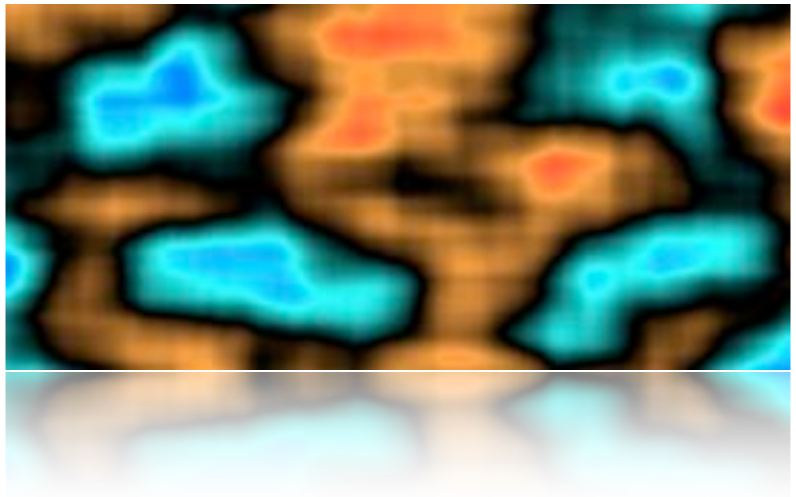
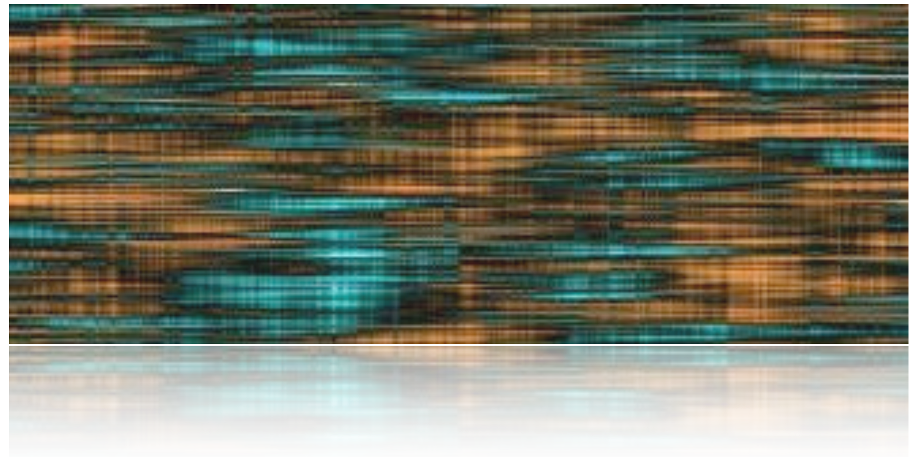
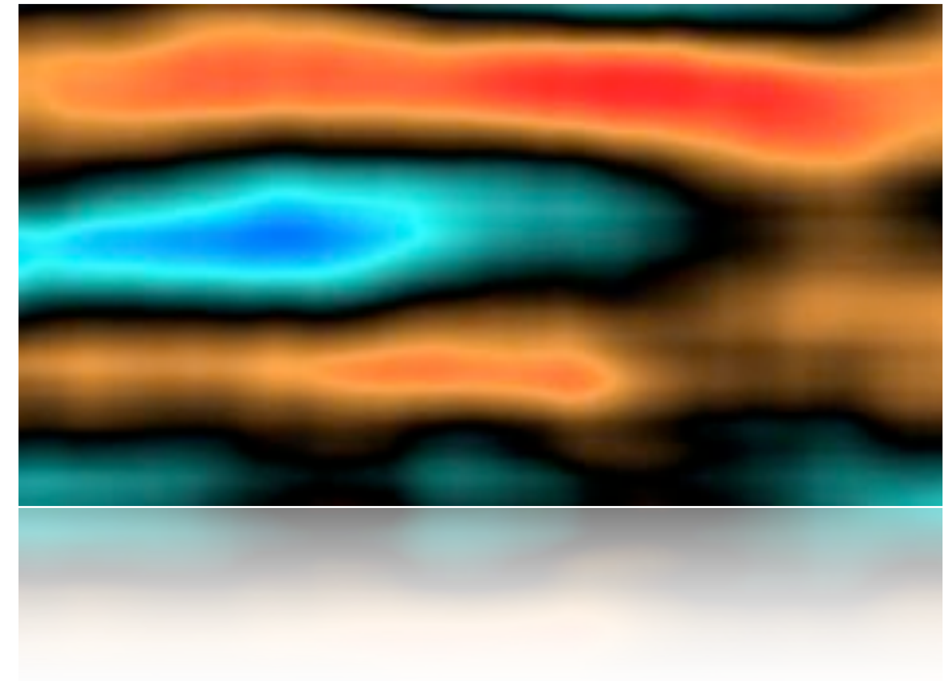
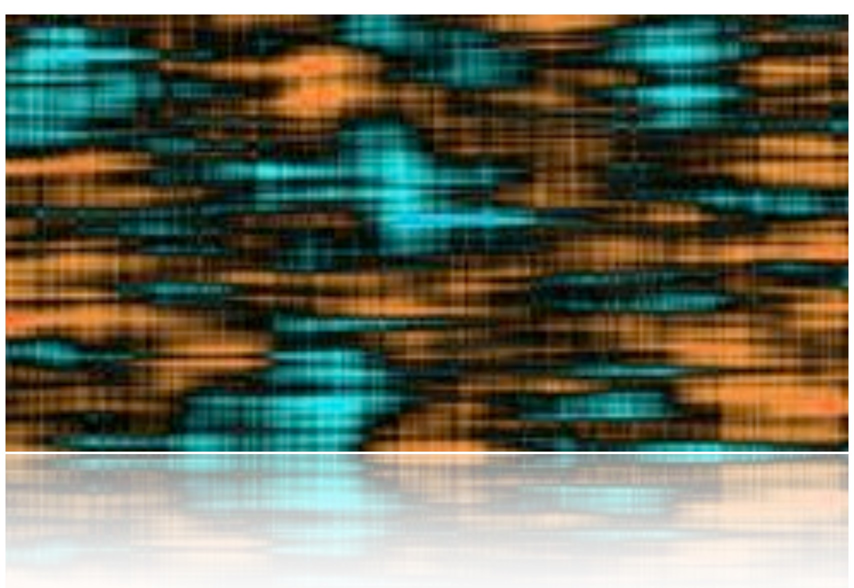


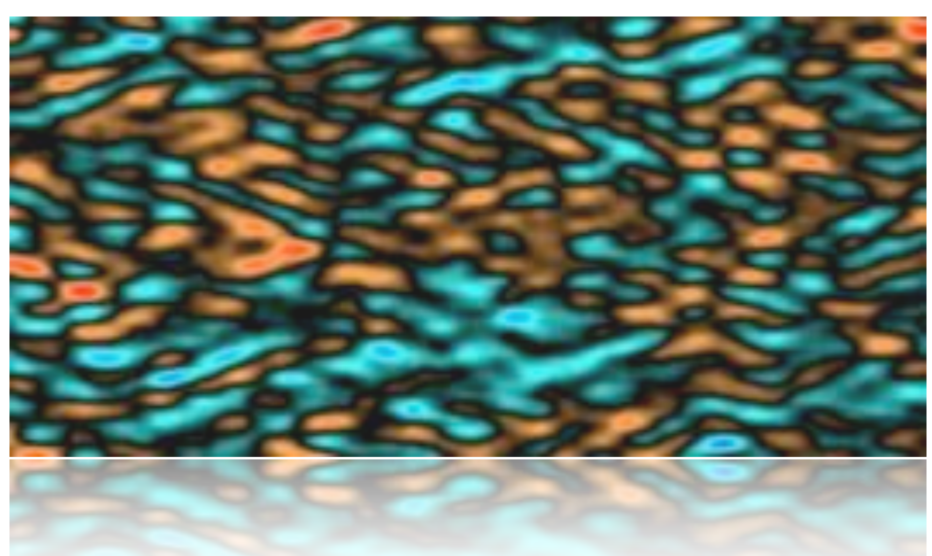


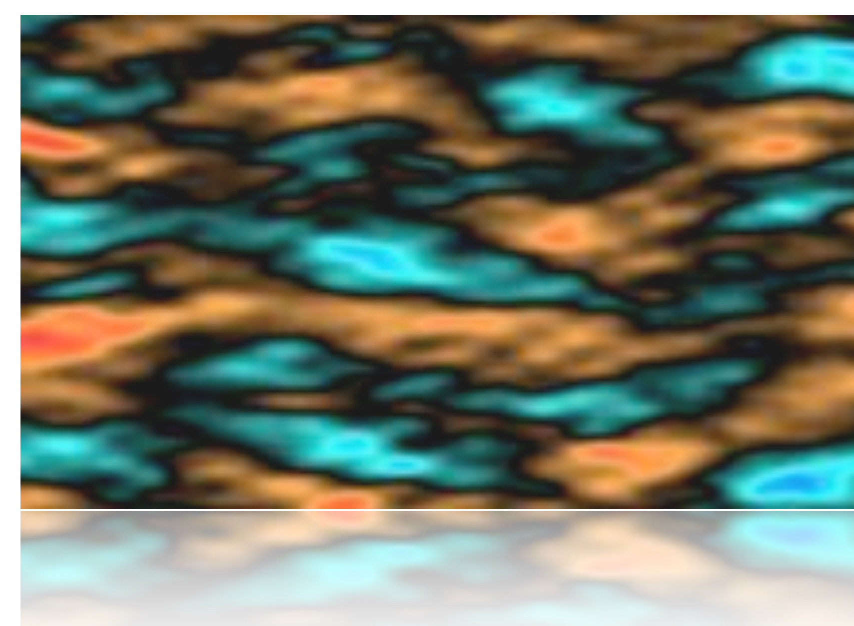


Feedback control of fluid flow and stochastic methods

Jérôme Hœpffner
Post-Doc, Fukagata Lab.
Keio university







Stochastic disturbances

Flow systems of engineering interest are often exposed to disturbances that are erratic, unpredictable, and thus conveniently described by their statistics.

- wall roughness
- Free-stream turbulence
- Acoustic waves

Statistics

Random vector $w = \begin{pmatrix} w_1(t) \\ w_2(t) \\ \vdots \\ w_N(t) \end{pmatrix}$, *White noise* if: $E w_i(t)\overline{w_j(t')} = W_{ij}\delta(t-t')$

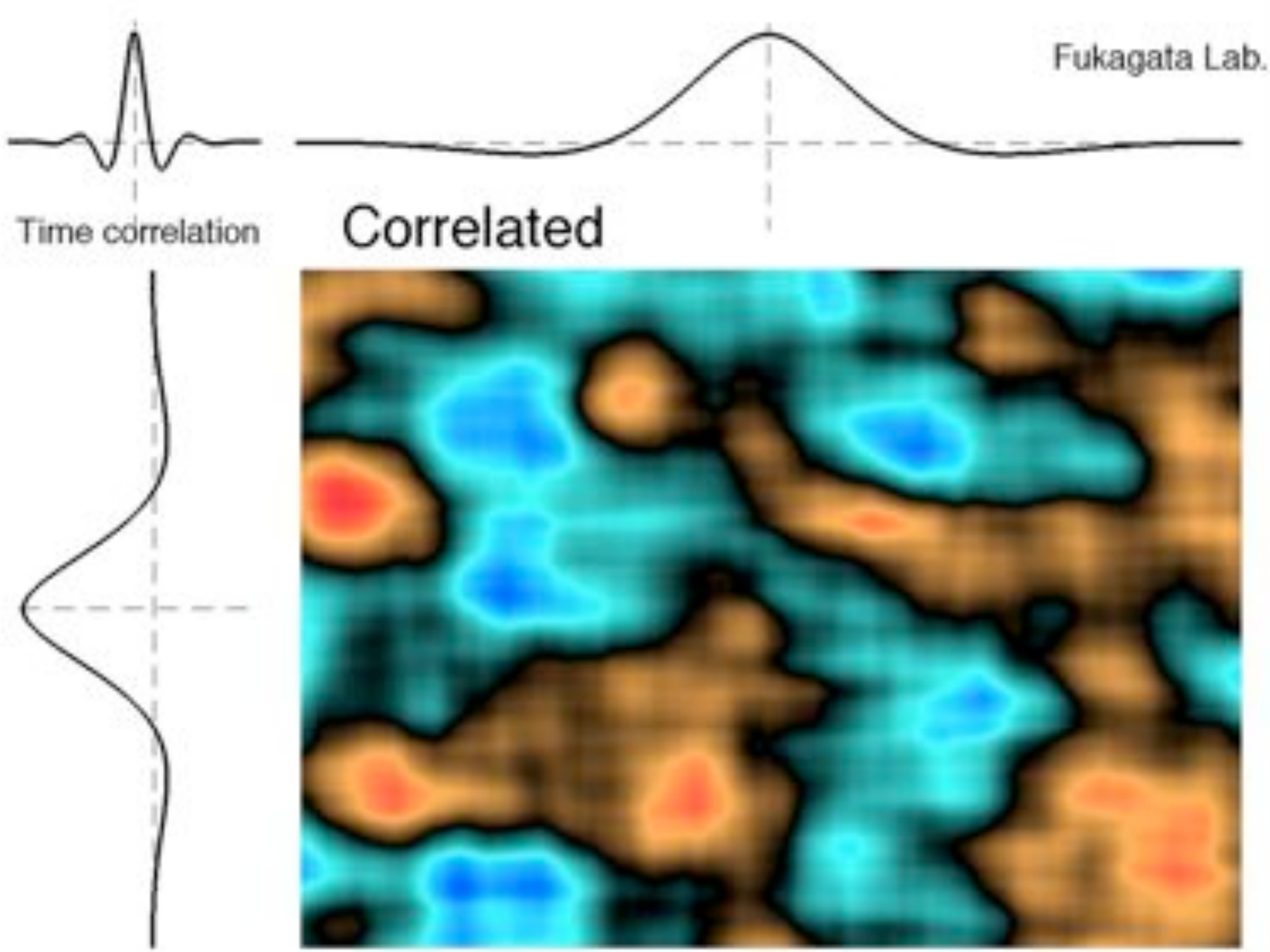
Covariance matrix:

$$W \triangleq E w w^H = \begin{pmatrix} E|w_1|^2 & E w_1\overline{w_2} & \dots & E w_1\overline{w_N} \\ E w_2\overline{w_1} & E|w_2|^2 & & \vdots \\ \vdots & & \ddots & \vdots \\ E w_N\overline{w_1} & \dots & \dots & E|w_N|^2 \end{pmatrix}$$

Diagonal elements: variance

Off-diagonal elements: covariance

Illustration of spatial and temporal statistics



1) Stochastic flow systems

$$\dot{q} = Aq + w, \quad \mathsf{cov}(w) = W$$

stochastic excitation→ stochastic state

q should now be described by its covariance matrix P.

