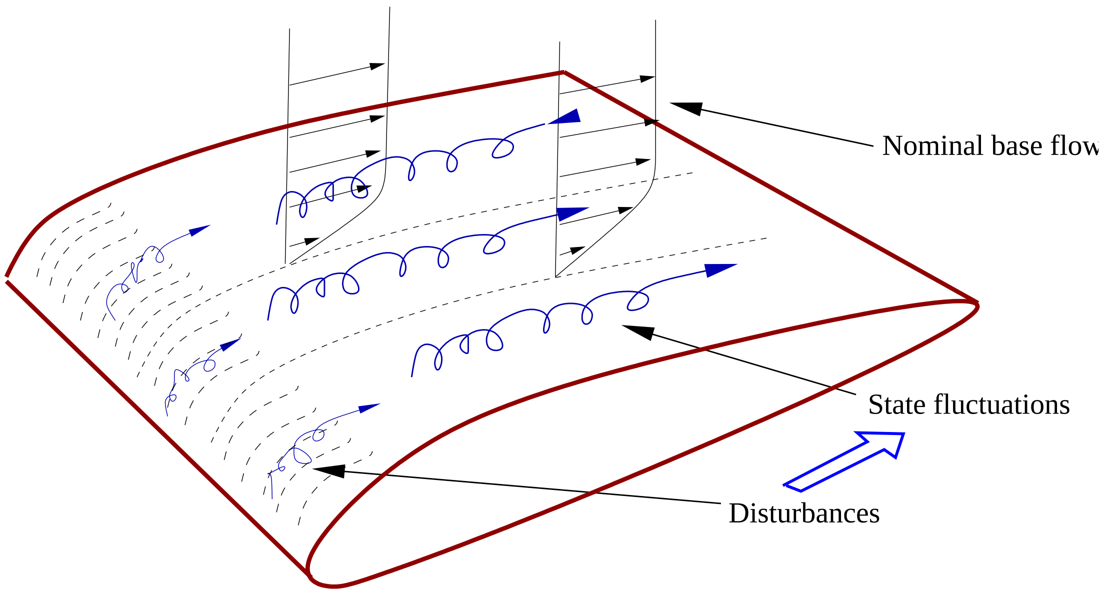
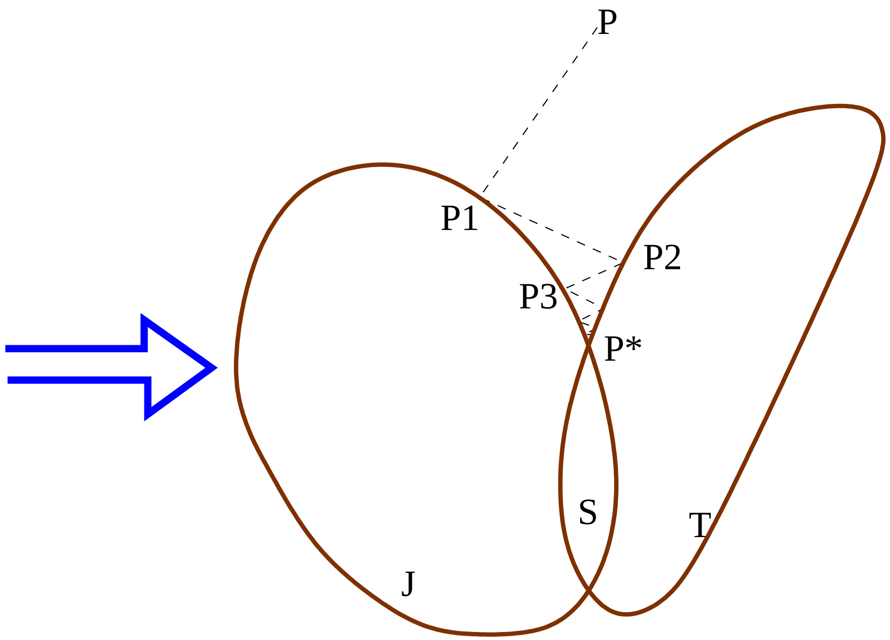
KTH Mechanics

Modeling flow statistics using convex optimization





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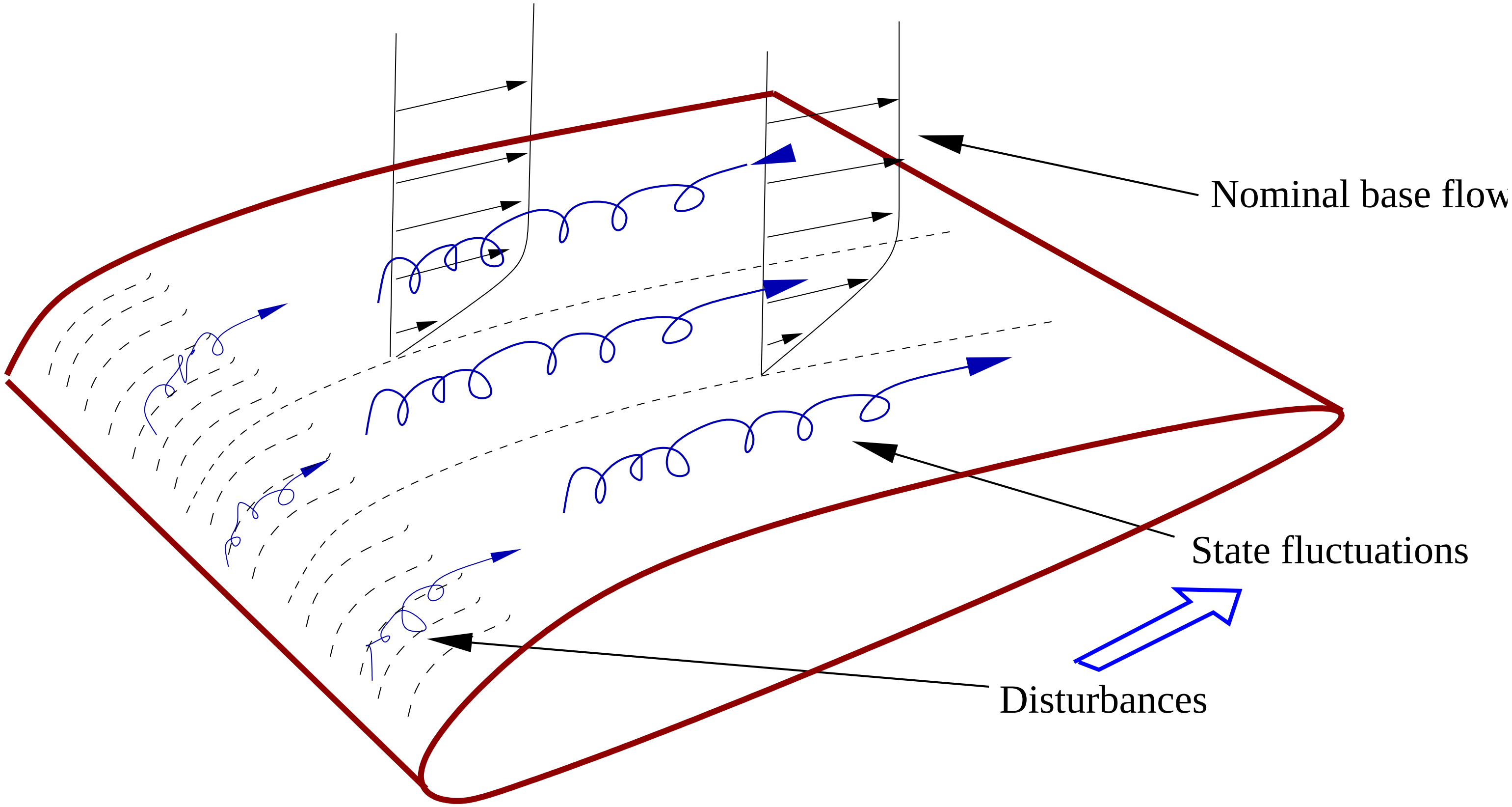
Motivations

