\partial_x) = \sum L_{jk}\partial_j\partial_k$ second-order skew-symmetric operator with constant coefficients, $E \in C^\infty$ vanishes at the origin.

$$\begin{cases} \partial_t a^\varepsilon + v\cdot\nabla a^\varepsilon - i\frac{\varepsilon}{2}\Delta a^\varepsilon = -\frac{1}{2}a^\varepsilon \operatorname{div} v - i g^\varepsilon q^\varepsilon a^\varepsilon, \\ \partial_t \psi^\varepsilon + v\cdot\nabla\psi^\varepsilon + i a^\varepsilon g^\varepsilon \nabla q^\varepsilon - i\frac{\varepsilon}{2}\Delta\psi^\varepsilon \\ \qquad = -\frac{1}{2}\psi^\varepsilon \operatorname{div} v - \psi^\varepsilon\cdot\nabla v - \frac{1}{2}a^\varepsilon\nabla \operatorname{div} v - i q^\varepsilon \nabla\left(a^\varepsilon g^\varepsilon\right), \\ \partial_t q^\varepsilon + v\cdot\nabla q^\varepsilon + \operatorname{Im}(g^\varepsilon \overline{a}^\varepsilon \operatorname{div}\psi^\varepsilon) = -\frac{\sigma+1}{2}q^\varepsilon \operatorname{div} v. \end{cases}$$

Corollary. $U^\varepsilon := (2q^\varepsilon, a^\varepsilon, \overline{a}^\varepsilon, \psi^\varepsilon, \overline{\psi}^\varepsilon)$ satisfies an equation of the form

$$\partial_t U^\varepsilon + \sum_{1\leqslant j\leqslant n} A_j(v, a^\varepsilon g^\varepsilon, \overline{a}^\varepsilon g^\varepsilon)\partial_j U^\varepsilon + \varepsilon\mathcal{L}(\partial_x)U^\varepsilon = E(\Phi, U^\varepsilon, a^\varepsilon g^\varepsilon, \nabla(a^\varepsilon g^\varepsilon)),$$

with $\Phi = (\nabla\phi, \nabla^2\phi, \nabla^3\phi)$, A_j hermitian and lineair in their arguments, $\mathcal{L}(\partial_x) = \sum L_{jk}\partial_j\partial_k$ second-order skew-symmetric operator with constant coefficients, $E \in C^\infty$ vanishes at the origin.

$$
\begin{cases}
\partial_t a^\varepsilon + v\cdot\nabla a^\varepsilon - i\frac{\varepsilon}{2}\Delta a^\varepsilon = -\frac{1}{2}a^\varepsilon \operatorname{div} v - i g^\varepsilon q^\varepsilon a^\varepsilon, \\
\partial_t \psi^\varepsilon + v\cdot\nabla\psi^\varepsilon + i a^\varepsilon g^\varepsilon \nabla q^\varepsilon - i\frac{\varepsilon}{2}\Delta\psi^\varepsilon \\
\qquad = -\frac{1}{2}\psi^\varepsilon \operatorname{div} v - \psi^\varepsilon\cdot\nabla v - \frac{1}{2}a^\varepsilon \nabla \operatorname{div} v - i q^\varepsilon \nabla(a^\varepsilon g^\varepsilon), \\
\partial_t q^\varepsilon + v\cdot\nabla q^\varepsilon + \operatorname{Im}(g^\varepsilon \overline{a}^\varepsilon \operatorname{div}\psi^\varepsilon) = -\frac{\sigma+1}{2} q^\varepsilon \operatorname{div} v.
\end{cases}
$$

Corollary. $U^\varepsilon := (2q^\varepsilon, a^\varepsilon, \overline{a}^\varepsilon, \psi^\varepsilon, \overline{\psi}^\varepsilon)$ satisfies an equation of the form

$$
\partial_t U^\varepsilon + \sum_{1\leqslant j\leqslant n} A_j(v, a^\varepsilon g^\varepsilon, \overline{a}^\varepsilon g^\varepsilon)\partial_j U^\varepsilon + \varepsilon\mathcal{L}(\partial_x)U^\varepsilon = E(\Phi, U^\varepsilon, a^\varepsilon g^\varepsilon, \nabla(a^\varepsilon g^\varepsilon)),
$$

with $\Phi = (\nabla\phi, \nabla^2\phi, \nabla^3\phi)$, A_j hermitian and lineair in their arguments, $\mathcal{L}(\partial_x) = \sum L_{jk}\partial_j\partial_k$ second-order skew-symmetric operator with constant coefficients, $E \in C^\infty$ vanishes at the origin.

