# Model reduction strategies for linear feedback control of boundary layer transition

Jérôme Hœpffner

*Royal Institute of Technology (KTH)*

*Stockholm, Sweden*

# Orr-Sommerfeld-Squire

- Incompressible
- Linearization about a mean profile
- Parallel boundary layer
- Wave-like perturbation
- $\tilde{v}$ $\tilde{\eta}$ formulation

$$\underbrace{\begin{pmatrix} -\Delta & 0 \\ 0 & I \end{pmatrix}}_{M} \underbrace{\begin{pmatrix} \dot{\tilde{v}} \\ \dot{\tilde{\eta}} \end{pmatrix}}_{\dot{\mathsf{q}}} = \underbrace{\begin{pmatrix} \mathcal{L}_{OS} & 0 \\ \mathcal{L}_C & \mathcal{L}_{SQ} \end{pmatrix}}_{L} \underbrace{\begin{pmatrix} \tilde{v} \\ \tilde{\eta} \end{pmatrix}}_{\mathsf{q}}$$

# Lifting of boundary conditions

- Splitting

$$\underbrace{\begin{pmatrix} \tilde{v} \\ \tilde{\eta} \end{pmatrix}}_{\mathsf{q}} = \underbrace{\begin{pmatrix} \tilde{v}_h \\ \tilde{\eta}_h \end{pmatrix}}_{\mathsf{q}_h} + \tilde{\phi} \underbrace{\begin{pmatrix} \tilde{v}_p \\ \tilde{\eta}_p \end{pmatrix}}_{\mathsf{q}_p}$$

- Controlled dynamic

$$\underbrace{\dot{\mathsf{q}}_h = M^{-1} L \mathsf{q}_h}_{Dynamic} + \underbrace{\tilde{\phi} M^{-1} L \mathsf{q}_p - \dot{\tilde{\phi}} \mathsf{q}_p}_{Control}$$

- Embed $\tilde{\phi}$ in the system

$$\underbrace{\begin{pmatrix} \dot{\mathsf{q}}_h \\ \dot{\tilde{\phi}} \end{pmatrix}}_{\dot{\mathsf{x}}} = \underbrace{\begin{pmatrix} M^{-1} L & M^{-1} L \mathsf{q}_p \\ 0 & 0 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} \mathsf{q}_h \\ \tilde{\phi} \end{pmatrix}}_{\mathsf{x}} + \underbrace{\begin{pmatrix} -\mathsf{q}_p \\ 1 \end{pmatrix}}_{B_2} \underbrace{\dot{\tilde{\phi}}}_{\tilde{u}}$$

- Choice of $\mathsf{q}_p$

$$L\mathsf{q}_p = 0 \qquad \text{and non-homogeneous boundary conditions}$$

# The plant and the estimator

- The plant

$$\begin{cases} \dot{\mathsf{x}} &= A\mathsf{x} + B_1 w_1 + B_2 \tilde{u} & \text{State} \\ z &= C_1 \mathsf{x} + D_{12} \tilde{u} & \text{Output} \\ y &= C_2 \mathsf{x} + D_{21} w_2 & \text{Measurement} \end{cases}$$

- The estimator

$$\begin{cases} \dot{\hat{\mathsf{x}}} &= A\hat{\mathsf{x}} + B_1 w_1 + B_2 \tilde{u} - \tilde{v} & \text{Estimated state} \\ \hat{y} &= C_2 \hat{\mathsf{x}} & \text{Measurement} \end{cases}$$

- The feedback laws

$$\tilde{u} = K\hat{\mathsf{x}} \qquad , \qquad \tilde{v} = L(y - \hat{y}) = L\delta y$$

# Objective functions

$$z = C_1 \mathsf{x} + D_{12}\tilde{u} \qquad C_1 = \begin{pmatrix} M^{\frac{1}{2}} \\ 0 \end{pmatrix} \qquad D_{12} = \begin{pmatrix} 0 \\ \ell I \end{pmatrix}$$

$$\begin{aligned} &Controller \quad \mathcal{J}_c = \mathcal{E}[\mathsf{x}^* M \mathsf{x} + \ell^2 \tilde{u}^* \tilde{u}] = \mathcal{E}[z^* z] \\ &Estimator \quad \mathcal{J}_e = \mathcal{E}\big[(\mathsf{x} - \hat{\mathsf{x}})^* (\mathsf{x} - \hat{\mathsf{x}})\big] = \mathcal{E}[\delta \mathsf{x}^* \delta \mathsf{x}] \end{aligned}$$