We can observe the flow fluctuations, but what are the sources of disturbances?



Wall roughness, Accoustic waves, Free-stream turbulence, ...

Lead to a great variety of state fluctuations

Hope for a quantitative, statistical description of sources of disturbances

Inspiration

1) Modeling flow statistics using the linearized Navier–Stokes equations,
Jovanovič & Bamieh, CDC 2001
→ Presentation and justification of the modeling problem in fluid mechanics. Model the covariance of disturbances.

2) A unified algebraic approach to linear control design,
Skelton, Iwasaki & Grigoriadis, Taylor & Francis 1998
→ LMI problem formulation, solution by alternating projection.

Our aim: Add an optimal flavour to 1) ,
show the limitations of the modeling,
use methods from 2)

Idea: Lyapunov equation

Assume a dynamic model A is available: linear, stable.

Stochastic description of system's state and external disturbances

$$\dot{x} = Ax + w \qquad \begin{cases} P = Exx^H \\ M = Eww^H \end{cases}$$

$$\text{At steady state, Lyapunov equation:} \begin{cases} AP + PA^H + M = 0 \\ A : \text{Dynamic operator} \\ P : \text{State covariance} \\ M : \text{Disturbance covariance} \end{cases}$$

Knowing the state covariance and with a dynamic model,

$\rightarrow$ recover covariance of disturbances

The Lyapunov cone

P and M are covariance matrices

$$P \geq 0, \quad M \geq 0, \Rightarrow AP + PA^H \leq 0$$

The operator A generates a **convex** cone.

Lyapunov theorem:

$$\forall M \geq 0, \exists ! P \geq 0 / AP + PA^H + M = 0$$

but $\exists P \geq 0 / AP + PA^H$ is indefinite

Problem: P **might be out of the cone of our model** A **...**

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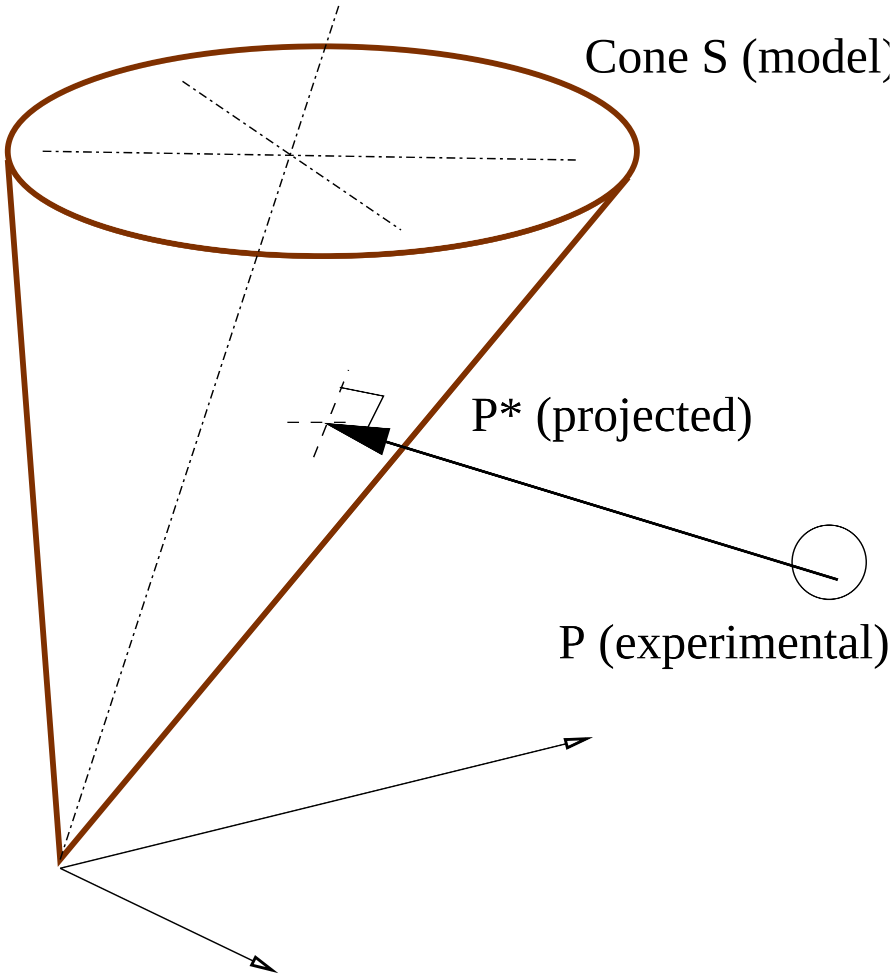
Find the closest one