How to get P from A and W ?

Lyapunov equation

Explicit state solution:

$$\dot{q} = Aq + w \quad \Rightarrow \quad q(t) = \int_{\tau=0}^{\infty} e^{A(t-\tau)} w(\tau) \mathsf{d}\tau + e^{At} q_0$$

State covariance:

$$\underbrace{Eq(t)q(t)^H}_{P(t,t)} = \int_0^{\infty} \int_0^{\infty} e^{A(t-\tau)} \overbrace{Ew(\tau)w(\tau')^H}^{W\delta(\tau-\tau')} e^{A^H(t-\tau')} \mathsf{d}\tau \mathsf{d}\tau'$$

$$= \int_0^{\infty} e^{A(t-\tau)} W e^{A^H(t-\tau)} \mathsf{d}\tau$$

Differentiating this convolution integral:

$$\dot{P} = AP + PA^H + W$$

Numerical solution of the Lyapunov equation

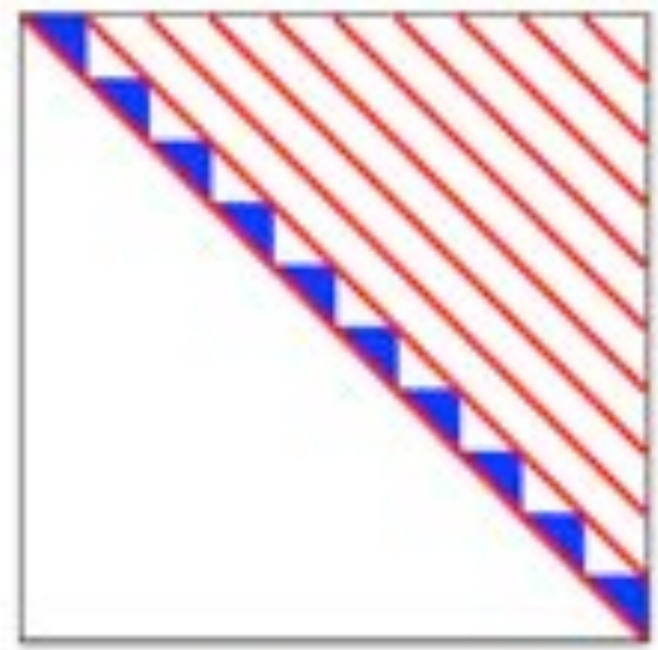
Solve: $AX + XA^H + W = 0$

1. Schur decomposition $A = UA'U^H$, $\rightarrow A'$ upper diagonal, U orthogonal.
2. Resulting equation $A'\overbrace{U^H X U}^{X'} + \overbrace{U^H X U}^{X'} A'^H + \overbrace{U^H W U}^{W'} = 0$
3. Use Kronecker product $\otimes$

$$A \otimes B \triangleq \begin{pmatrix} a_{11}B & a_{12}B & \dots & a_{1m}B \\ a_{21}B & a_{22}B & \dots & a_{2m}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}B & a_{n2}B & \dots & a_{nm}B \end{pmatrix}$$

$$\text{vec}(A'X' + X'A'^H + W') = 0$$

$$= \underbrace{(I \otimes A' + \overline{A}' \otimes I)}_{\mathcal{F}} \text{vec}(X') + \text{vec}(W')$$



$\mathcal{F}$ has upper diagonal structure

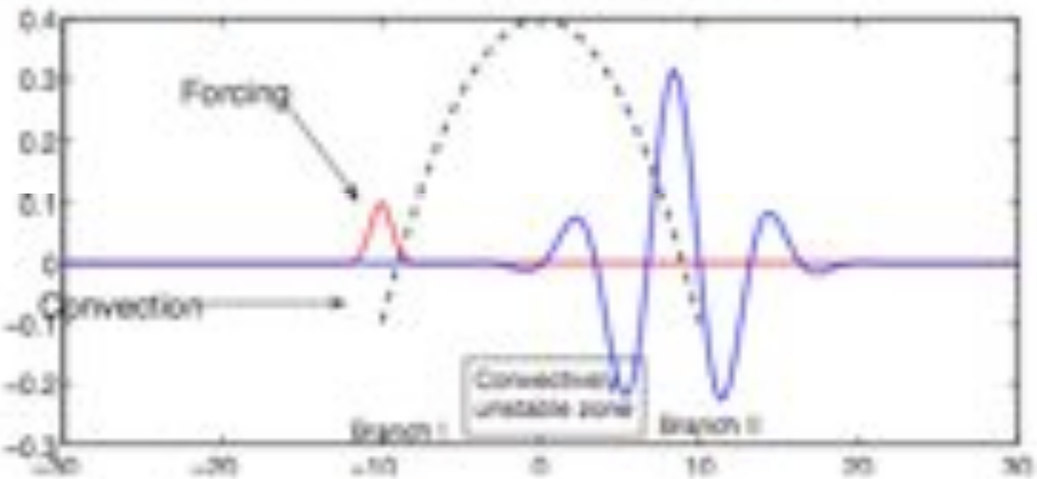
4. Solve by backward substitution

1D example: Ginzburg-Landau

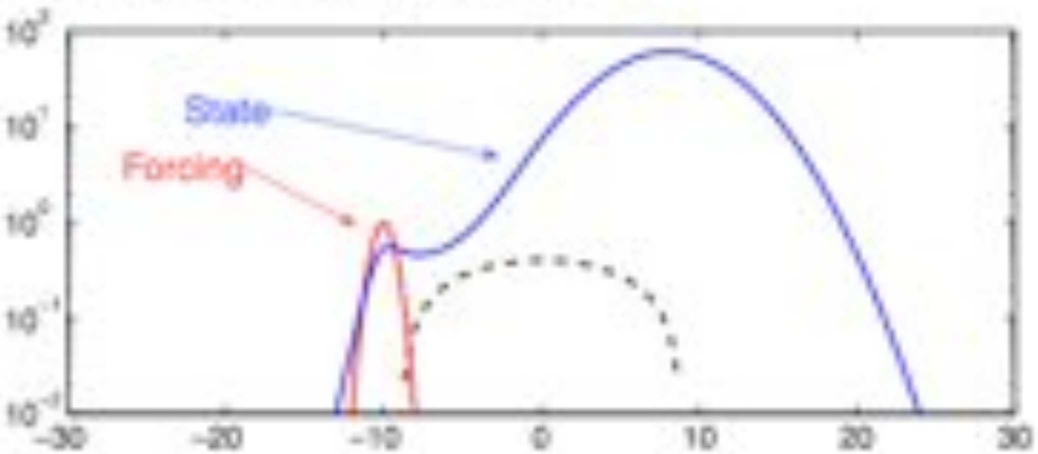
$$\dot{q} + Uq_x = \gamma q_{xx} + \mu(x)q$$

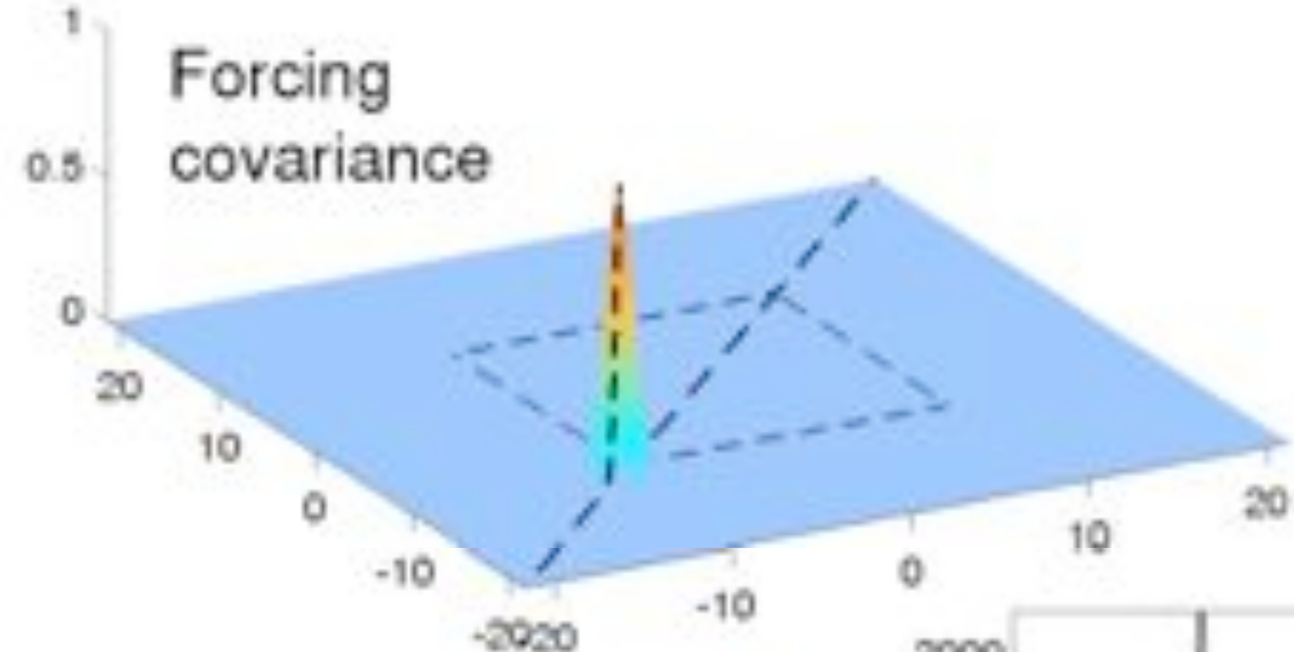
Excitations: $w(x,t) = f(x)\lambda(t)$,
$\lambda \in \mathbb{R}$ is white noise, $E|w|^2 = 1$.

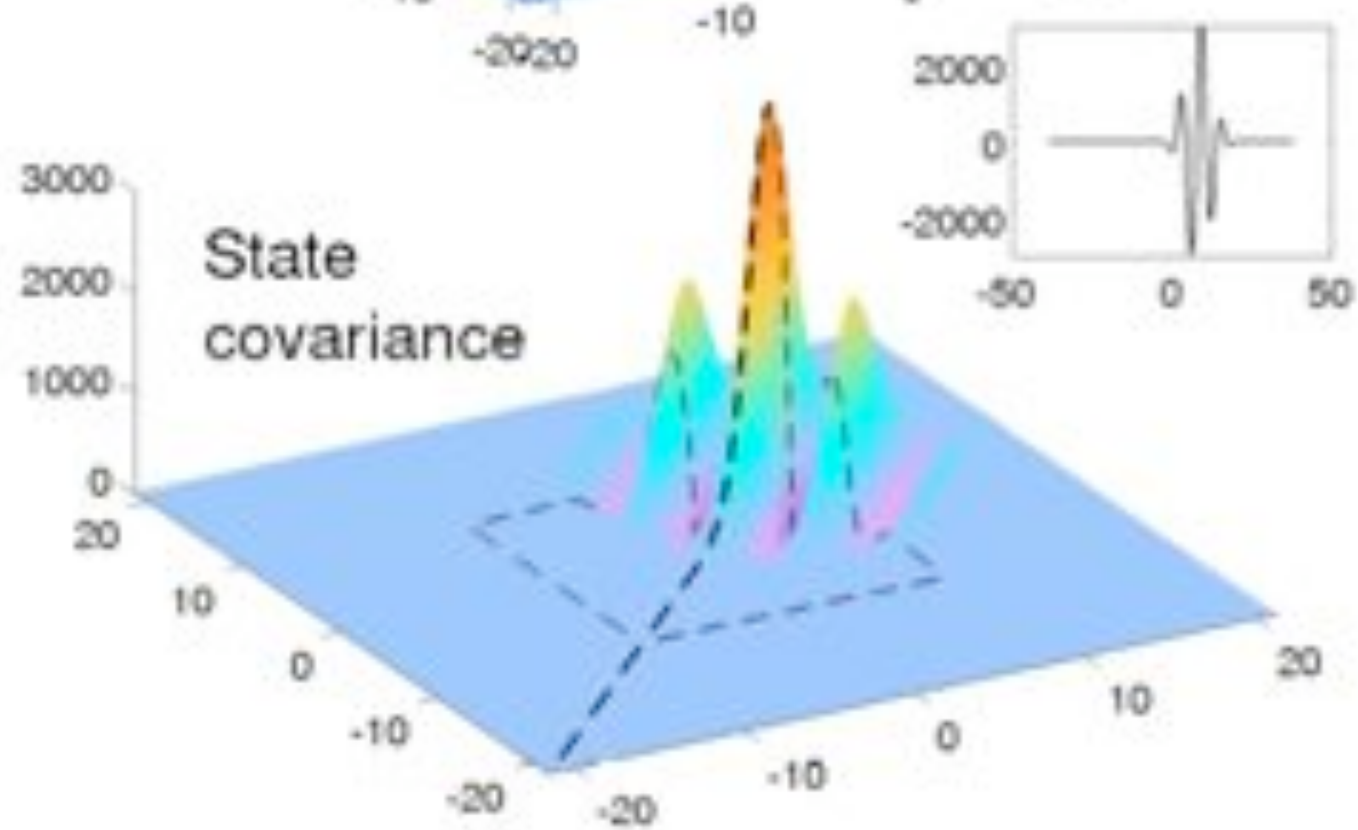
Convectively unstable region:



