

# “Wunderlich, meet Kirchhoff”: A general and unified description of elastic ribbons and thin rods

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**Abstract** The equations for the equilibrium of a thin elastic ribbon are derived by adapting the classical theory of thin elastic rods. Previously established ribbon models are extended to handle geodesic curvature, natural out-of-plane curvature, and a variable width. Both the case of a finite width (Wunderlich’s model) and the limit of small width (Sadowksy’s model) are recovered. The ribbon is assumed to remain developable as it deforms, and the direction of the generatrices is used as an internal variable. Internal constraints expressing inextensibility are identified. The equilibrium of the ribbon is found to be governed by an equation of equilibrium for the internal variable involving its second-gradient, by the classical Kirchhoff equations for thin rods, and by specific, thin-rod-like constitutive laws; this extends the results of Starostin and van der Heijden (2007) to a general ribbon model. Our equations are applicable in particular to ribbons having geodesic curvature, such as an annulus cut out in a piece of paper. Other examples of application are discussed. By making use of a material frame rather than the Frenet–Serret’s frame, the present work unifies the description of thin ribbons and thin rods.

**Keywords** Elastic plates 74K20 · Elastic rods 74K10 · Energy minimization 74G65

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## 1 Introduction

A ribbon is an elastic body whose dimensions (typical length  $L$ , width  $w$  and thickness  $h$ ) are all very different,  $L \gg w \gg h$ . While previous work has been focussed on the case of rectangular ribbons, we consider the general case of ribbons having non-zero natural curvatures, both in the out-of-plane and the in-plane directions. This extension includes ribbon geometries such as those obtained by cutting a piece of paper along two arbitrary curves.

From a mechanical perspective, elastic ribbons lie halfway between the 1D case of thin rods (for which  $w \sim h$ ), and the 2D case of thin elastic plates or shells (for which  $w \sim L$ ). On one hand, their elastic energy is given by the theory of thin elastic plates or shells. On the other hand, ribbons look like 1D structures (thin rods) when observed from the large scale  $L$ : this suggests that they can be described by the classical equations for thin elastic rods. This article is concerned with the following problem of dimensional reduction: starting from a thin, developable shell model, can one recover the 1D equations of equilibrium applicable to thin rods?

This work builds up on a few seminal articles. The dimensional reduction has already been carried out at the energy level and in the particular case of rectangular, naturally flat ribbons: Sadowsky [21] derived a 1D energy functional for a narrow ribbon (small  $w$ ), and his work was later generalized by Wunderlich [28] to a finite width  $w$ . Their dimensional reduction was made possible by focussing on developable configurations of the ribbon, which are preferred energetically in the thin limit,  $h \ll w$ . Developable surfaces are special cases of ruled surfaces, *i.e.* they are spanned by a set of straight lines called generatrices or rulings: the 1D elastic energy of Wunderlich is based on a reconstruction of the surface of the ribbon in terms of its center-line and of the angle between the generatrices and the center-line tangent. We use a similar parameterization here and derive the 1D energy functional for a developable, but not necessarily rectangular, ribbon.

Next comes the question of minimizing this 1D energy to solve the equilibrium problem. Upper bounds for the energy have been obtained by inserting trial forms of the ribbon into the 1D energy, as was done in the context of the elastic Möbius strip [28, 19]. Finding equilibrium solutions, however, requires one to derive the equations of equilibrium by a variational method. This has been done in a beautiful article by Starostin and van der Heijden [25] for naturally flat and rectangular ribbons. They found equilibrium equations that bear a striking resemblance with the Kirchhoff equations governing the equilibrium of thin rods. Their result was later extended to helical ribbons [24], which is another case where geodesic curvature is absent. Here, we want to revisit and extend their work in the following ways.

First, the derivation of Starostin and van der Heijden, based on the variational bicomplex formalism, uses a different approach than the classical theory of thin rods. The final equations, however, look similar to the Kirchhoff equations for the equilibrium of thin rods. In fact, previous work on thin ribbons has developed as a field largely independent from the vast literature on thin

rods. This is unfortunate in view of their deep similarities. Here, we advocate the viewpoint that a ribbon is just a special kind of a thin rod, having an internal parameter and being subjected to kinematical constraints — this is quite similar to the way the incompressibility constraint is handled in 3D elasticity. These specificities can be incorporated naturally into the classical theory of thin rods, as we show. Doing this allows one to recycle much of the existing knowledge on thin rods. In particular, the equations of equilibrium for ribbons are derived in close analogy with those for rods, and in a straightforward way.