→ Minimization problem



Consider the cone :

$$\mathcal{S} = \{P \geq 0 / AP + PA^H \leq 0\}$$

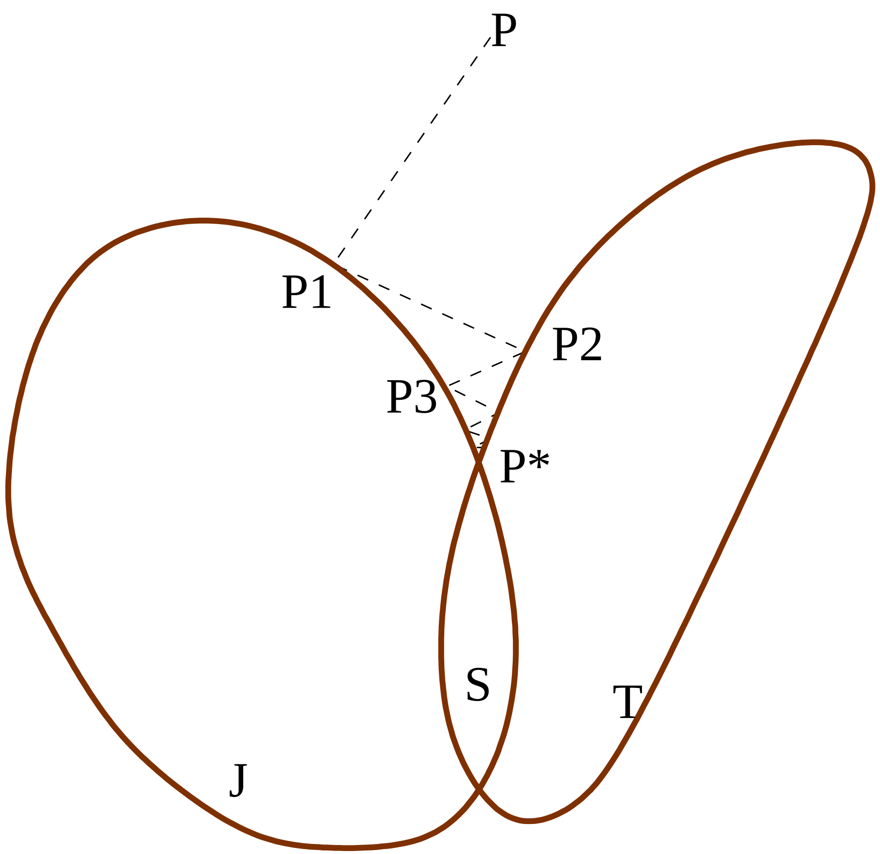
Find $P^* \in \mathcal{S}$ closest to our experimental P

P^* is the orthogonal projection of P on $\mathcal{S}$

Solution by alternating projection

Convex minimization problem, large dimension: $P, M, A,$ have $n(n-1)/2$ elements

Too big for central path method. Can we use alternating projection?



We can decompose $\mathcal{S}$ into the intersection of two simpler sets $\mathcal{J} \bigcap \mathcal{T}$:

$\rightarrow$ Derive simple analytical projection formula on sets $\mathcal{J}$ and $\mathcal{T}$

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Intersection of the sets $\mathcal{J}$ and $\mathcal{T}$

$$\mathcal{S} = \mathcal{J} \bigcap \mathcal{T}, \qquad \begin{aligned} \mathcal{J} &= \left\{ W \in \mathcal{H}_{2n} / \begin{pmatrix} A, I \end{pmatrix} W \begin{pmatrix} A^H \\ I \end{pmatrix} \leq 0 \right\} \\ \mathcal{T} &= \left\{ W \in \mathcal{H}_{2n} / W = \begin{pmatrix} 0 & W_{12} \\ W_{12}^H & 0 \end{pmatrix}, W_{12} \in \mathcal{H}_n \right\} \end{aligned}$$

Projection on $\mathcal{J}$:

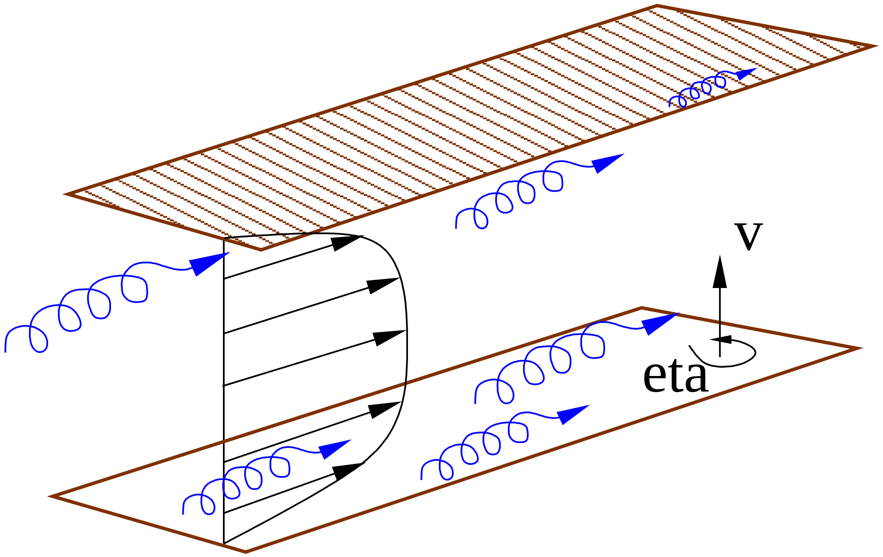
Comes down to a projection on negativity set $\{P \in \mathcal{H}_n / P \leq 0\}$ in the rank subspace of $\begin{pmatrix} A, I \end{pmatrix}$

Projection on $\mathcal{T}$:

$$V^* = \begin{pmatrix} 0 & \frac{1}{2}(V_{12} + V_{12}^H) \\ \frac{1}{2}(V_{12} + V_{12}^H) & 0 \end{pmatrix}$$

It costs one eigendecomposition in $\mathcal{H}_n$ per iteration.

Example: Channel flow



State variable is wall-normal velocity/wall-normal vorticity.

$$L_{OS} = -\mathrm{i}k_x U\Delta + \mathrm{i}k_x \mathrm{D}^2 U + \Delta^2/Re,$$
$$L_{SQ} = -\mathrm{i}k_x U + \Delta/Re,$$
$$L_C = -\mathrm{i}k_z \mathrm{D}U$$

Spatial invariance in horizontal direction $\longrightarrow$ work in spatial frequency space. Orr-Sommerfeld/squire equation for small state perturbations at each frequency pair:

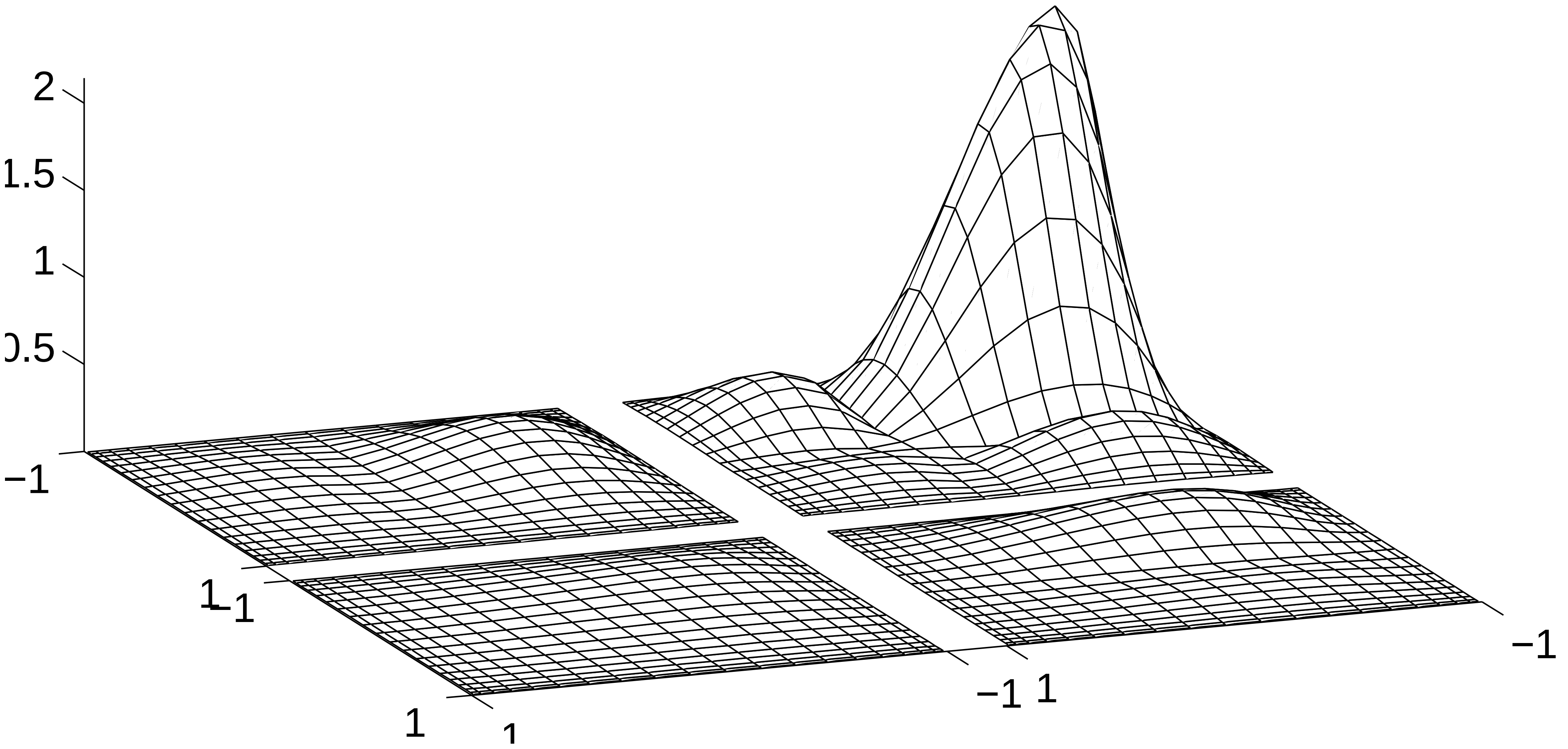
$$\underbrace{\begin{pmatrix}\dot{v}\\ \dot{\eta}\end{pmatrix}}_{\dot{x}} = \underbrace{\begin{pmatrix}\Delta^{-1}L_{OS} & 0\\ L_C & L_{SQ}\end{pmatrix}}_{A}\underbrace{\begin{pmatrix}v\\ \eta\end{pmatrix}}_{x} + \underbrace{\begin{pmatrix}d_v\\ d_\eta\end{pmatrix}}_{d}$$

plant/Model: Parametric mismatch in the Reynold number:

$$\mu = \left|\frac{Re - Re_{model}}{Re_{model}}\right|$$

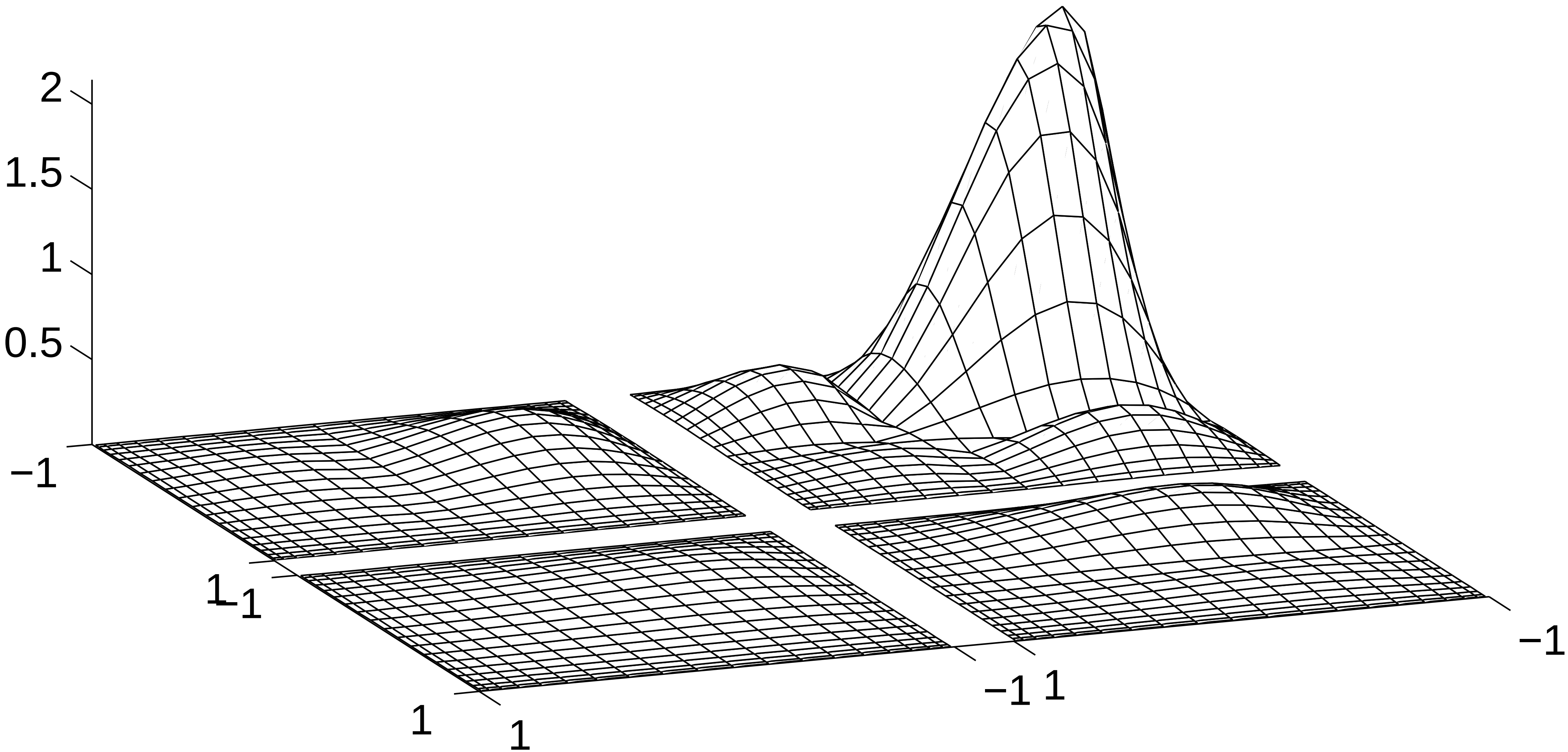
Low $Re \longrightarrow$ dominating viscous effects.