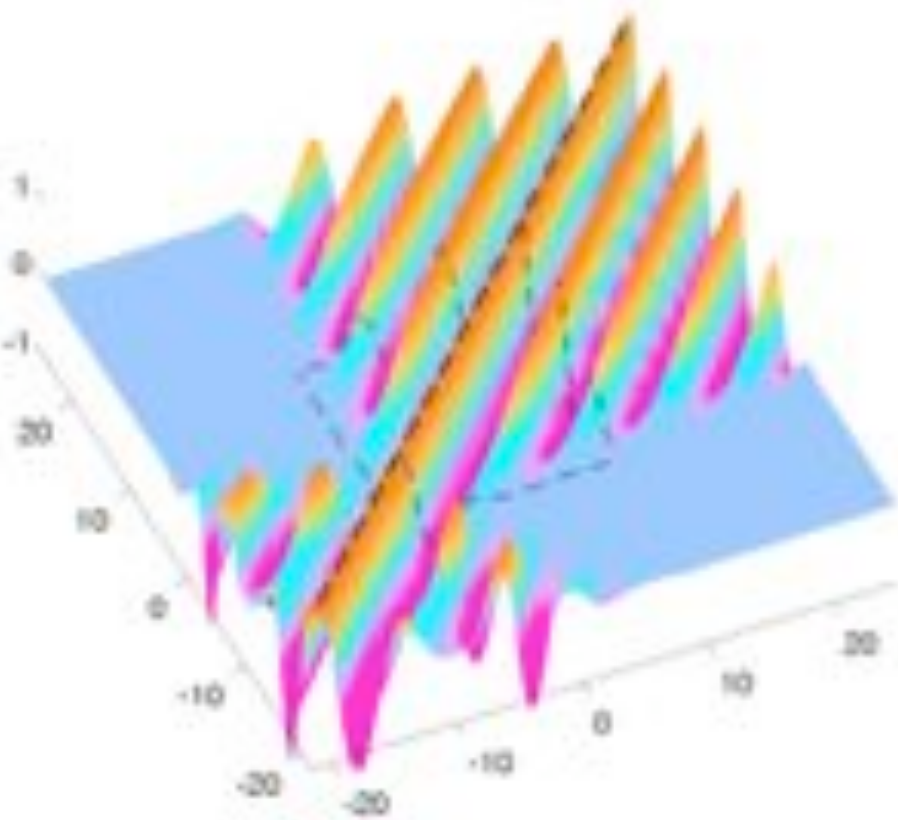
Forcing and State *rms*:







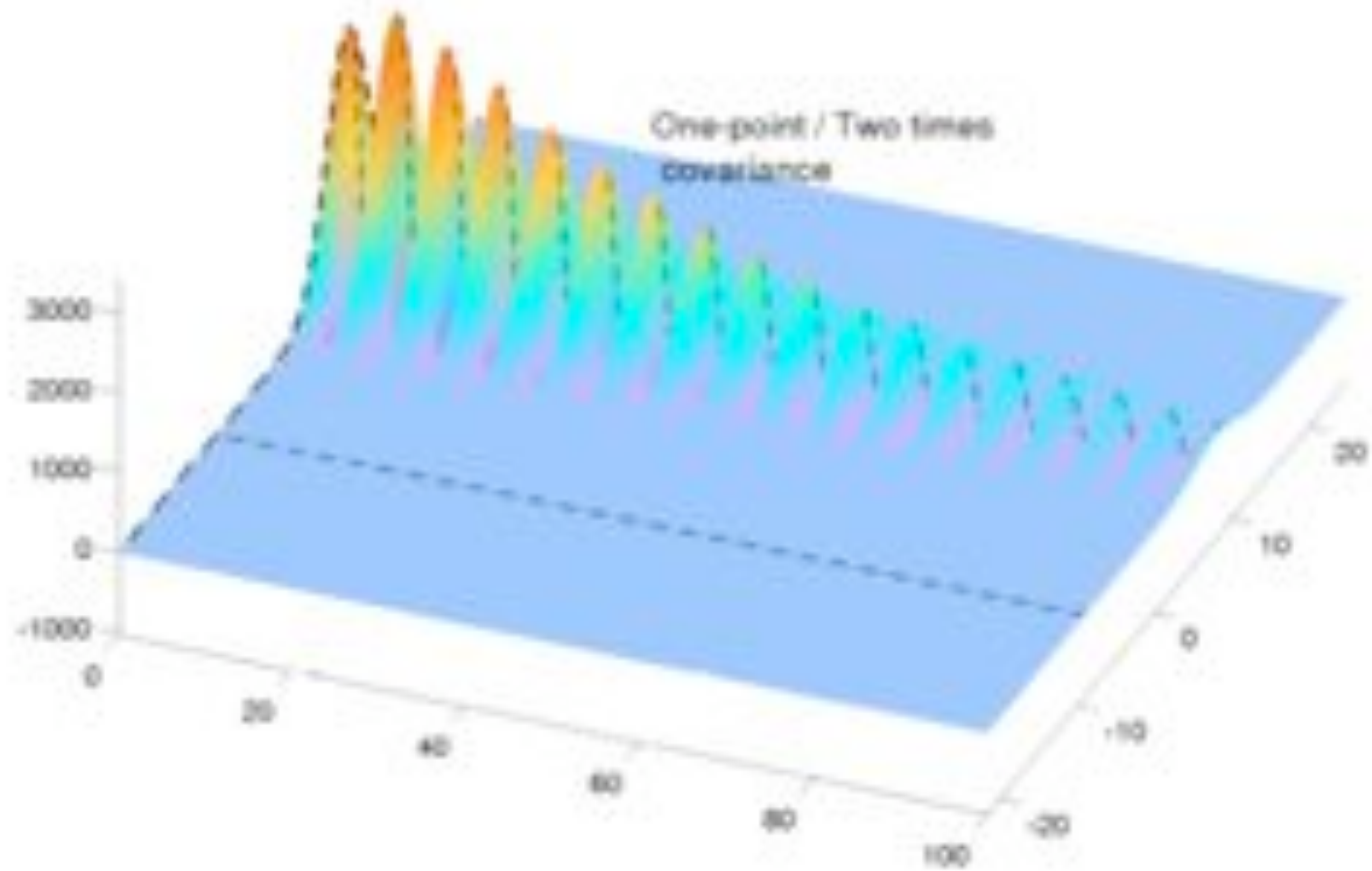
1D example: Ginzburg-Landau



One point/Two times covariance:

$$\mathrm{cov}(q(t), q(t')) = P(t, t') = \mathrm{e}^{A(t'-t)} P(t, t),$$

$$0 = AP + PA^{+} + W.$$

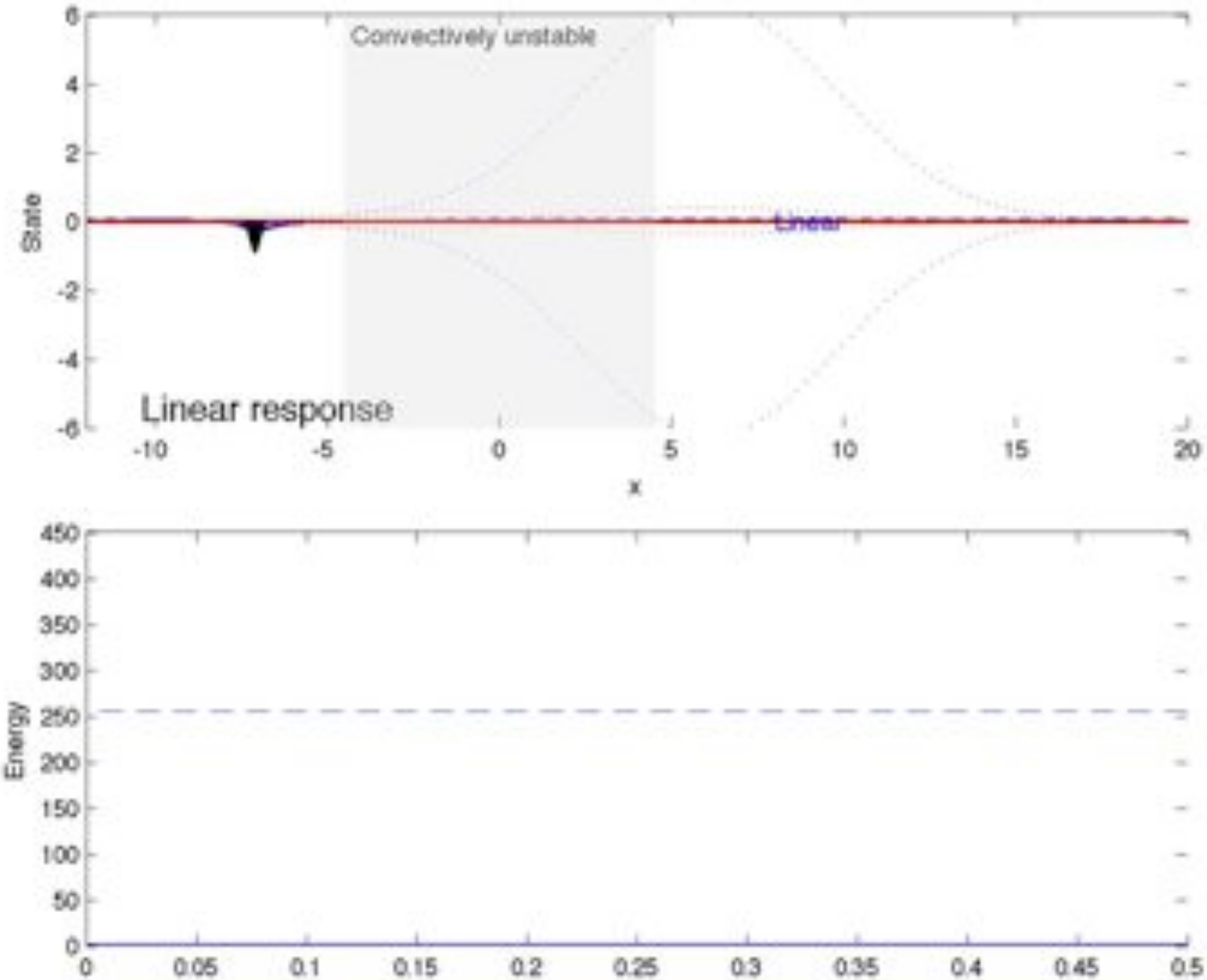


Two points correlation

(normalized to unit *rms*):

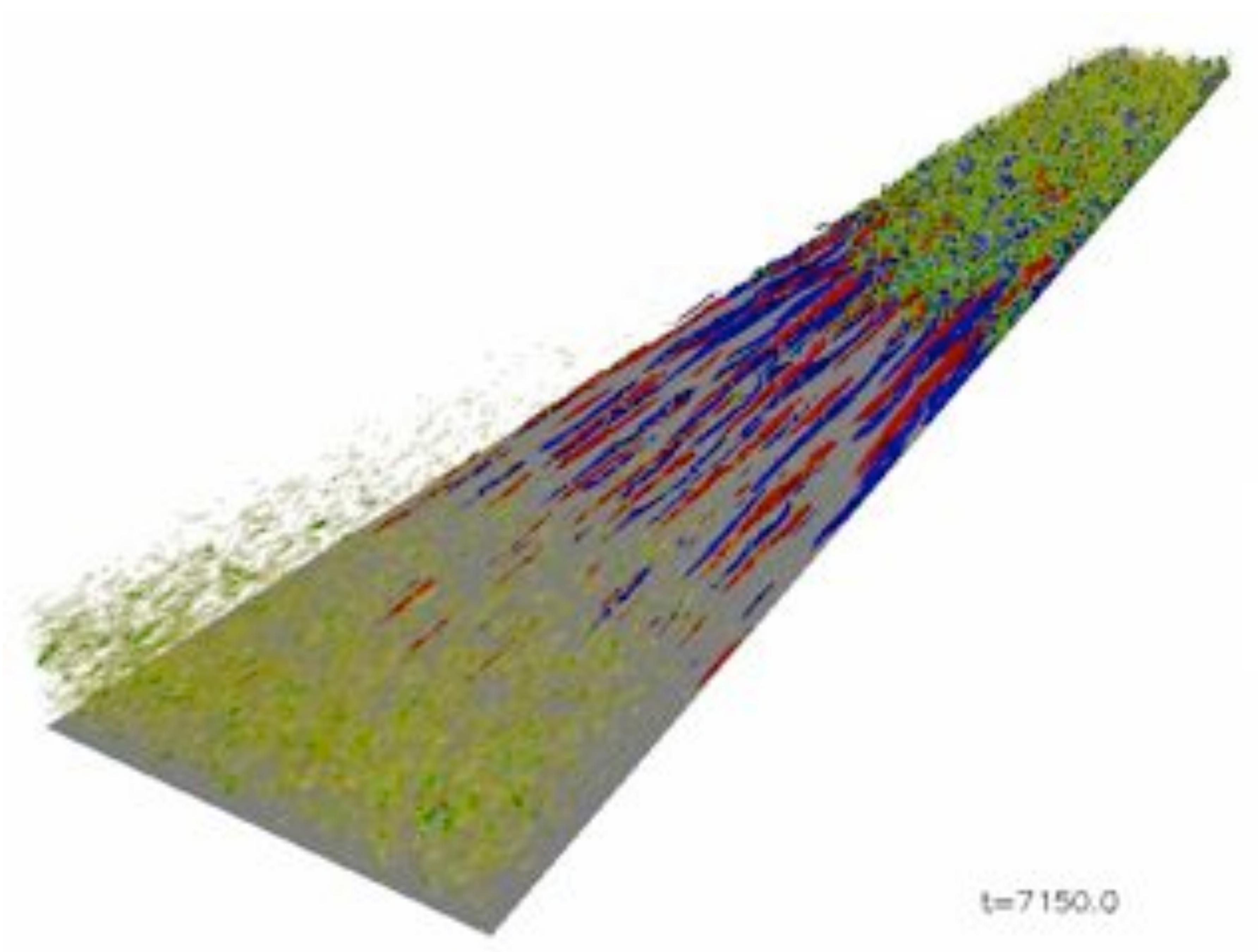
$$\mathrm{corr}(q_i, q_j) = E\frac{q_i\overline{q_j}}{|q_i||q_j|} = \tilde{P}_{ij}$$