Second, we make use of directors, as in the classical theory of rods. By contrast, Wunderlich has introduced a parameterization of the mid-surface of the ribbon based on the Frenet–Serret frame associated with the center-line. Wunderlich’s energy, in particular, is defined in terms of Frenet–Serret’s notions of torsion and curvature. This parameterization has a drawback: it is specific to the case where the center-line is a geodesic, as we show. By working instead with directors, we can naturally extend Wunderlich’s model to ribbons that have geodesic curvature, *i.e.* to ribbons curved in their own plane such as an annulus cut out from a piece of paper.

In the present work we make use of several ideas introduced in a recent article [10], where we have shown that the buckling of a curved strip cut out from a piece of paper and folded along its central circle [11] can be analyzed using the language of thin rods. This was done by identifying the relevant geometrical constraints and constitutive laws. Here, we do not consider any fold but allow for more general geometries (non-uniform width and geodesic curvature).

This article is organized as follows. In section 2, we extend the parameterization of developable surfaces introduced by Wunderlich: making use of the frame of directors, we account for the geodesic curvature of the center-line and a variable width. In section 3, Wunderlich’s energy functional is extended. In section 4, the equilibrium equations of a general ribbon are derived by a variational method adapted from the theory of thin rods. In section 5, we recover known ribbon models in the special case of a geodesic center-line ( $\kappa_g = 0$ ) and constant width  $w$ . In section 6, we present some equilibrium problems for ribbons having geodesic curvature as possible illustrations of our theory.

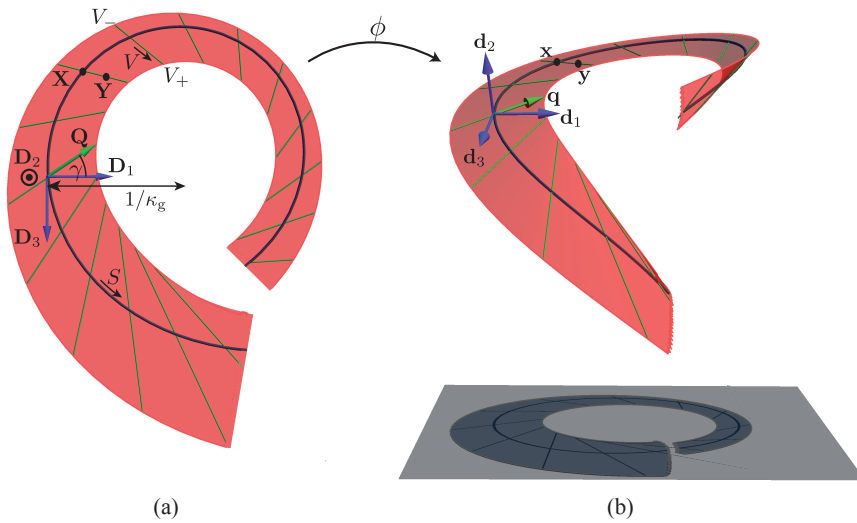
## 2 Geometry of a developable ribbon

### 2.1 Developable transformation from reference to current configuration

As we consider developable ribbons, we can assume that the reference configuration is planar<sup>1</sup>. This planar reference configuration is not necessarily stress-free (we shall address the case of naturally curved ribbons). In the reference configuration, a material line  $\mathbf{X}(S)$ , called the *center-line*, is traced out

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<sup>1</sup> For a closed developable ribbon, there may not exist any *global* planar configurations — see the example in section 6.2. In that case, we introduce an arbitrary cut in the planar configuration of reference.