Local modulated energy

$$\begin{cases} \partial_t \psi^\varepsilon + v \cdot \nabla \psi^\varepsilon + i a^\varepsilon g^\varepsilon \nabla q^\varepsilon - i \frac{\varepsilon}{2} \Delta \psi^\varepsilon \\ \qquad = -\frac{1}{2} \psi^\varepsilon \operatorname{div} v - \psi^\varepsilon \cdot \nabla v - \frac{1}{2} a^\varepsilon \nabla \operatorname{div} v - i q^\varepsilon \nabla \left(a^\varepsilon g^\varepsilon \right), \\ \partial_t q^\varepsilon + v \cdot \nabla q^\varepsilon + \operatorname{Im}(g^\varepsilon \overline{a}^\varepsilon \operatorname{div} \psi^\varepsilon) = -\frac{\sigma + 1}{2} q^\varepsilon \operatorname{div} v. \end{cases}$$

Corollary. The modulated energy $e^\varepsilon := |\psi^\varepsilon|^2 + (q^\varepsilon)^2$, solves

$$\partial_t e^\varepsilon + \operatorname{div}(e^\varepsilon \nabla \phi) + \operatorname{div}\big(2 \operatorname{Im}(q^\varepsilon \overline{a}^\varepsilon \psi^\varepsilon)\big) + \operatorname{div}\left(\varepsilon \operatorname{Im}\left(\overline{\psi}^\varepsilon \cdot \nabla \psi^\varepsilon\right)\right)$$
$$= -(q^\varepsilon)^2 \Delta \phi - \operatorname{Re}\left(\left(2 \psi^\varepsilon \cdot \nabla \nabla \phi + a^\varepsilon \nabla \Delta \phi\right) \cdot \overline{\psi}^\varepsilon\right).$$

This yields the desired modulated energy estimate ($k = 1$) by integration and Gronwall's lemma.

Consider the equation

$$\partial_t U^\varepsilon + \sum_{1\leqslant j\leqslant n} A_j(v, a^\varepsilon g^\varepsilon, \overline{a}^\varepsilon g^\varepsilon)\partial_j U^\varepsilon + \varepsilon\mathcal{L}(\partial_x)U^\varepsilon = E(\Phi, U^\varepsilon, a^\varepsilon g^\varepsilon, \nabla(a^\varepsilon g^\varepsilon)),$$

The “trick” is that g^ε is a zero-order term, and we dont form an evolution equation for g^ε. Yet, g^ε is given by an expression of the form $g^\varepsilon = G(|a^\varepsilon|^2, |a|^2)$ where G is not smooth.

If $\sigma \geqslant 3$ or if $\sigma \geqslant 2$ and $n = 1$, then

$$\begin{aligned}\left\|[A_j, \Lambda^{\sigma-1}]\partial_j U^\varepsilon\right\|_{L^2} &\leqslant K \left\|A_j\right\|_{H^\sigma} \left\|U^\varepsilon\right\|_{H^{\sigma-1}} \\ &\leqslant C \left(\|v\|_{H^\sigma} + \|a^\varepsilon g^\varepsilon\|_{H^\sigma}\right) \|U^\varepsilon\|_{H^{\sigma-1}} .\end{aligned}$$

Since $U^\varepsilon = (\dots, \nabla a^\varepsilon, \dots)$, to conclude, it is enough to estimate $a^\varepsilon g^\varepsilon$ in H^σ. Set

$$F_\sigma(z,z') = zG_\sigma\left(|z|^2, |z'|^2\right), \quad \text{so that} \quad a^\varepsilon g^\varepsilon = F_\sigma\left(a^\varepsilon, a\right)$$

One has $F_\sigma \in C^{\sigma-1}$ but $F_\sigma \notin C^\sigma$.