Example: Ginzburg landau excited by random perturbations



A flow example: transition to turbulence in boundary layers

With Luca Brandt



Fully turbulent inflow and flat plate

Turbulent free-stream
→ receptivity
→ streaks
→ streak instability
→ turbulent spots
→ turbulent boundary layer

LES: Schlatter & Brandt

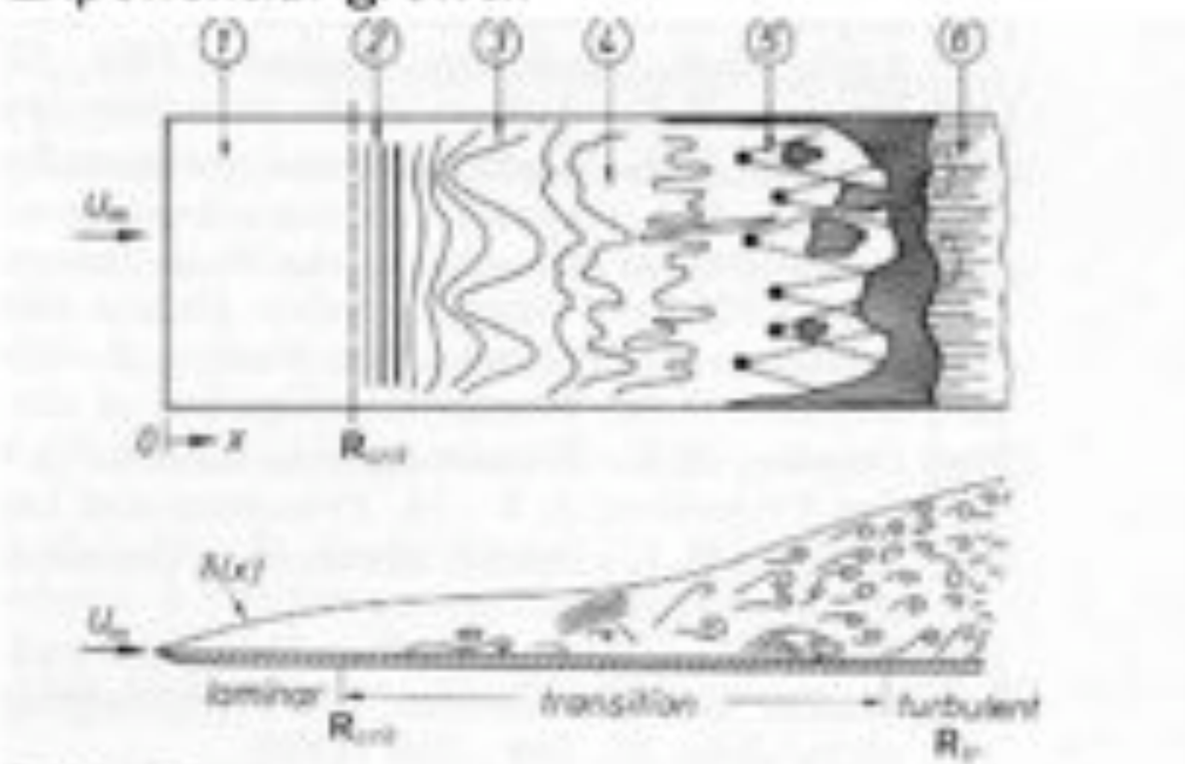
Boundary layer stability

TS waves:

Large Reynolds

2D waves

Exponential growth

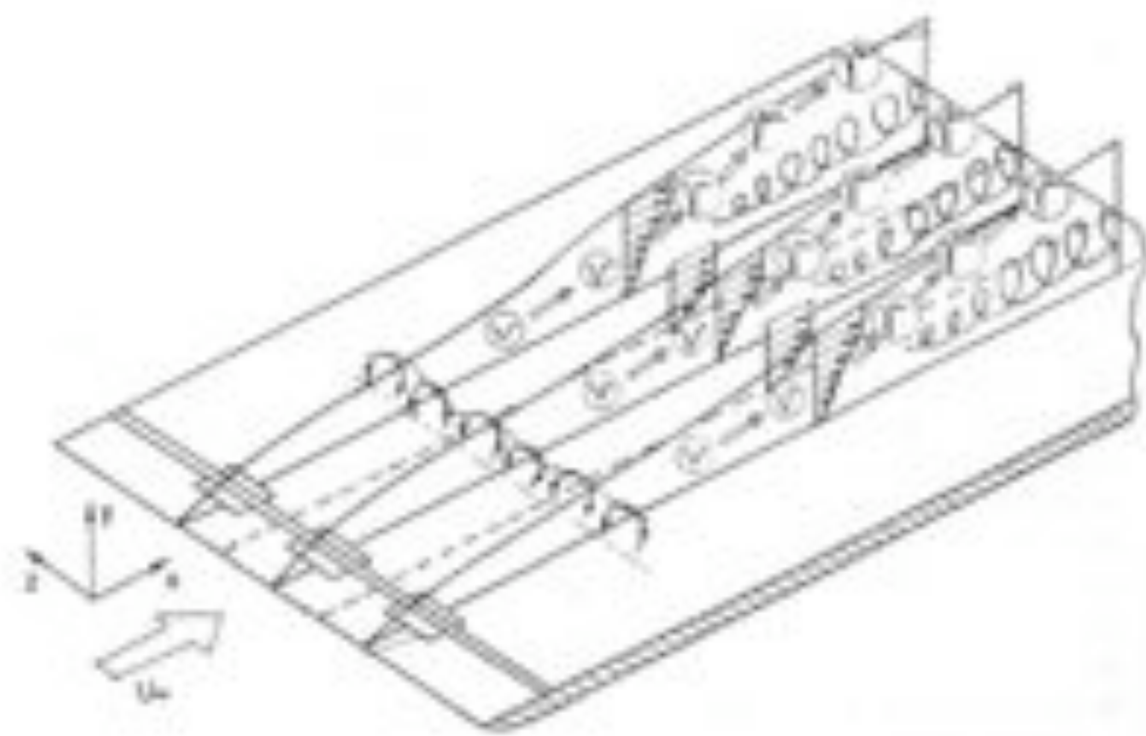


Streaks:

Subcritical Reynolds

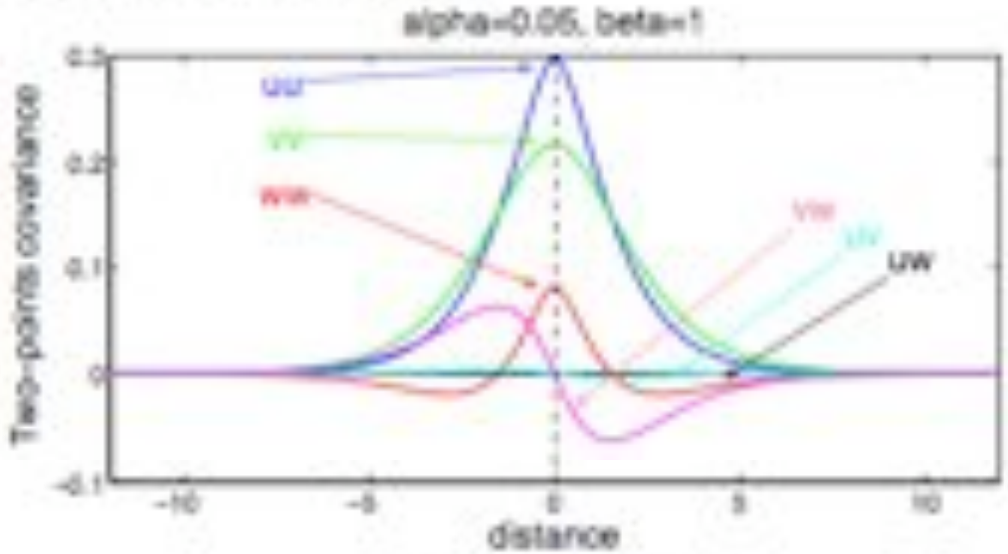
Large external disturbances

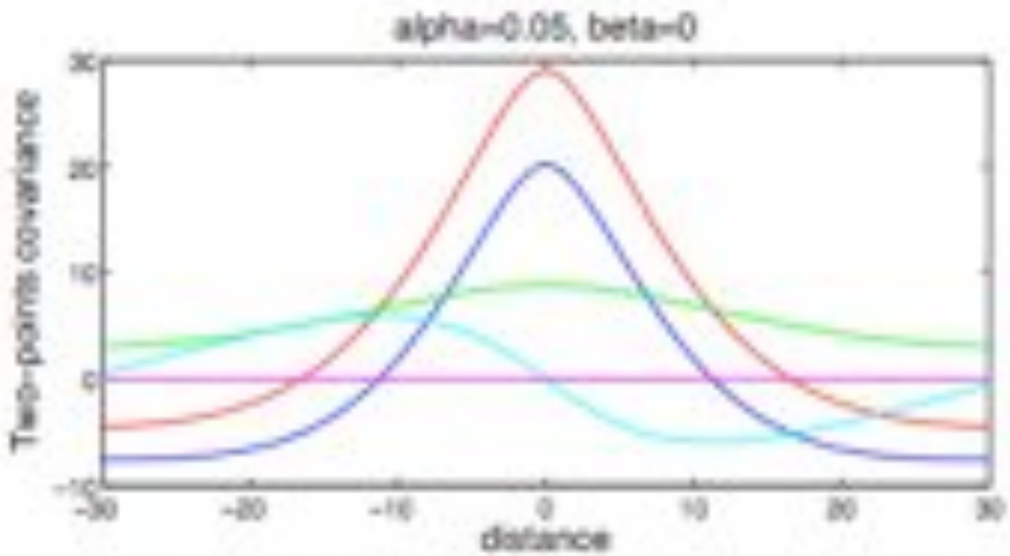
Transient growth



Two-point correlation of FST

Covariance in y:

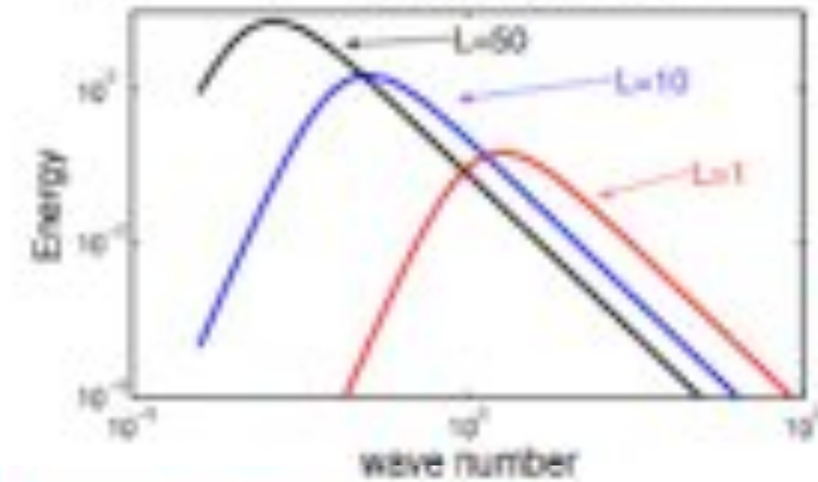




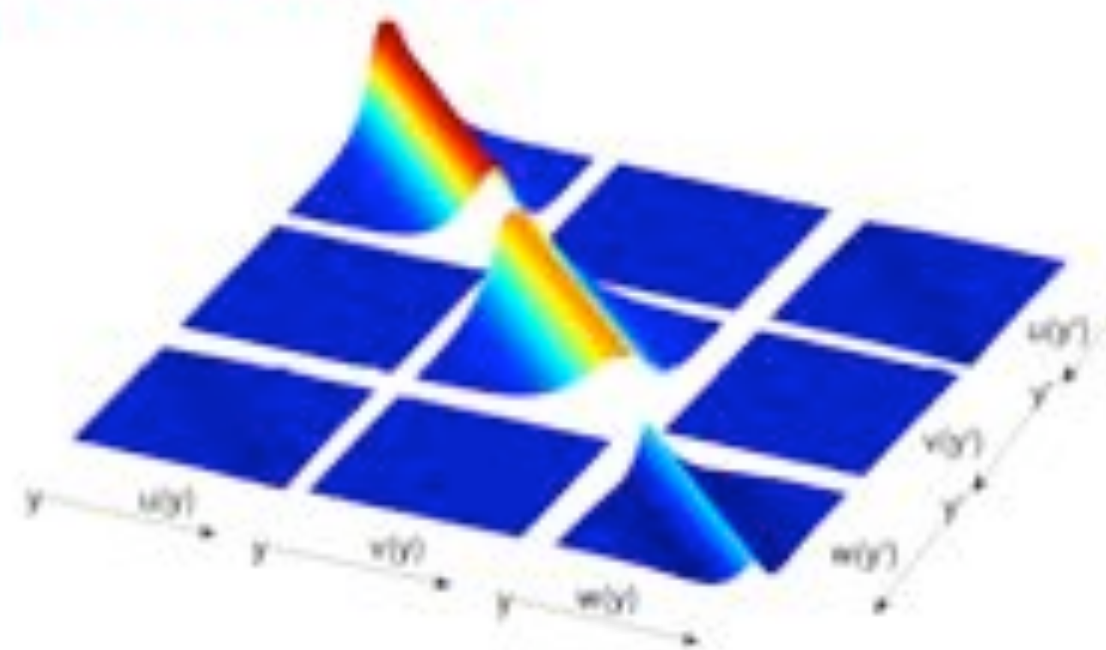
Isotropic turbulence: Von-Karman spectrum