Given an experimental state covariance



State covariance is assimetric → something happens at one wall!

Projected state covariance ($\mu = 0.5$)



Projected using alternating convex projection

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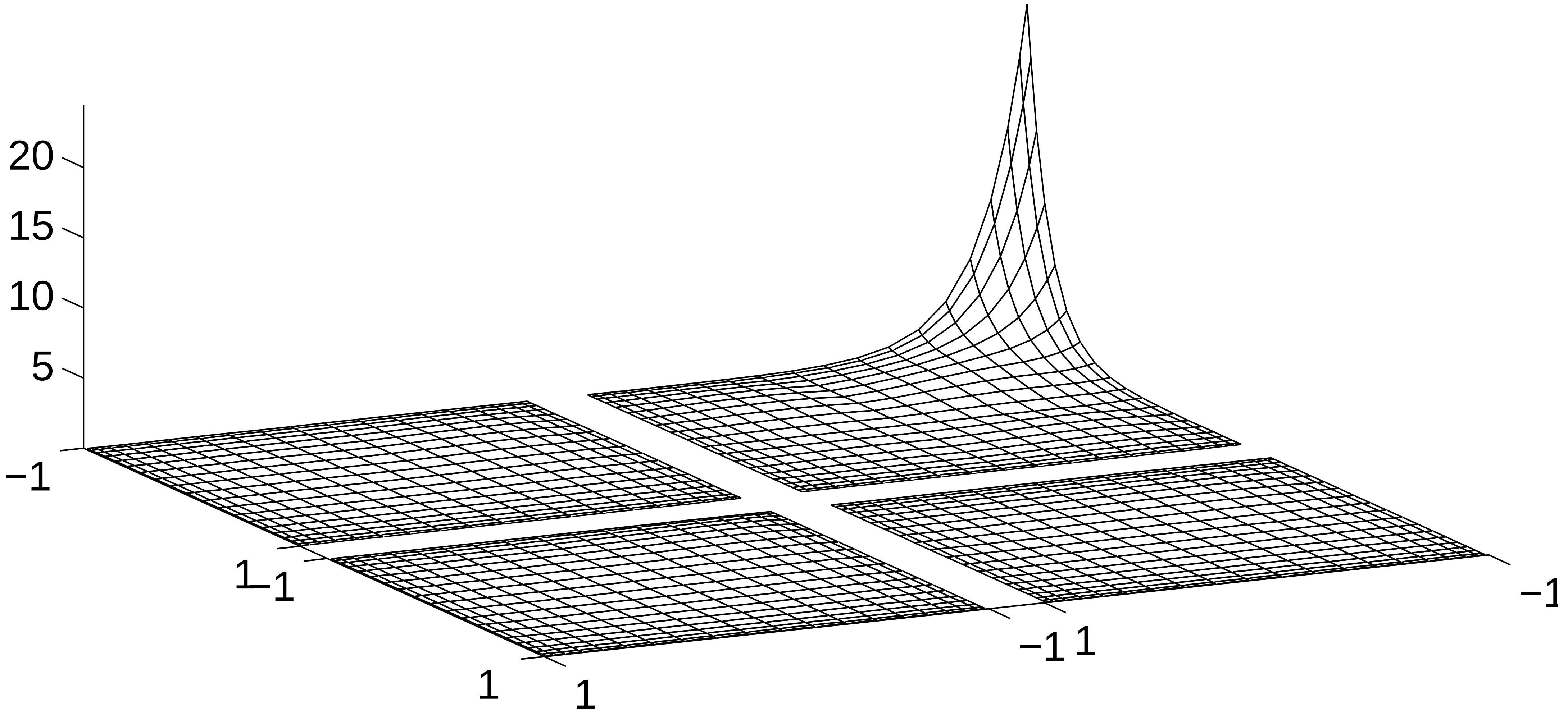
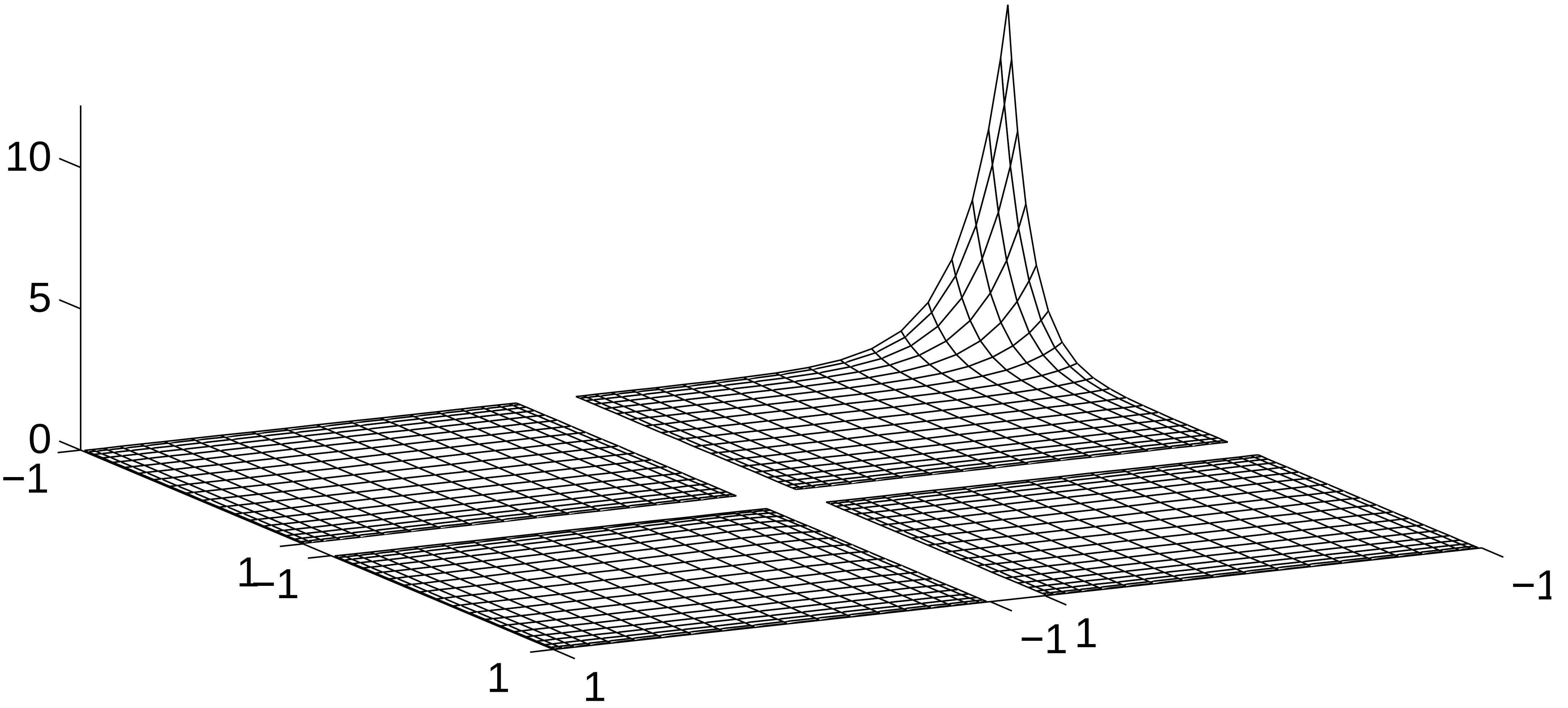
Corresponding disturbance covariance ($\mu = 0.5$)