# Solution of the optimisation

$$L = -\frac{1}{\alpha^2} Y C_2^* \quad where \quad Y = Ric \begin{pmatrix} A^* & -\frac{1}{\alpha^2} C_2^* C_2 \\ -B_1 B_1^* & -A \end{pmatrix}$$

$$K = -\frac{1}{\ell^2} B_2^* X \quad where \quad X = Ric \begin{pmatrix} A & -\frac{1}{\ell^2} B_2 B_2^* \\ -C_1^* C_1 & -A^* \end{pmatrix}$$

# Adjoint and bi-orthogonality

- definition of adjoint

$$\langle \mathcal{L}\mathsf{q}, \xi \rangle = \langle \mathsf{q}, \mathcal{L}^{+}\xi \rangle + boundary\ terms$$

- Inner product

$$\langle \mathsf{q}, \xi \rangle_M = \langle M\mathsf{q}, \xi \rangle = \int_0^\infty \xi^* M \mathsf{q}\ dy$$

- adjoint Orr-Sommerfeld-Squire

$$\underbrace{\begin{pmatrix} -\Delta & 0 \\ 0 & I \end{pmatrix}}_{M^+} \underbrace{\begin{pmatrix} \dot{\tilde{v}}^{+} \\ \dot{\tilde{\eta}}^{+} \end{pmatrix}}_{\dot{\mathsf{q}}^+} = \underbrace{\begin{pmatrix} \mathcal{L}^{+}_{OS} & \mathcal{L}^{+}_{C} \\ 0 & \mathcal{L}^{+}_{SQ} \end{pmatrix}}_{L^+} \underbrace{\begin{pmatrix} \tilde{v}^{+} \\ \tilde{\eta}^{+} \end{pmatrix}}_{\mathsf{q}^+}$$

- Bi-orthonormality

$$\lambda_n = \sigma_n^* \qquad and \qquad \langle \mathsf{q}_n, \mathsf{q}_m^{+} \rangle_M = \delta_{nm}$$

# The modal canonical form

- The state projection

$$\begin{pmatrix} \dot{\mathsf{k}}_1 \\ \dot{\mathsf{k}}_2 \\ \vdots \\ \dot{\mathsf{k}}_N \\ \dot{\tilde{\phi}} \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 & \dots & 0 & 0 \\ 0 & \lambda_2 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & 0 & \vdots \\ 0 & 0 & 0 & \lambda_N & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \\ \vdots \\ k_N \\ \tilde{\phi} \end{pmatrix} + \dot{\tilde{\phi}} \begin{pmatrix} -\langle \mathsf{q}_p, \mathsf{q}_1^+ \rangle_M \\ -\langle \mathsf{q}_p, \mathsf{q}_2^+ \rangle_M \\ \vdots \\ -\langle \mathsf{q}_p, \mathsf{q}_N^+ \rangle_M \\ 1 \end{pmatrix}$$

- The measurement projection

$$y = \begin{pmatrix} C_2\mathsf{q}_1^+, C_2\mathsf{q}_2^+, \dots, C_2\mathsf{q}_N^+ \end{pmatrix} \begin{pmatrix} \mathsf{k}_1 \\ \mathsf{k}_2 \\ \vdots \\ \mathsf{k}_N \end{pmatrix}$$

- The modal residuals

$$f_n = \| - \langle \mathsf{q}_p, \mathsf{q}_n^+ \rangle_M \|_2 \qquad q_n = \|C_2\mathsf{q}_n^+\|_2$$

# Reducing the order of the system

**Full order system**

$$\begin{cases} \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} \tilde{u} \\ y = \begin{pmatrix} C_1, C_2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + D\tilde{u} \end{cases}$$

**Truncation** Set $x_2 = 0$

$$\begin{cases} \dot{x}_1 = A_{11}x_1 + B_1\tilde{u} \\ y = C_1 x_1 + D\tilde{u} \end{cases}$$