Energy spectra:

$$E(k) = \frac{2}{3}kL\frac{a(kL)^4}{(b+(kL)^2)^{(17/6)}}$$

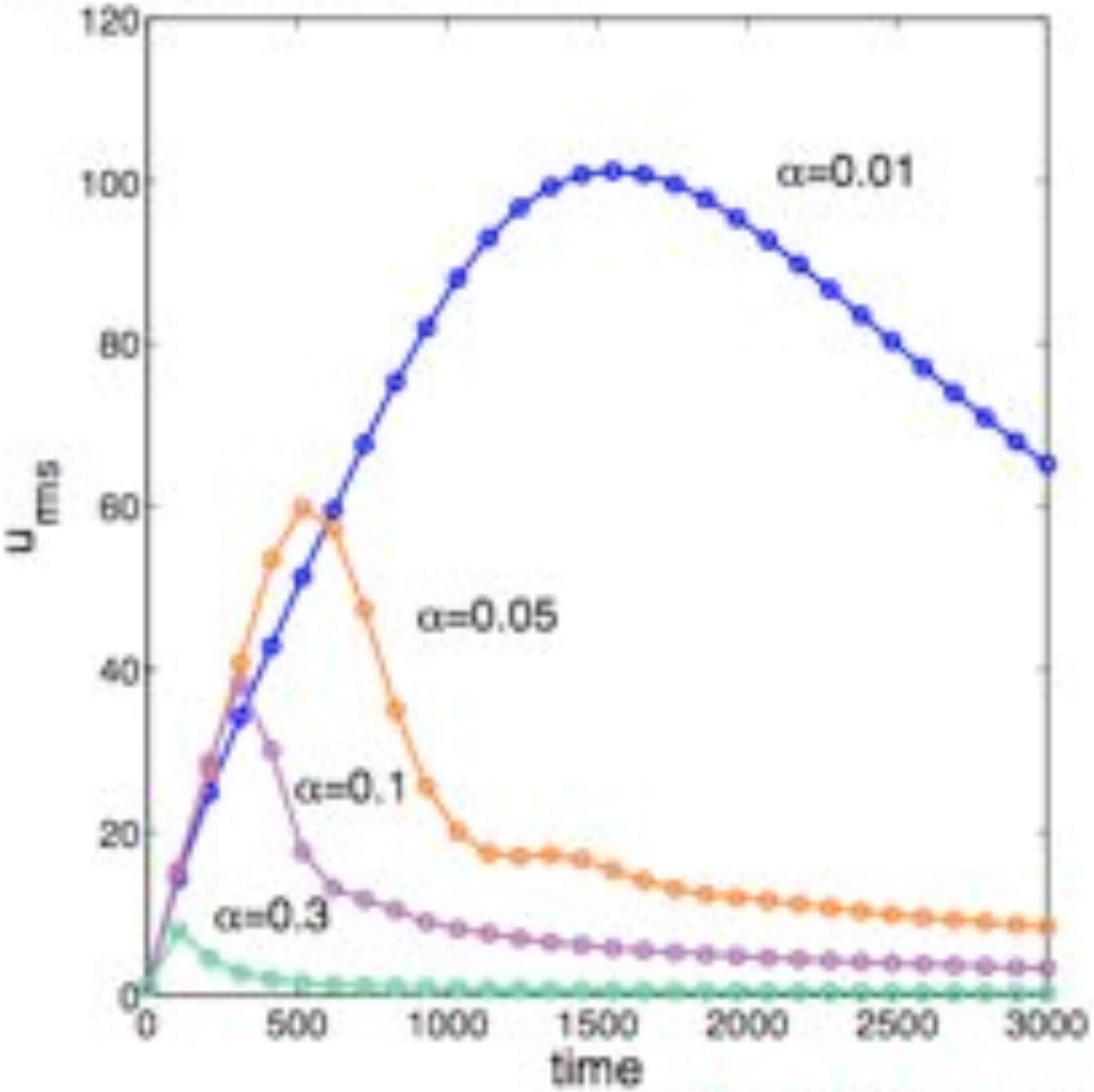


Covariance matrix:

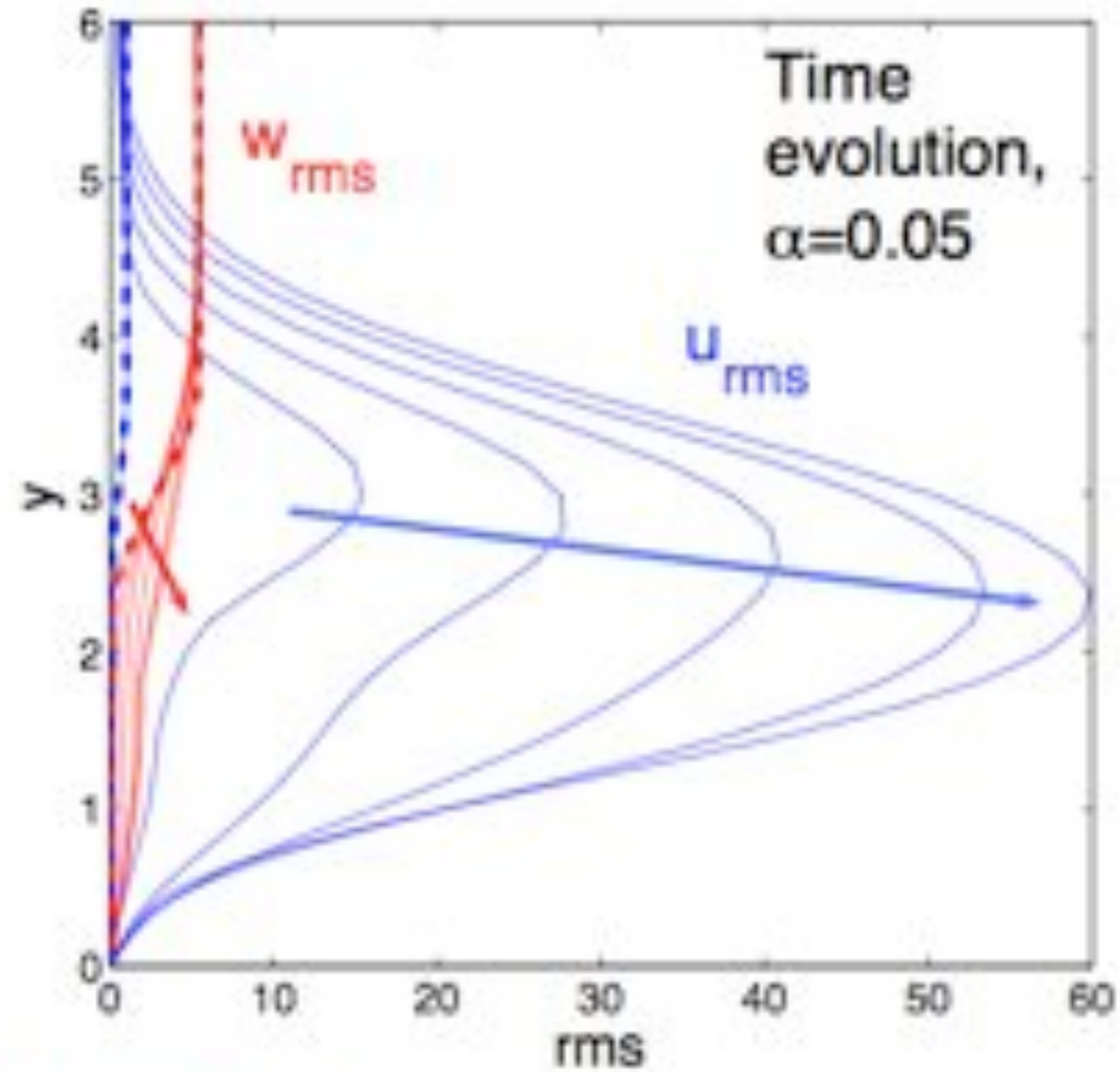


Streaky flow excited by FST: Stochastic initial value problem

Time evolution of u_{rms}



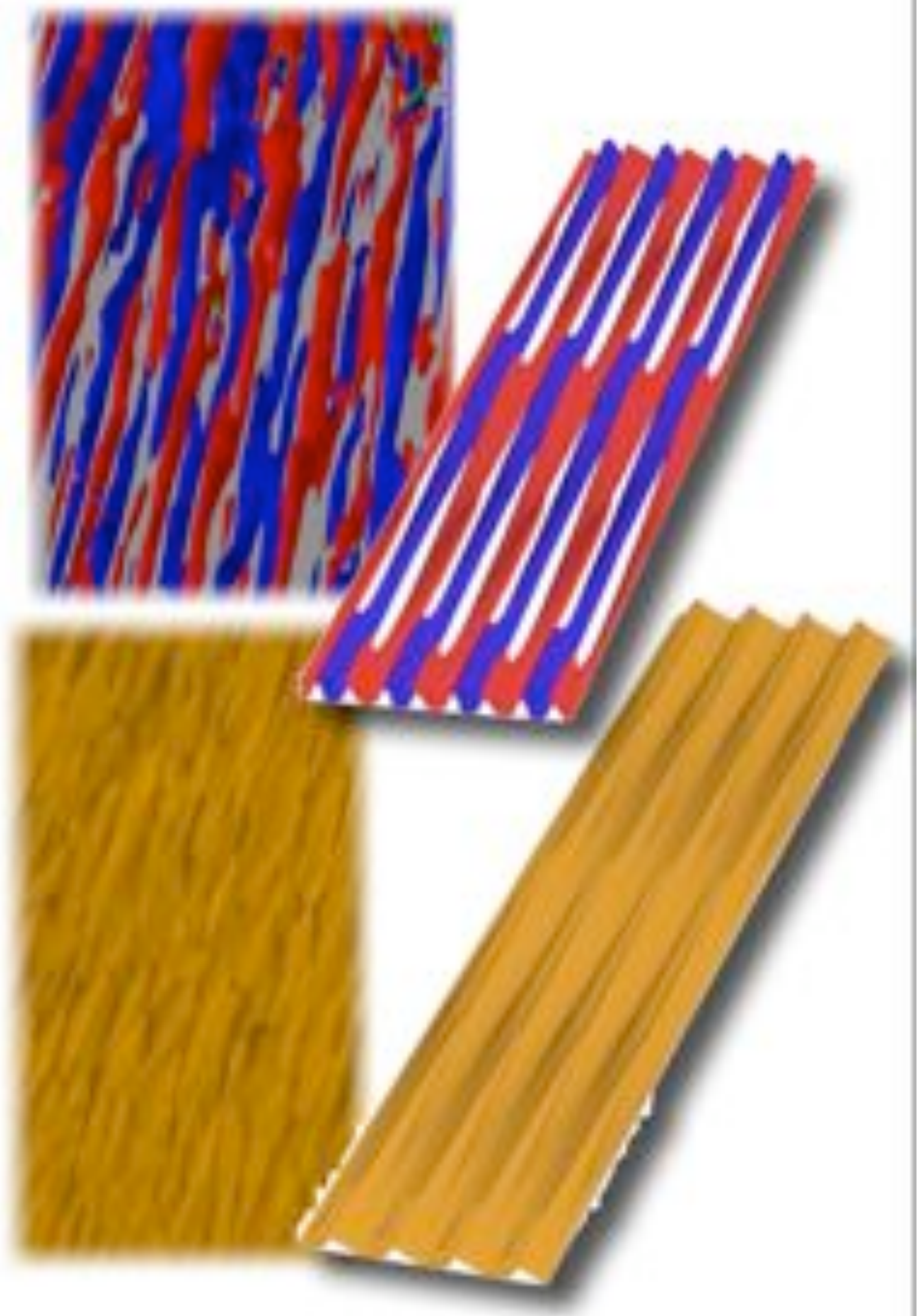
rms profile:



The FST is the flow initial condition

Comparison of flow structures: Streamwise velocity

Comparison of flow structures: Streamwise shear



An other example of flow system with statistical analysis:

Channel with compliant walls

With Bottaro & Favier (Genova, Italy)

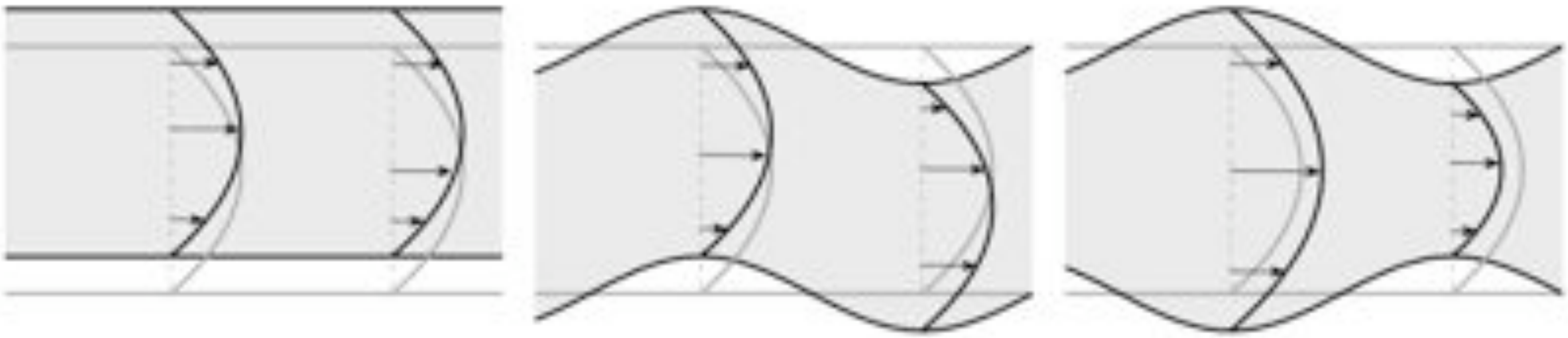


FIGURE 6. Sketch of the flow deformation for the sinuous mechanism at infinite (left) and finite (center) wavelengths, On the right the varicose mechanism is sketched.