from Lyapunov equation $M = -(AP + PA^H)$

Compare to “true” disturbance covariance



Used to force the plant

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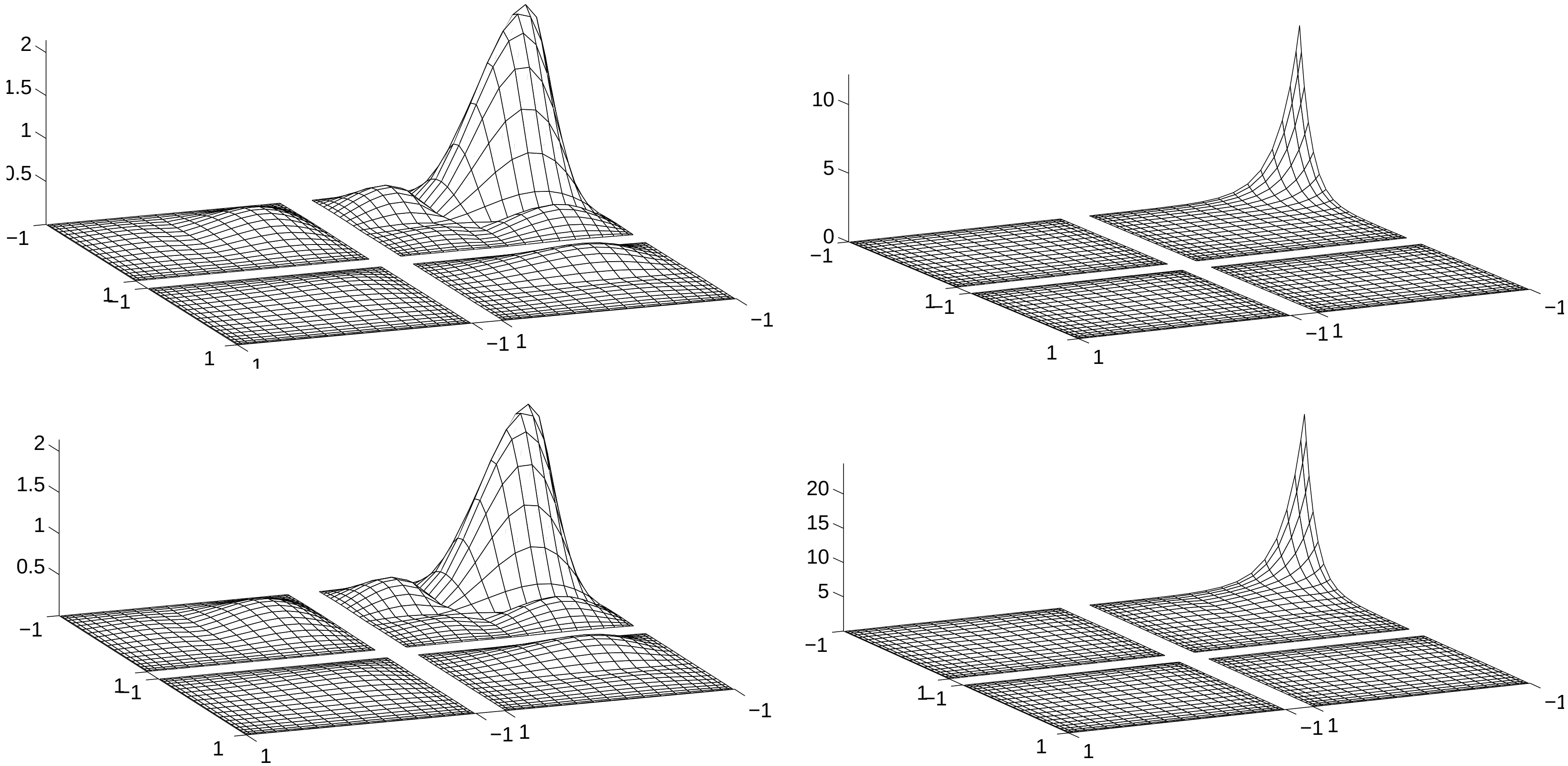
(parametric mismatch $\mu = 0.5$)



$Re_{model} = Re/2$. Lower sensitivity $\rightarrow$ need larger forcing.