**Fig. 1** Geometry of a developable ribbon (a) in the planar, undeformed configuration, and (b) in the actual configuration. The direction of the generatrices is measured by the parameter  $\eta$  in the model, which is the tangent of the angle  $\gamma$  between the director  $\mathbf{d}_1$  and the generatrix direction  $\mathbf{q}$  (the angle  $\gamma$  shown in the figure is negative, and  $\eta < 0$  here).

on the ribbon, see figure 1a. Here,  $S$  is the arc-length along the center-line as measured in reference configuration,  $|\mathbf{X}'(S)| = 1$ . Primes denote derivation with respect to arc-length, and boldface characters denote vectors. The surface of the ribbon is oriented by prescribing a constant unit vector  $\mathbf{D}_2$ , perpendicular to the plane of the ribbon. Let  $\mathbf{D}_3(S) = \mathbf{X}'(S)$  be the unit tangent to the center-line. The vectors  $\mathbf{D}_1(S) = \mathbf{D}_2 \times \mathbf{D}_3(S)$ ,  $\mathbf{D}_2$  and  $\mathbf{D}_3(S)$  then form an orthonormal frame.

A deformed, developable configuration of the ribbon is specified by the functions

$$(\mathbf{x}(S), \mathbf{d}_1(S), \mathbf{d}_2(S), \mathbf{d}_3(S), \eta(S)), \quad (1)$$

which are subjected to geometrical constraints derived later. Here  $\mathbf{x}(S)$  is the deformed center-line,  $\mathbf{d}_i(S)$  (for  $i = 1, 2, 3$ ) define the *frame of directors* (also called the *material frame*) and  $\eta(S)$ , defined below, encodes the definition of the generatrices.

The third director is chosen to be the tangent to the deformed center-line,

$$\mathbf{x}'(S) = \mathbf{d}_3(S), \quad (2^*)$$

and the second director  $\mathbf{d}_2(S)$  is defined to be normal to the ribbon at  $\mathbf{x}(S)$  as in the reference configuration, see figure 1b. The directors are defined to be orthonormal,

$$\mathbf{d}_i(S) \cdot \mathbf{d}_j(S) = \delta_{ij}, \quad (3^*)$$

where  $\delta_{ij}$  stands for Kronecker's symbol. This implies that  $|\mathbf{x}'| = |\mathbf{d}_3| = 1 = |\mathbf{X}'(S)|$ . By construction, the directors  $\mathbf{d}_i$  are *material* vectors: contrary to the

Frenet–Serret frame associated with the center-line, they follow the rotation of the ribbon.

As the ribbon is inextensible, it remains developable by Gauss’ *Theorema egregium*. Smooth, developable surfaces are ruled [23]: there exists a one-parameter family of straight lines, called *generatrices*, that sweeps out over the entire surface. As in previous work [28,25], we define  $\eta(S)$  to be the tangent of the angle  $\gamma$  between  $\mathbf{d}_1$  and the generatrix. Then, the vector

$$\mathbf{q}(\eta, S) = \eta(S) \mathbf{d}_3(S) + \mathbf{d}_1(S) \quad (4)$$

spans the generatrix<sup>2</sup>. Therefore, the transformation from the reference to the deformed configuration can be expressed as the mapping  $\phi$ :

$$\phi: \mathbf{Y} = \mathbf{X}(S) + V \mathbf{Q}(\eta, S) \mapsto \mathbf{y} = \mathbf{x}(S) + V \mathbf{q}(\eta, S). \quad (5)$$

Here,  $V$  is a coordinate along the generatrix,  $\mathbf{Y}$  and  $\mathbf{y}$  denote a current point along the ribbon in reference and actual configurations, respectively. The vector  $\mathbf{Q}$  is defined as  $\mathbf{Q}(\eta, S) = \eta(S) \mathbf{D}_3(S) + \mathbf{D}_1(S)$ : it defines the direction of the generatrix brought back in the reference configuration.

We use the longitudinal and transverse coordinates  $(S, V)$  to parameterize the ribbon’s surface.  $S$  varies in the interval  $0 \leq S \leq L$ , where  $L$  is the curvilinear length of the center-line. The transverse coordinate  $V$  varies in a domain  $V_-(\eta, S) \leq V \leq V_+(\eta, S)$ . The endpoints  $V_\pm(\eta, S)$  of the interval are such that the points  $\mathbf{Y}_\pm(S) = \mathbf{X}(S) + V_\pm(\eta, S) \mathbf{Q}(\eta, S)$  lie on the edges of the ribbon. The functions  $V_\pm(\eta, S)$  capture the relative position of the edges and of the center-line, and are called the *edge functions*. Explicit expressions are derived in section 2.2 for some ribbon geometries.