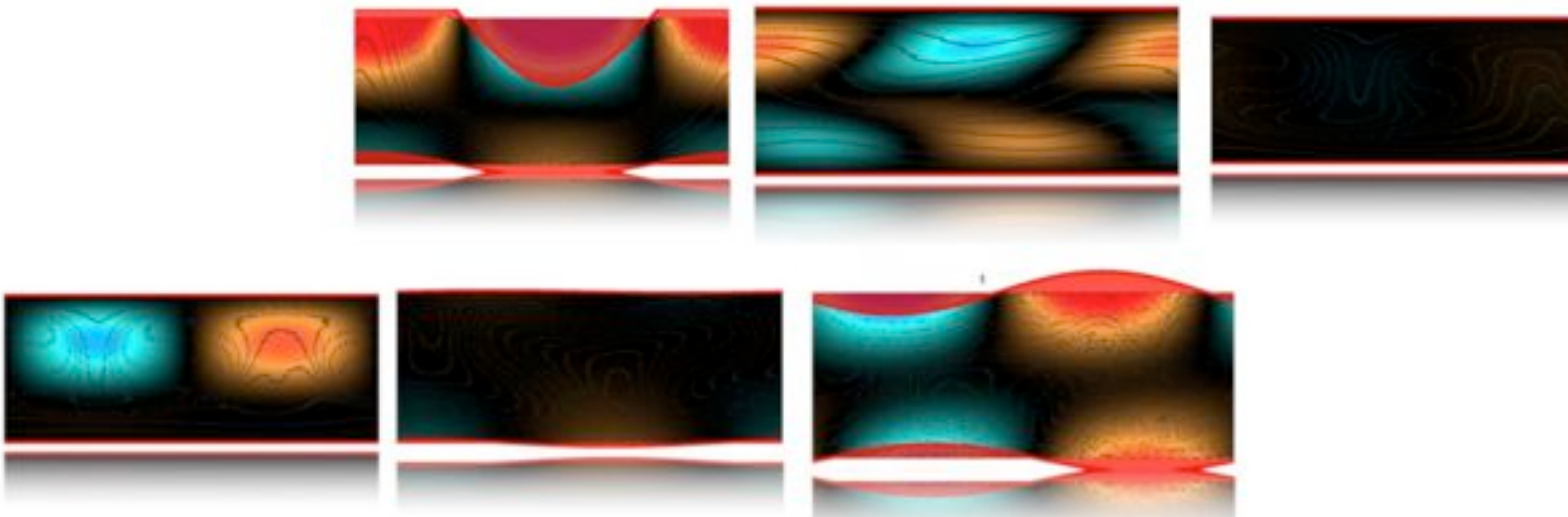
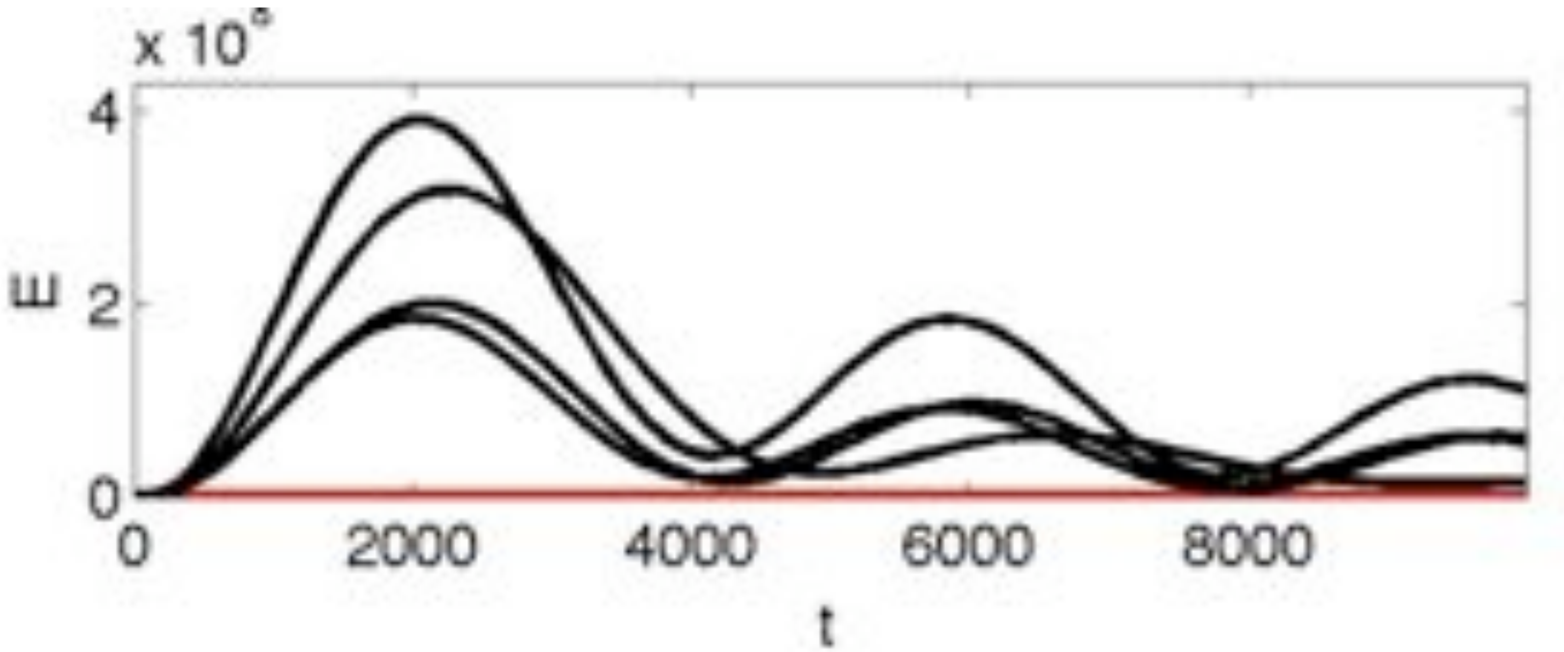


Illustration of response to random initial conditions



$K=10^1$

Flexible channel: slow oscillations

Large amplitude

Now, using these methods to do control

2) Control of stochastic flow systems

Control to reduce flow *rms*

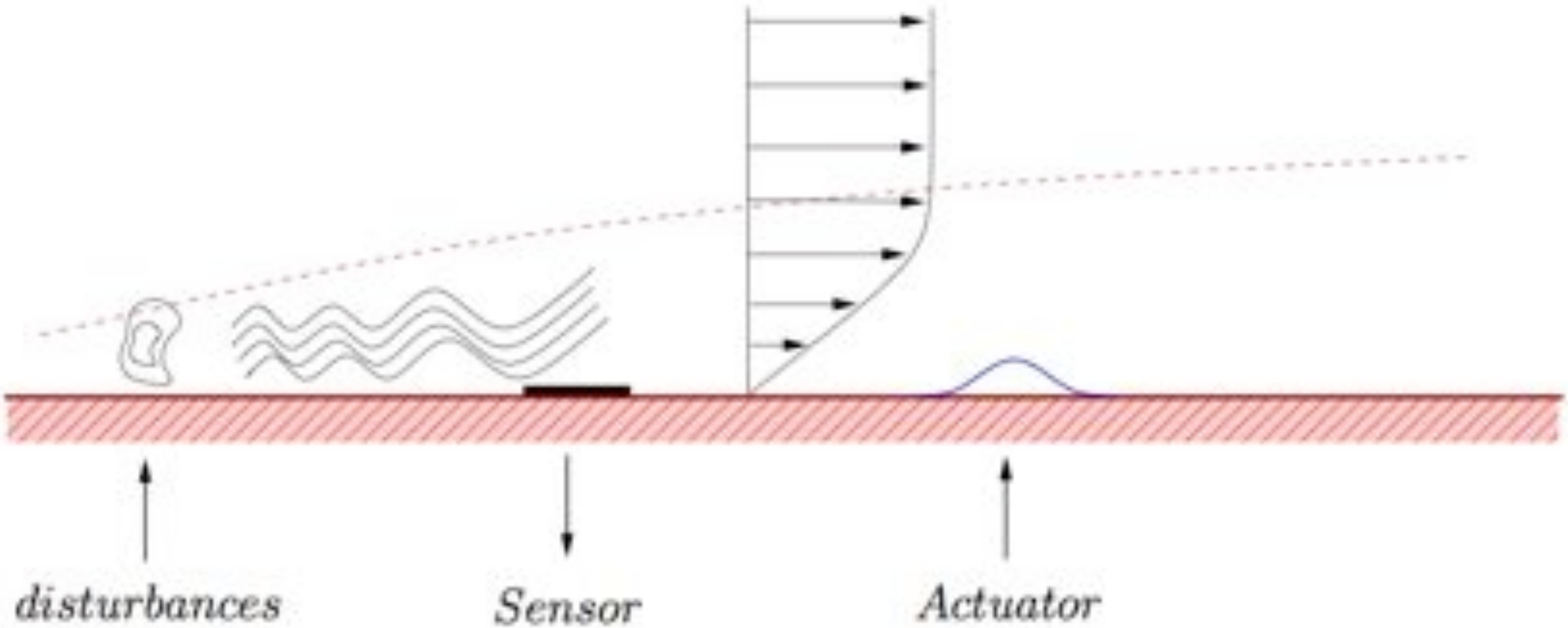
→ Actuators, sensors, feedback law

Minimize for stochastic properties

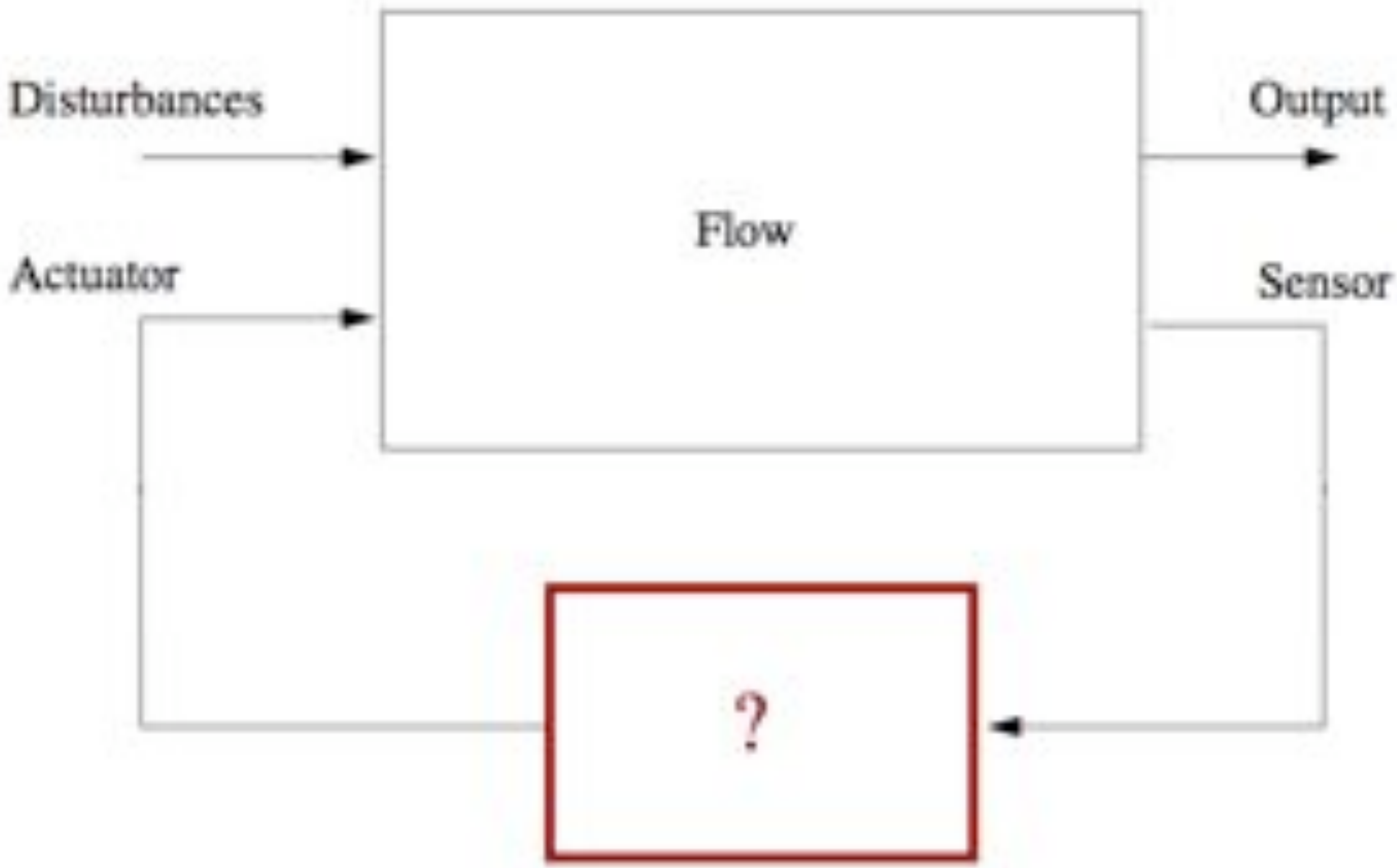
Actuators and sensors

Actuators to act on the flow state:
- Blowing and suction at the wall
- Wall deformation
- ...