**Residualization** Set $x_2 = cst$

$$\begin{cases} \dot{x}_1 = (A_{11} - A_{12}A_{22}^{-1}A_{21})x_1 + (B_1 - A_{12}A_{22}^{-1}B_2)\tilde{u} \\ y = (C_1 - C_2A_{22}^{-1}A_{21})x_1 + (D - C_2A_{22}^{-1}B_2)\tilde{u} \end{cases}$$

# Which realisation to reduce?

**Modal realisation** • Bound on the error

$$\|G - G_r\|_\infty \leq \sum_{n \in Disc} \frac{|f_n g_n|}{|\Re(\lambda_n)|}$$

**Balanced realisation** • Controllability and observability gramians

$$\begin{cases} A\Sigma + \Sigma A^* + BB^* = 0 \\ A^*\Sigma + \Sigma A + C^*C = 0 \end{cases}$$

- The Hankel singular values $\sigma_n$

$$\Sigma = diag(\sigma_1, \sigma_2, ..., \sigma_N)$$

- Bound on the error

$$\|G - G_r\|_\infty \leq 2 \sum_{n \in Disc} \sigma_n$$

(Twice the sum of the tail)

# Optimal Hankel norm approximation

- Hankel norm

$$\|G(s)\|_H^2 = \max_{u(t)} \frac{\int_0^\infty \|z(\tau)\|_2^2 \, d\tau}{\int_{-\infty}^0 \|u(\tau)\|_2^2 \, d\tau}$$

- Optimisation :

  Search $G_r$ of order $k$ such that $\|G - G_r\|_H$ is minimum

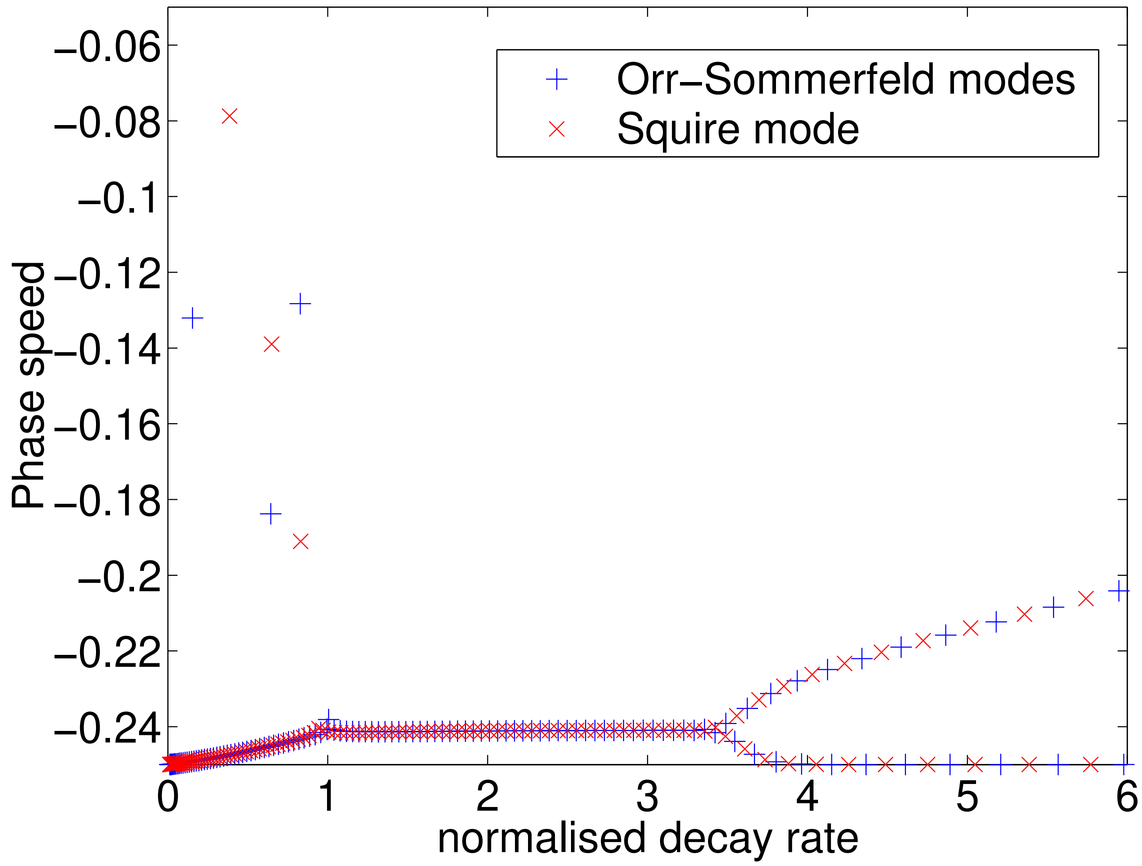
- Infinity norm bound

$$\|G - G_r\|_\infty \leq \sum_{n \in Disc} \sigma_n$$

(The sum of the tail)