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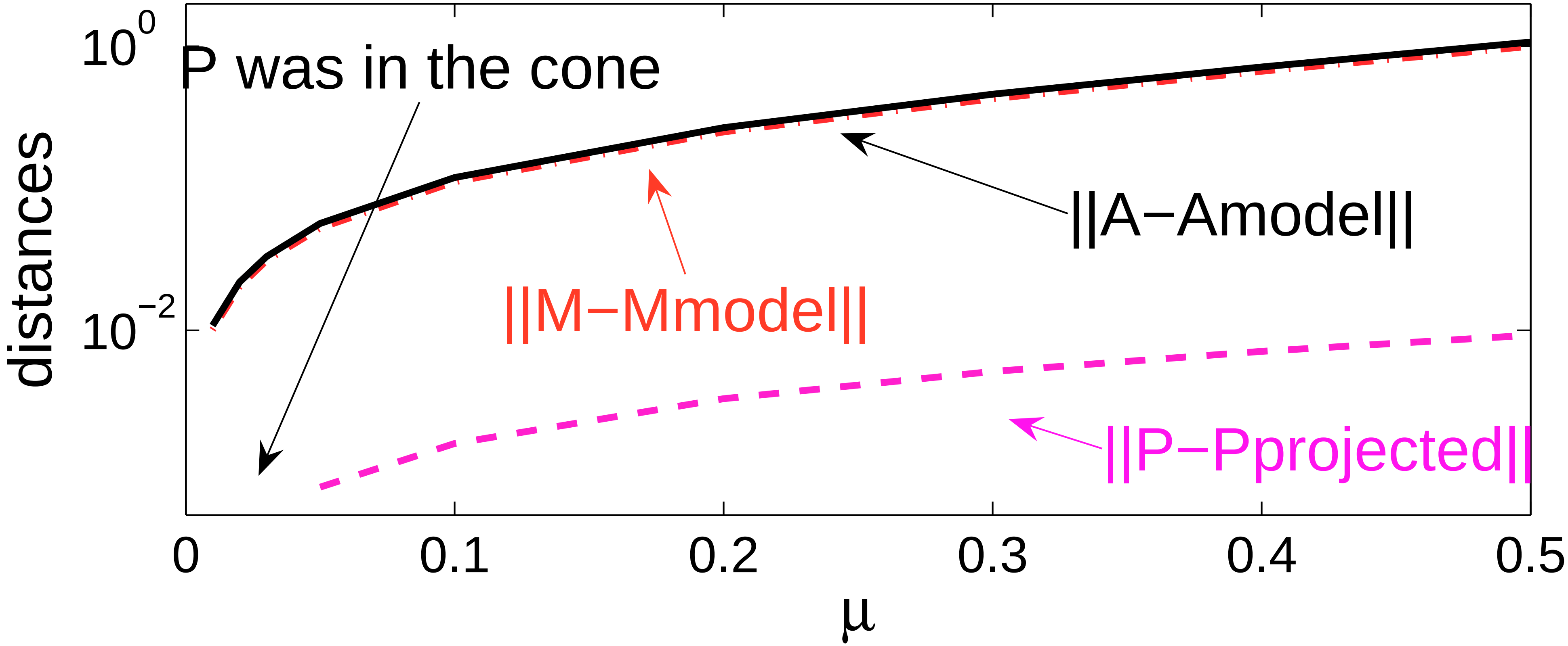
Distance with mismatch μ

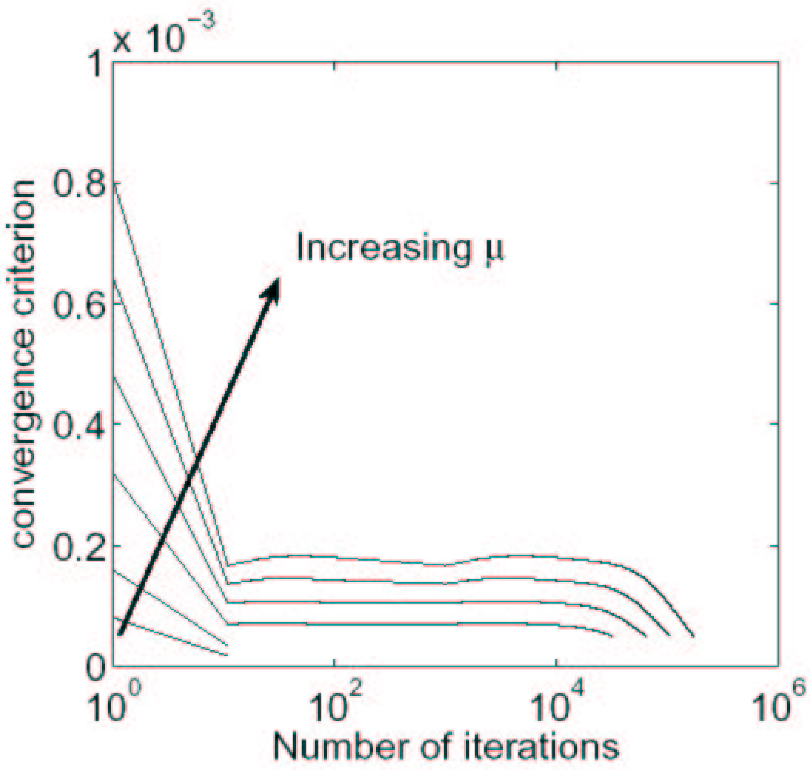
experimental/projected distance:

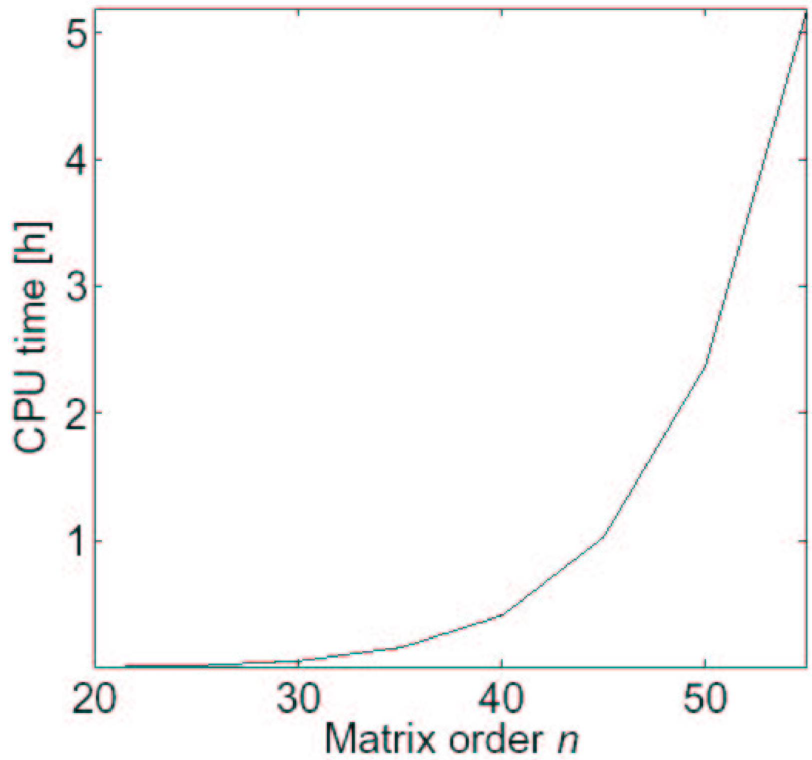


P is not in the cone, but the projected one is close.

Matrix distance measured in Frobenius norm.