Sensors to measure the flow state:
- Skin friction
- Pressure
- ...


disturbances
Sensor
Actuator

Feedback



Use optimization for the feedback law

Estimation

Sensor information

+ → **estimate full 3D flow state**

Dynamic model

Case 1:

No disturbances,

Known initial condition

→ Need good model

$$\text{Flow: } \begin{cases} \dot{q} = Aq \\ y = Cq \end{cases}, \quad q(0) = q_0$$

$$\text{Estimator: } \begin{cases} \dot{\hat{q}} = A\hat{q} \\ \hat{y} = C\hat{q} \end{cases}, \quad \hat{q}(0) = q_0$$

Case 2:

Disturbances,

Unknown initial condition

→ Need feedback

$$\text{Flow: } \begin{cases} \dot{q} = Aq + w \\ y = Cq + g \end{cases}, \quad q(0) = q_0$$

$$\text{Estimator: } \begin{cases} \dot{\hat{q}} = A\hat{q} - L(y - \hat{y}) \\ \hat{y} = C\hat{q} \end{cases}, \quad \hat{q}(0) = 0$$

Control and estimation

$$\text{system}\begin{cases}\dot{q} = Aq + w + Bu, \\ y = Cq + g\end{cases}, \quad \text{estimator}\begin{cases}\dot{\hat{q}} = A\hat{q} - L(y - \hat{y}), \\ \hat{y} = C\hat{q}\end{cases}$$

Full information control:

Feedback: $u = Kq$

Closed loop: $\dot{q} = \underbrace{(A+BK)}_{A_c} q + w$

$A_c = A + BK$ **is stable?**

Estimation:

Estimation error $\tilde{q} = q - \hat{q}$:

$\dot{\tilde{q}} = \underbrace{(A+LC)}_{A_e} \tilde{q} + w - Lg$

$A_e = A + LC$ **is stable?**

Output feedback control: $u = K\hat{q}$.

Lyapunov equations
for control and estimation systems

Mean energy= integral of *rms*

$$E_K = \mathrm{Tr}(P)$$

System is sensitive or unstable → large energetic response to external disturbances

Full information Control:

$$\dot{q} = \underbrace{(A + BK)}_{A_c} + w$$

Lyapunov:

$$\underbrace{(A + BK)^{+}}_{A_c^{+}} P + P\underbrace{(A + BK)}_{A_c} + W = 0$$

Estimation:

$$\dot{\tilde{q}} = \underbrace{(A + LC)}_{A_e}\tilde{q} + w - Lg$$

Lyapunov:

$$\underbrace{(A + LC)}_{A_e}\tilde{P} + \tilde{P}\underbrace{(A + LC)^{+}}_{A_e^{+}} + W + \alpha^2 LL^{+} = 0$$

Now: find optimal feedback K and L

Optimization

Constrained minimisation → Lagrange multiplier Λ

Minimax problem for Lagrangians $\mathscr{L}_c$ and $\mathscr{L}_e$.

Control:

minimize

$$E(\|q\|^2 + \ell^2 \underbrace{\|u\|^2}_{\|Kq\|^2}) = \mathrm{Tr}(PQ + \ell^2 KPK^+)$$

$$\mathscr{L}_c = \overbrace{\mathrm{Tr}(PQ + KPK^+)}^{\text{Objective}} + \mathrm{Tr}[\Lambda(\overbrace{(A+BK)P + P(A+BK)^+ + W}^{\text{Constraint}})$$

$$\left.\begin{array}{l} \nabla_\Lambda \mathscr{L}_c = 0 \\ \nabla_P \mathscr{L}_c = 0 \\ \nabla_K \mathscr{L}_c = 0 \end{array}\right\} \Rightarrow \begin{cases} 0 = A^+\Lambda + \Lambda A - \Lambda BB^+\Lambda/\ell^2 + Q, \\ K = B^+\Lambda/\ell^2. \end{cases}$$

Estimation:

minimize

$$E(\underbrace{\|q - \hat{q}\|^2}_{\|\tilde{q}\|^2}) = \mathrm{Tr}(\tilde{P})$$

$$\mathscr{L}_e = \overbrace{\mathrm{Tr}(\tilde{P})}^{\text{Objective}} + \mathrm{Tr}[\Lambda(\overbrace{(A+LC)\tilde{P} + \tilde{P}(A+LC)^+ + \alpha^2 LL^+ + W}^{\text{Constraint}})]$$

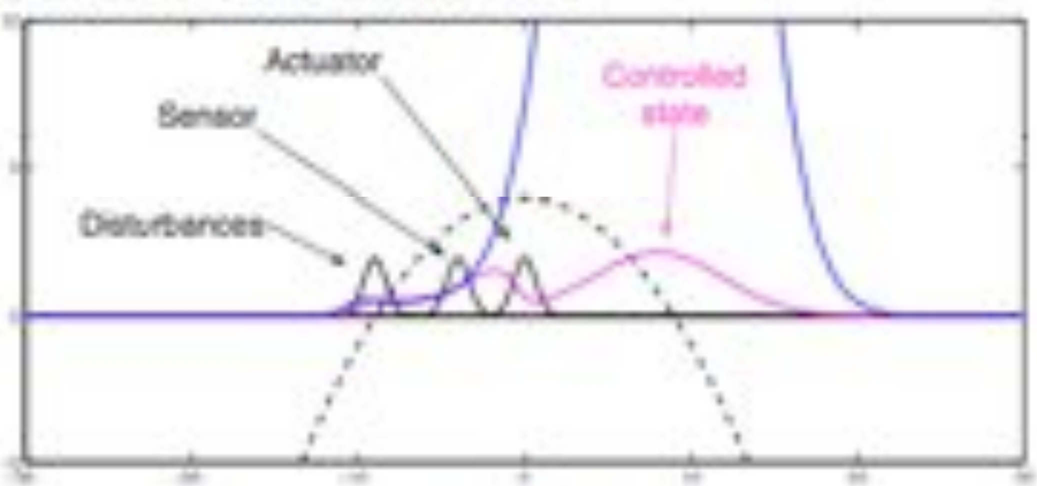
$$\left.\begin{array}{l} \nabla_\Lambda \mathscr{L}_e = 0 \\ \nabla_{\tilde{P}} \mathscr{L}_e = 0 \\ \nabla_L \mathscr{L}_e = 0 \end{array}\right\} \Rightarrow \begin{cases} 0 = A\tilde{P} + \tilde{P}A^+ - \tilde{P}C^+C\tilde{P}/\alpha^2 + W \\ L = -\tilde{P}C^+/\alpha^2. \end{cases}$$

Same structure for control and estimation → two Riccati equations

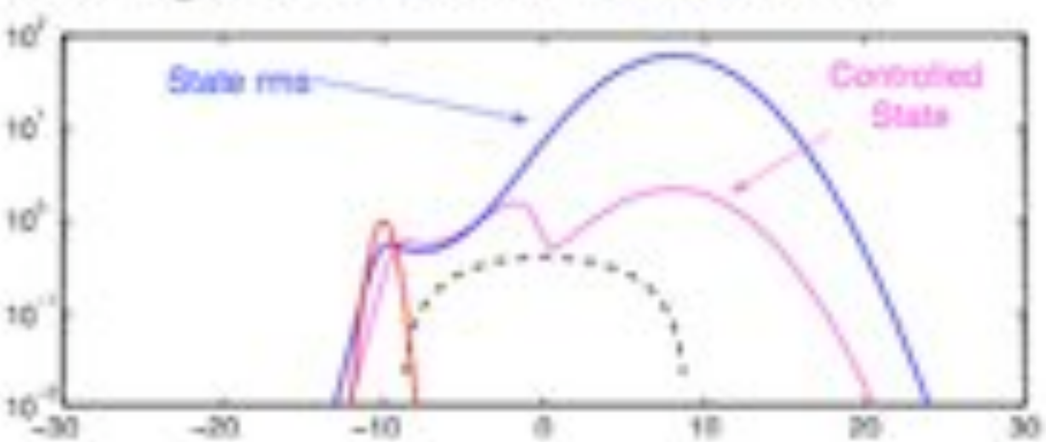
1D example: Controlled Ginzburg-Landau

$$\begin{cases} \dot{q} + U q_x = \gamma q_{xx} + \mu(x) q + b(x) u(t) \\ y(t) = \int_x c(x) q(x) \mathrm{d}x \end{cases}$$

Sensors and actuators



Forcing and controlled state *rms*:



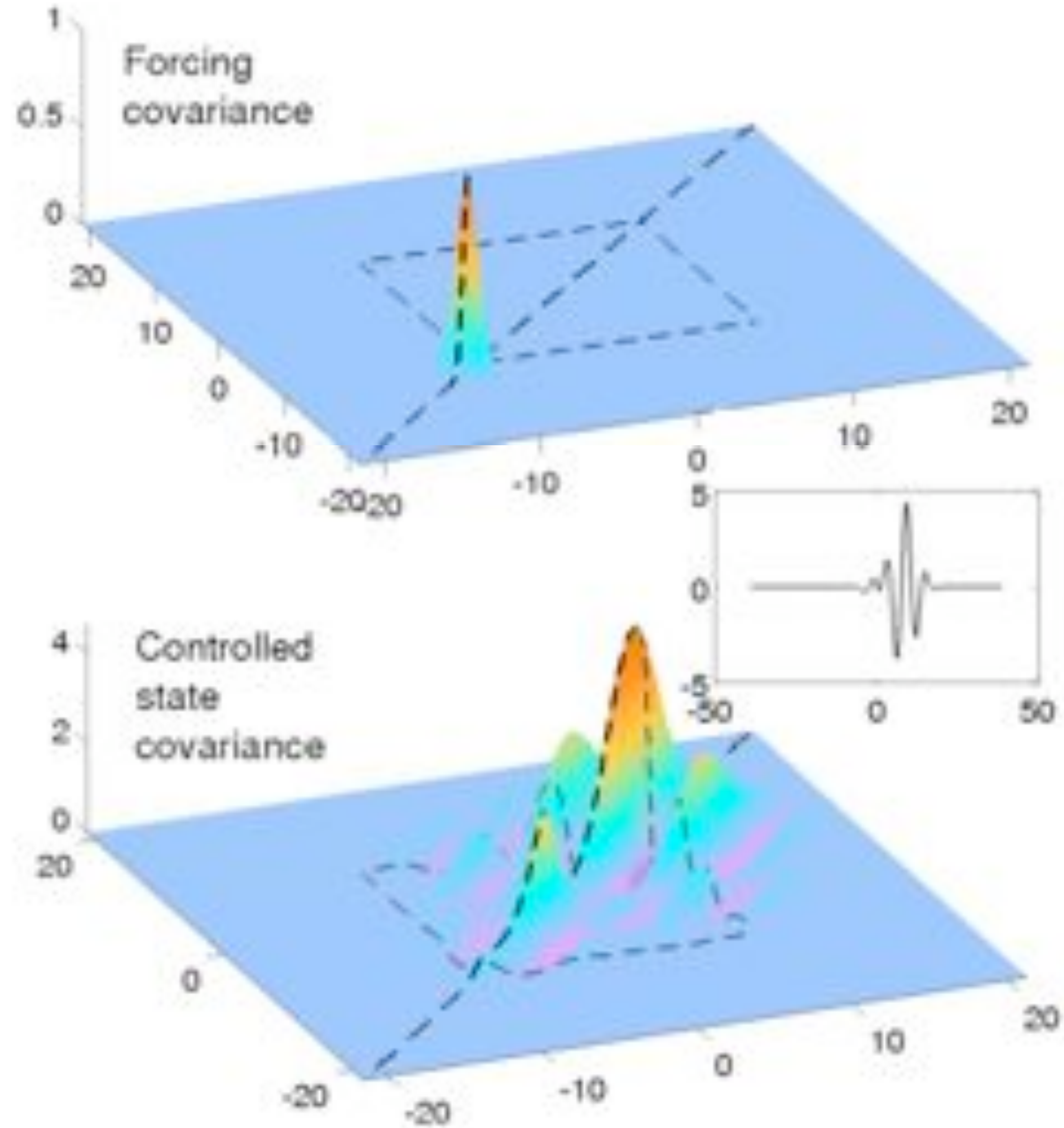
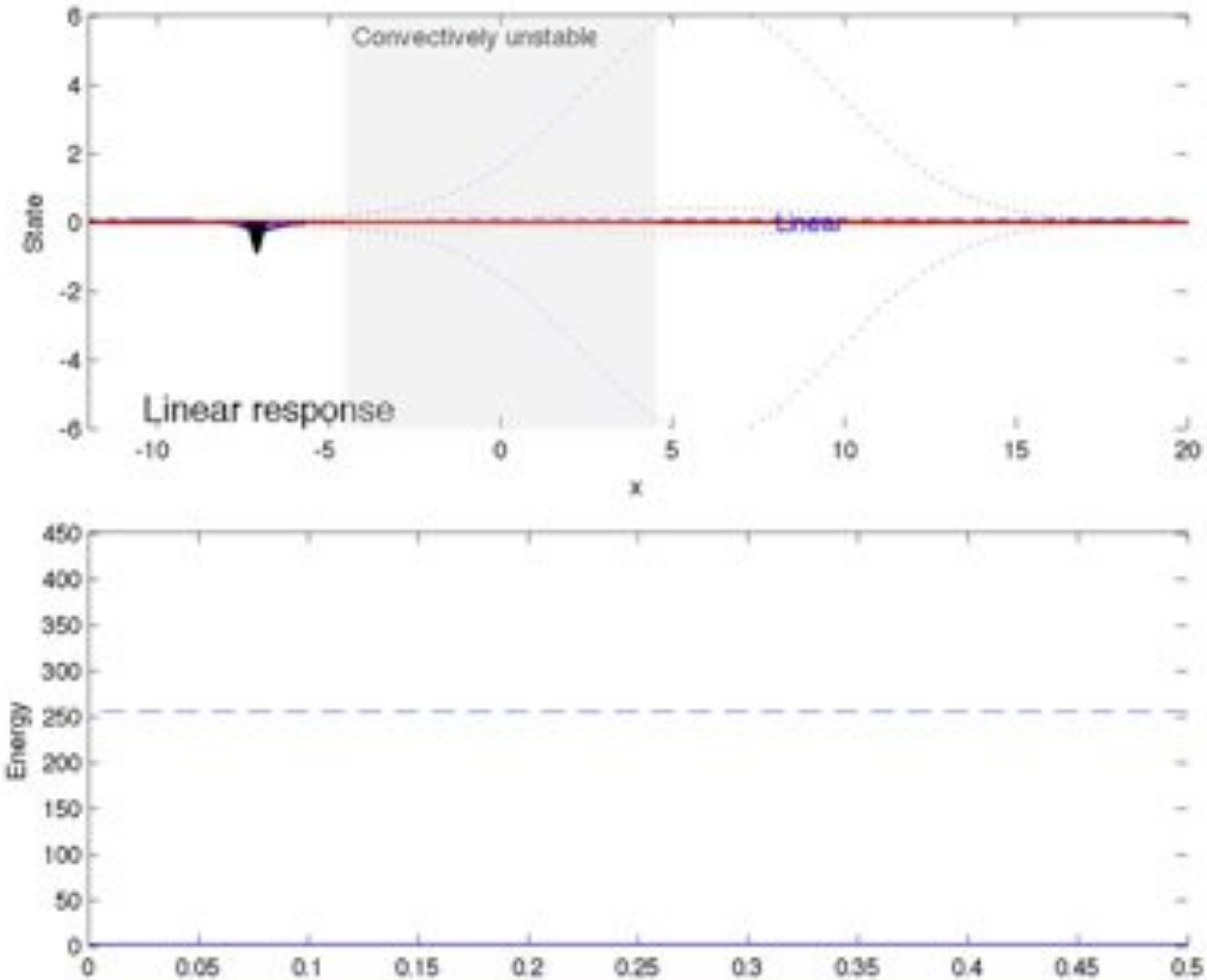
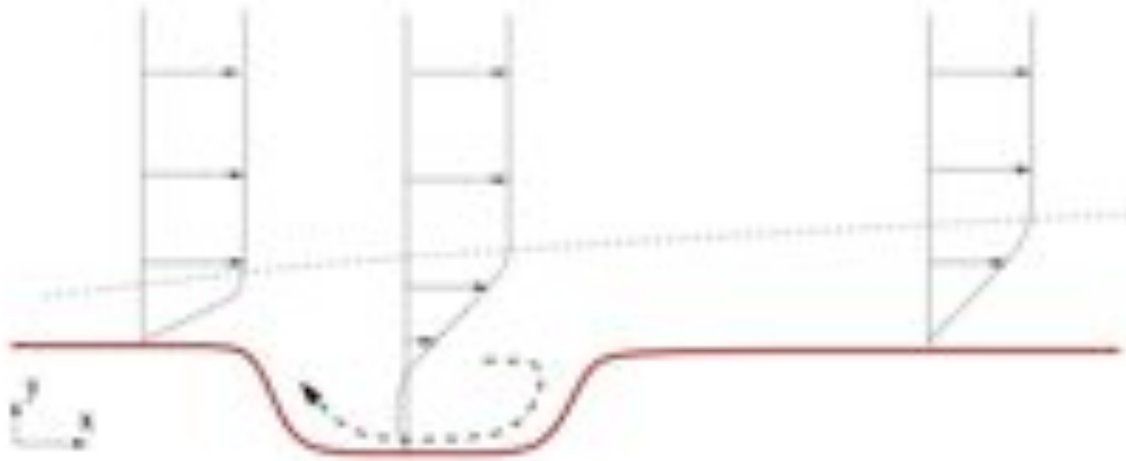