# Orr-Sommerfeld-Squire spectrum

Reynolds = 350, $(\alpha, \beta) = (0.25, 0.649)$

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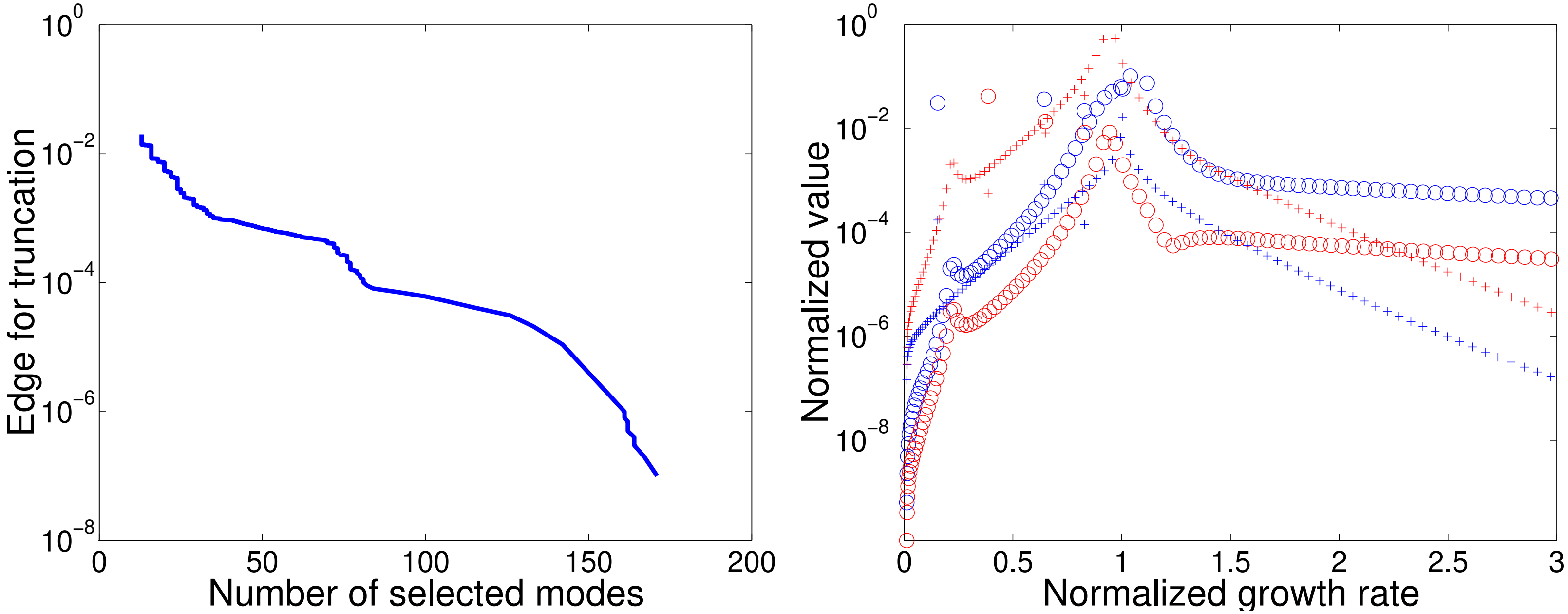
# Truncation criterion

Red : Squire modes

Blue : Orr-Sommerfeld modes

$\bigcirc$ : truncation criterion $\frac{|f_n g_n|}{|\Re(\lambda_n)|}$

$+$ : Coefficient of initial condition for maximum transient energy growth



# Conclusion

- Bad controller and reduction
- Truncation / residualization
- Modal instability and reduction
- Modes Interaction and reduction
- Balanced realisation reduction / optimal Hankel norm approximation