To estimate $a^\varepsilon g^\varepsilon$ in H^σ, one cannot use usual nonlinear estimates. We use that F_σ is homogeneous of degree σ and

Proposition. Let $p \geqslant 1$ and $m \geqslant 2$ be integers and consider $F\colon \mathbf{R}^p \to \mathbf{C}$. Assume that $F \in C^\infty(\mathbf{R}^p \setminus \{0\})$ is homogeneous of degree m, that is:

$$F(\lambda y) = \lambda^m F(y), \qquad \forall \lambda \geqslant 0, \forall y \in \mathbf{R}^p.$$

Then, for $n \leqslant 3$, there exists $K > 0$ such that, for all $u \in H^m(\mathbf{R}^n)$ with values in $\mathbf{R}^p$, $F(u) \in H^m(\mathbf{R}^n)$ and

$$\|F(u)\|_{H^m} \leqslant K \|u\|_{H^m}^m .$$

The same is true when $m = 1$ and $n \in \mathbf{N}$.

This allows to estimate the initial data for q^ε since

$$q^\varepsilon = \frac{|z|^2 - |z'|^2}{\varepsilon} \mathcal{Q}_\sigma(z, z')\Big|_{(z,z')=(a^\varepsilon, a)}, \qquad \mathcal{Q}_\sigma \text{ homogeneous of degree } \sigma - 1.$$

Link between supercriticals

Following Christ-Colliander-Tao, consider initial data that concentrate at the origin

$$u_{h,0}(x) = h^{s-\frac{n}{2}} a_0\left(\frac{x}{h}\right).$$

Introduce the change of variable (already introduced by R. Carles when $\sigma = 1$)

$$u^\varepsilon(t,x) = h^{\frac{n}{2}-s} u_h\left(h^2 \varepsilon t, hx\right), \quad \varepsilon = h^{\sigma(s_c - s)},$$

which solves

$$i\varepsilon\partial_t u^\varepsilon + \frac{\varepsilon^2}{2}\Delta u^\varepsilon = |u^\varepsilon|^{2\sigma} u^\varepsilon \quad ; \quad u^\varepsilon(0,x) = a_0(x).$$

Since,

$$\|u_h(t)\|_{\dot{H}^m} = h^{s-m}\left\| u^\varepsilon\left(\frac{t}{h^2\varepsilon}\right)\right\|_{\dot{H}^m},$$

it is enough to prove that u^ε becomes ε-oscillatory for times of order $O(1)$.

The linear equation

Set

$$u^\varepsilon(t) = e^{-itH^\varepsilon/\varepsilon} a_0, \quad H^\varepsilon := -(\varepsilon^2/2)\Delta + V(x).$$

Then (Egorov)

$$\| \operatorname{Op}_\varepsilon(q) u^\varepsilon(t)\|_{L^2} = \|e^{itH^\varepsilon/\varepsilon} \operatorname{Op}_\varepsilon(q) e^{-itH^\varepsilon/\varepsilon} a_0\|_{L^2} = \| \operatorname{Op}_\varepsilon(q \circ \Phi_t) a_0\|_{L^2} + O(\varepsilon),$$

with

$$\Phi_t(x,\xi) = \big(X(t,x) + t\xi\, ,\, \xi + (\nabla\phi)(t, X(t,x))\big),$$

$$\partial_t \phi + \frac{1}{2}|\nabla\phi|^2 + V(x) = 0, \quad \phi(0,x) = 0,$$

$$\partial_t X(t,x) = (\nabla\phi)(t, X(t,x)), \quad X(0,x) = x.$$

With $q(x,\xi) = i\xi$, we obtain

$$\|\varepsilon\nabla u^\varepsilon(t)\|_{L^2} = \|(\nabla\phi)(t, X(t,x)) a_0\|_{L^2} + O(\varepsilon).$$

The solution becomes ε-oscillatory for $t > 0$.

Proposition. $\beta^\varepsilon \in C^1_{t,x}$ and $g^\varepsilon \in C^0_{t,x}$. Moreover,

$$\partial_t \beta^\varepsilon + \varepsilon g^\varepsilon \operatorname{div}(\operatorname{Im}(\overline{a}^\varepsilon \nabla a^\varepsilon)) + v \cdot \nabla \beta^\varepsilon + \frac{\sigma+1}{2} \beta^\varepsilon \operatorname{div} v = 0.$$

Proof. We have found

$$\beta^\varepsilon \Big(\partial_t \beta^\varepsilon + \varepsilon g^\varepsilon \operatorname{div}(\operatorname{Im}(\overline{a}^\varepsilon \nabla a^\varepsilon)) + v \cdot \nabla \beta^\varepsilon + \frac{\sigma+1}{2} \beta^\varepsilon \operatorname{div} v \Big) = 0.$$