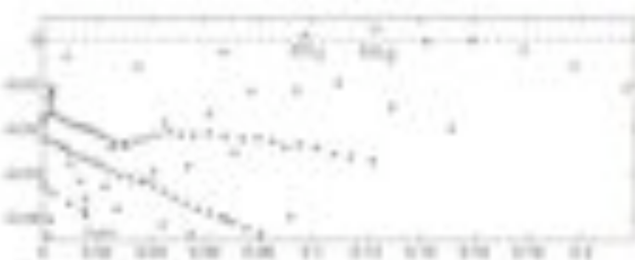
Illustration: control of system subject to random perturbations



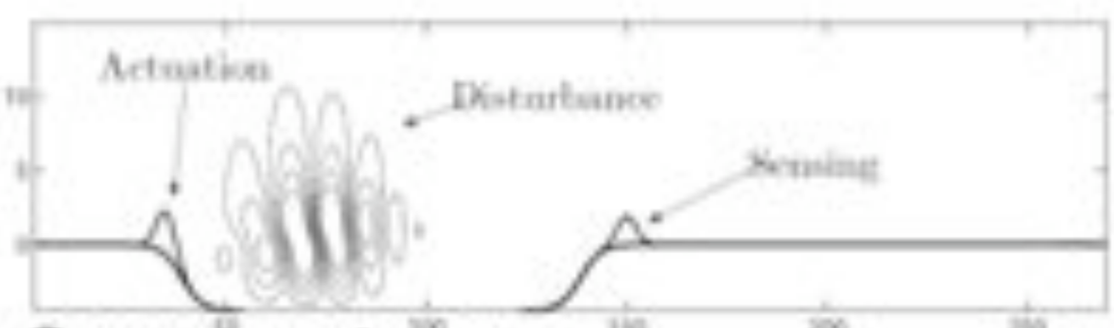
Controle retroactif:

ecoulement fortement non-parallele





Modes globaux et spectre

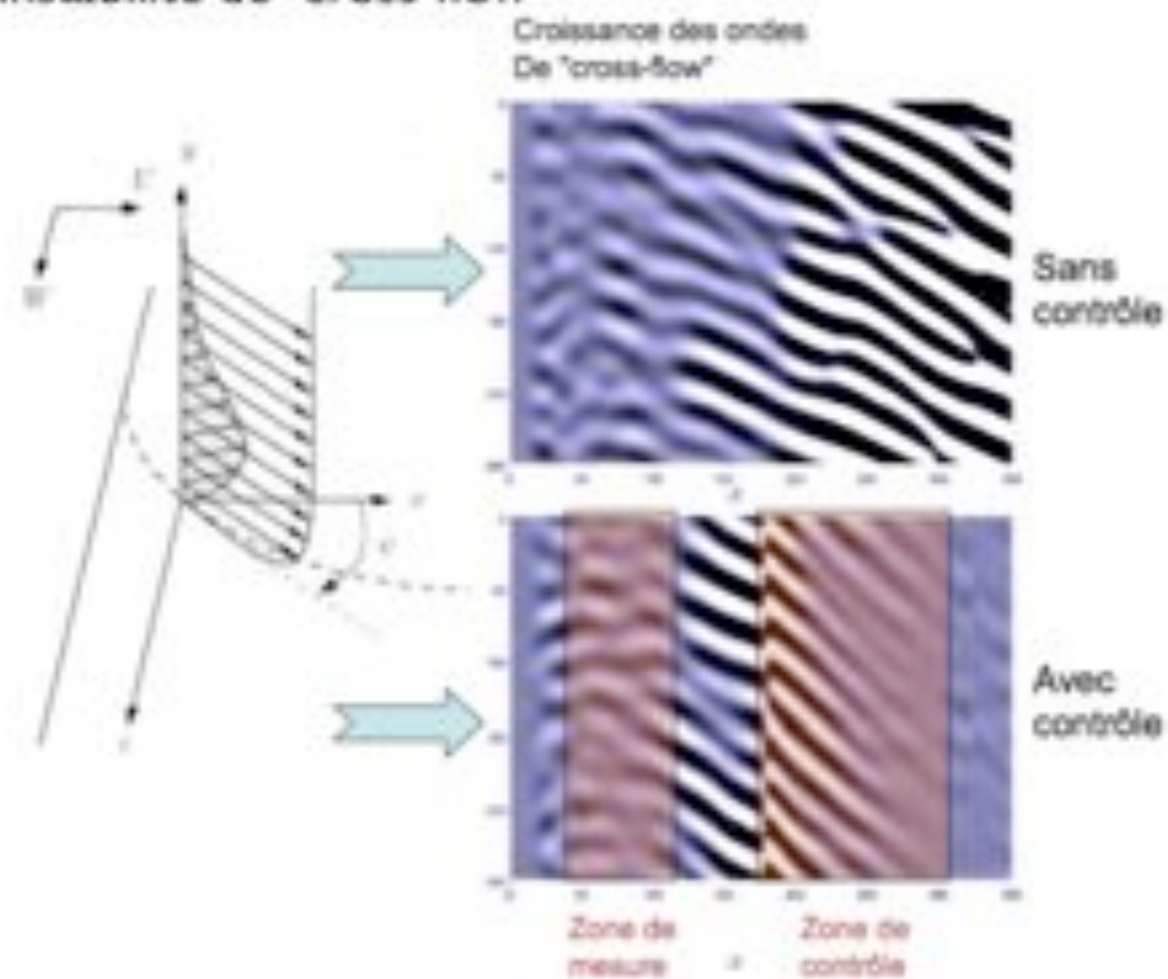


Capteur et actioneur

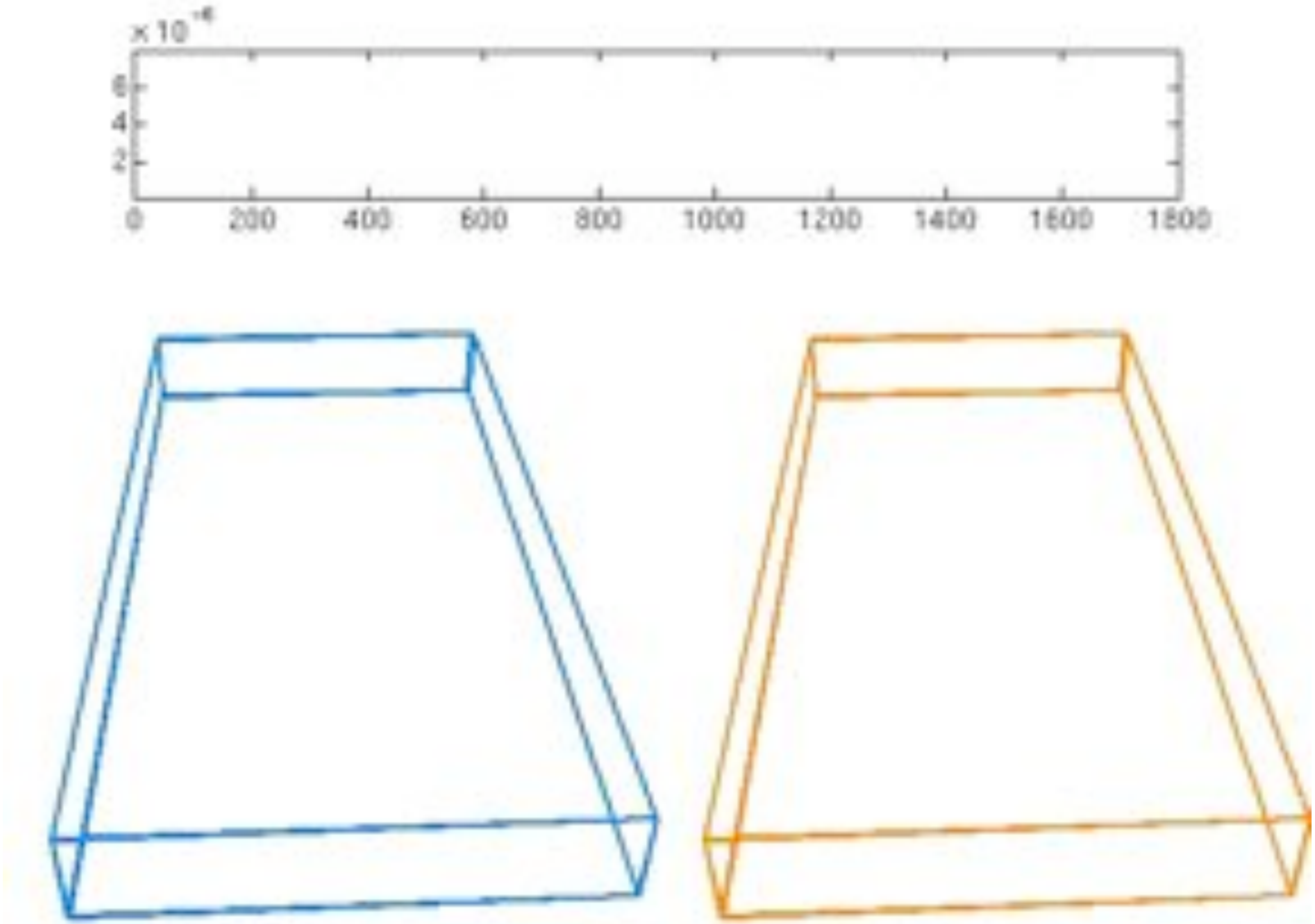
Illustration of control result

Couche limite 3D:

Instabilite de "cross-flow"



Boundary layer subject to harmonic point-source excitation



Animation: Hogberg, Chevalier, Henningson

Conclusions

- Stochastic methods to study flow response to complex perturbations

- For linear system: solve directly for the statistics: Lyapunov equation

- The response properties should be used for control:

  mixed dynamic/statistical approach