Computation





Iterations **computation time**

More iteration when larger mismatch. Slower when reaching solution. ☹

Conclusions

Have a model and a experimental state covariance $\longrightarrow$ recover disturbance covariance. Illustration on channel flow.

Observations

- Need projection if P is not in the cone
- Can use alternating convex projection, defining cone as intersection

Remains

- Too slow computations $\longrightarrow$ should use directional alternating projection

Extra slides

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Alternating projection for optimality problem

We recall here the alternating projection algorithm for the optimality problem.

Consider the family of closed, convex sets $\{\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_m\}$ and a given matrix X_0. The sequence of matrices $\{X_i\}$, $i = 1, 2, \ldots, \infty$ computed as follow:

$$
\begin{aligned}
& X_1 = \mathcal{P}_{\mathcal{C}_1} X_0, \ Z_1 = X_1 - X_0 \\
& X_2 = \mathcal{P}_{\mathcal{C}_2} X_1, \ Z_2 = X_2 - X_1 \\
& \quad \vdots \\
& X_m = \mathcal{P}_{\mathcal{C}_m} X_{m-1}, \ Z_m = X_m - X_{m-1} \\
& X_{m+1} = \mathcal{P}_{\mathcal{C}_1} (X_m - Z_1), \ Z_{m+1} = Z_1 + X_{m+1} - X_m \\
& X_{m+2} = \mathcal{P}_{\mathcal{C}_2} (X_{m+1} - Z_2), \ Z_{m+2} = Z_2 + X_{m+2} - X_{m+1} \\
& \quad \vdots \\
& X_{2m} = \mathcal{P}_{\mathcal{C}_m} (X_{2m-1} - Z_m), \ Z_{2m} = Z_m + X_{2m} - X_{2m-1} \\
& X_{2m+1} = \mathcal{P}_{\mathcal{C}_1} (X_{2m} - Z_{m+1}), \ Z_{2m+1} = Z_{m+1} + X_{2m+1} - X_{2m} \\
& \quad \vdots
\end{aligned}
$$

converges to the orthogonal projection of X_0 on $\mathcal{C}_1 \bigcap \mathcal{C}_2 \bigcap \cdots \bigcap \mathcal{C}_m$.

Projection on negativity set

Let $X \in \mathcal{H}_n$, with eigenvalue-eigenvector decomposition $X = L\Lambda L^H$. The projection X^* of X onto the set of negative semidefinite matrices is

$$X^* = L\Lambda_- L^H,$$

where Λ_- is the diagonal matrix obtained by replacing the positive eigenvalues of X in Λ by zero.

Projection on $\mathcal{J}$

Let $W \in \mathcal{H}_{2n}$. Consider the singular value decomposition

$$\begin{pmatrix} A, I \end{pmatrix} F_2^{-1} = U \begin{pmatrix} \Sigma, 0 \end{pmatrix} V^H \tag{1}$$

where U and V are unitary matrices, and define

$$Y \triangleq V^H F_2 W F_2^H V = \begin{pmatrix} Y_{11} & Y_{12} \\ Y_{12}^H & Y_{22} \end{pmatrix}, \; Y_{11} \in \mathcal{H}_n \tag{2}$$

The projection $\mathcal{P}_{\mathcal{J}}^{Q_2} W$ of the matrix W onto the set $\mathcal{J}$ is

$$\mathcal{P}_{\mathcal{J}}^{Q_2} W = F_2^{-1} V \begin{pmatrix} Y_{11}^* & Y_{12} \\ Y_{12}^H & Y_{22} \end{pmatrix} V^H F_2^{-1H} \tag{3}$$

where Y_{11}^* is the projection of Y_{11} on the set of negative definite matrices for the unweighted Frobenius norm as in (20).

Let

$$\hat{W} = \begin{pmatrix} \hat{W}_{11} & \hat{W}_{12} \\ \hat{W}_{12}^H & \hat{W}_{22} \end{pmatrix} \in \mathcal{J} \tag{4}$$

be an arbitrary matrix in $\mathcal{J}$. We will show that the inner product $\langle W^* - W, W^* - \hat{W} \rangle$ is

(**??**), we have

$$\begin{aligned}
&\langle W^* - W, W^* - \hat{W} \rangle_{Q_1} \\
&= \langle F_2 W^* F_2^H - F_2 W F_2^H, F_2 W^* F_2^H - F_2 \hat{W} F_2^H \rangle_I \\
&= \langle Y^* - Y, Y^* - \hat{Y} \rangle_I,
\end{aligned} \tag{5}$$

with

$$\begin{aligned}
&Y^* = V^H F_2 W^* F_2^H V, \quad Y = V^H F_2 W F_2^H V, \\
&\hat{Y} = V^H F_2 \hat{W} F_2^H V.
\end{aligned} \tag{6}$$

since V is unitary. Partitioning the matrices as in (4) we obtain

$$\begin{aligned}
&\langle Y^* - Y, Y^* - \hat{Y} \rangle_I \\
&= \left\langle \begin{pmatrix} Y_{11}^* - Y_{11} & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} Y_{11}^* - \hat{Y}_{11} & Y_{12} - \hat{Y}_{12} \\ Y_{12}^H - \hat{Y}_{12}^H & Y_{22} - \hat{Y}_{22} \end{pmatrix} \right\rangle_I \\
&= \langle Y_{11}^* - Y_{11}, Y_{11}^* - \hat{Y}_{11} \rangle_I
\end{aligned} \tag{7}$$

Now observe that, since $\hat{W} \in \mathcal{J}$,we have

$$\begin{pmatrix} A, I \end{pmatrix} \hat{W} \begin{pmatrix} A^H \\ I \end{pmatrix} \leq 0, \tag{8}$$

and by substituting the singular value decomposition

$$U\left(\Sigma, 0\right)\underbrace{V^H F_2 \hat{W} F_2^H V}_{\hat{Y}}\begin{pmatrix}\Sigma^H \\ 0\end{pmatrix} U^H \leq 0, \tag{9}$$

then pre- and post- multiplying by $\Sigma^{-1}U^H$ and $(\Sigma^{-1}U^H)^H$ we obtain

$$\left(I, 0\right)\hat{Y}\begin{pmatrix}I \\ 0\end{pmatrix} \leq 0, \tag{10}$$

that is, $\hat{Y}_{11} \leq 0$. Note that, from lemma **??**, the orthogonal projection of the matrix Y_{11} on this set is given by (20). Hence, by construction of Y_{11}^* in (3), we have

$$\langle Y_{11}^* - Y_{11}, Y_{11}^* - \hat{Y}_{11}\rangle_I \leq 0, \tag{11}$$

that is, the inner product (5) is non-positive.