One has the equation in $\{\beta^\varepsilon \neq 0\}$. Since $\beta^\varepsilon \in C^1_{t,x}$, we need only prove that the equation is satisfied in the interior of $\omega^\varepsilon = ([0, \tau^\varepsilon[\times \mathbf{R}^n) \setminus \{\beta^\varepsilon \neq 0\}$. Note that $\omega^\varepsilon = \{\rho^\varepsilon = \rho\}$. Hence,

- In $\overset{\circ}{\omega}{}^\varepsilon$, one has $\beta^\varepsilon = 0$, $\partial_{t,x} \beta^\varepsilon = 0$.
- In $\overset{\circ}{\omega}{}^\varepsilon$, the conservation laws imply

$$\operatorname{div}(\operatorname{Im}(\overline{a}^\varepsilon \nabla a^\varepsilon)) = -\varepsilon^{-1} \Big(\partial_t (\rho^\varepsilon - \rho) + \operatorname{div}((\rho^\varepsilon - \rho) v) \Big) = 0.$$

Conservation laws for NLS, and blow-up

Pseudo-conformal law:

$$\frac{d}{dt}\left(\frac{1}{2}\|(x+i\varepsilon t\nabla_x)u^\varepsilon\|_{L^2}^2+\frac{t^2}{\sigma+1}\|u^\varepsilon\|_{L^{2\sigma+2}}^{2\sigma+2}\right)=\frac{t}{\sigma+1}(2-n\sigma)\|u^\varepsilon\|_{L^{2\sigma+2}}^{2\sigma+2}.$$

(via the scaling $\psi(t,x)=u(\varepsilon t,\varepsilon x)$.)

Write $u^\varepsilon = a^\varepsilon e^{i\phi/\varepsilon}$, and pass to the limit formally:

$$\frac{d}{dt}\int\left(\frac{1}{2}\left|(x-t\nabla\phi(t,x))\,a(t,x)\right|^2+\frac{t^2}{\sigma+1}|a(t,x)|^{2\sigma+2}\right)dx$$
$$=\frac{t}{\sigma+1}(2-n\sigma)\int|a(t,x)|^{2\sigma+2}dx.$$

Implies that $\|\rho(t)\|_{L^{2\sigma+2}}\to 0$.
Contradiction with

$$\partial_t\rho+\operatorname{div}(\rho v)=0.$$

Convergence of the quadratic observables

Wigner transform:

$$w^\varepsilon[f](x,\xi) = (2\pi)^{-n} \int_{\mathbf{R}^n} f\left(x - \varepsilon\frac{\eta}{2}\right) \overline{f}\left(x + \varepsilon\frac{\eta}{2}\right) e^{i\eta\cdot\xi}\, d\eta.$$

If ψ^ε is uniformly bounded in L^2, then $w^\varepsilon[\psi^\varepsilon]$ converges weakly as a tempered distribution.

Introduce

$$\rho := |a|^2, \quad v = \nabla\phi, \quad \rho^\varepsilon := |a^\varepsilon|^2 = |u^\varepsilon|^2 .$$

One has

$$\partial_t \rho + \operatorname{div}(\rho v) = 0,$$
$$\partial_t \rho^\varepsilon + \operatorname{div} \operatorname{Im} \left(\varepsilon \overline{u}^\varepsilon \nabla u^\varepsilon \right) = 0.$$

Hence, by defintion of $a^\varepsilon = u^\varepsilon e^{-i\phi/\varepsilon}$,

$$\partial_t \rho^\varepsilon + \operatorname{div} \left(\operatorname{Im}(\varepsilon \overline{a}^\varepsilon \nabla a^\varepsilon) + \rho^\varepsilon v \right) = 0.$$

Writing $\partial_t \beta^\varepsilon = (\partial_{r_1} B)(\rho^\varepsilon, \rho)\partial_t \rho^\varepsilon + (\partial_{r_2} B)(\rho^\varepsilon, \rho)\partial_t \rho$, we find

$$\partial_t \beta^\varepsilon + (\partial_{r_1} B)(\rho^\varepsilon, \rho) \operatorname{div}(\operatorname{Im}(\varepsilon \overline{a}^\varepsilon \nabla a^\varepsilon) + \rho^\varepsilon v) + (\partial_{r_2} B)(\rho^\varepsilon, \rho) \operatorname{div}(\rho v) = 0.$$