From equation (3), the directors define an orthonormal frame for any value of the arc-length parameter  $S$ . Therefore, there exists a vector  $\boldsymbol{\omega}(S)$  called the Darboux vector or the rotation gradient, such that

$$\mathbf{d}'_i(S) = \boldsymbol{\omega}(S) \times \mathbf{d}_i(S) \quad (6)$$

for  $i = 1, 2, 3$ . The operation  $\times$  denotes the cross product in the Euclidean space. The components of the rotation gradient in the basis of directors,  $\omega_i(S) = \boldsymbol{\omega}(S) \cdot \mathbf{d}_i(S)$  measure the amount of bending ( $i = 1, 2$ ) and twisting ( $i = 3$ ) of the center-line. An explicit expression is

$$\omega_i(S) = \frac{1}{2} \sum_{j=1}^3 \sum_{k=1}^3 \epsilon_{ijk} \mathbf{d}'_j(S) \cdot \mathbf{d}_k(S), \quad (7^*)$$

where  $\epsilon_{ijk}$  represents the permutation symbol:  $\epsilon_{ijk} = 1$  when  $(i, j, k)$  is an even permutation of the indices,  $\epsilon_{ijk} = -1$  when it is an odd permutation, and  $\epsilon_{ijk} = 0$  otherwise. In the language of the geometry of surfaces, the directors frame  $(\mathbf{d}_1, \mathbf{d}_2, \mathbf{d}_3)$  is called the Darboux frame associated with the center-line

<sup>2</sup> The vector  $\mathbf{q}$  depends both on the unknown *function*  $\eta(\cdot)$  and on the arc-length parameter  $S$ ; hence the arguments shown in the left-hand side of equation (4).

curve, and the strains  $\omega_1$ ,  $\omega_2$ , and  $\omega_3$  are respectively the normal curvature, the geodesic curvature, and the geodesic torsion. Note we use strain measures,  $\omega_i$ , that are based on the frame of directors (a material frame), while previous work used the Frenet–Serret frame associated with the center-line. Working with a frame of directors offers many advantages: it extends naturally to the case of non-geodesic center-lines, allows one to use the same language as in the theory of rods, and to remove the artificial singularities displayed by the Frenet–Serret frame near inflection points or straight segments.

## 2.2 Edge functions

The edge functions  $V_{\pm}(\eta, S)$  encode the relative positions of the edges of the ribbon with respect to the center-line. Expressions for  $V_{\pm}$  are derived below for the cases of a rectangular and an annular ribbon.

The case of a rectangular ribbon is quite simple. We use the central axis of the ribbon as the center-line. In reference configuration, the equation of the edges is  $(\mathbf{Y} - \mathbf{X}) \cdot \mathbf{D}_1 = \pm w/2$ , where  $w$  is the width of the ribbon. Inserting the parameterization of  $\mathbf{Y}$  from equation (5), this yields

$$V_{\pm}(\eta, S) = \pm \frac{w}{2} \quad (\text{rectangular ribbon}), \quad (8)$$

where the  $\pm$  labels the edges.

The case of an annular ribbon is studied next. We use an arc of a circle as the center-line, having its radius given by the inverse of the geodesic curvature,  $\kappa_g^{-1}$ , and a constant width  $w$  of the ribbon. In reference configuration, the center  $\mathbf{C}$  of the circular center-line is  $\mathbf{C} = \mathbf{X}(S) + \kappa_g^{-1} \mathbf{D}_1(S)$ . Therefore, the equation for the edges is  $(\mathbf{Y}(S, V) - \mathbf{C})^2 = (\kappa_g^{-1} \mp \frac{w}{2})^2$ . Using equation (5) and the definition of  $\mathbf{Q}$ , this shows that the edge functions  $V_{\pm}$  are the roots  $V$  of the second-order polynomial

$$\left(V - \frac{1}{\kappa_g}\right)^2 + (\eta V)^2 = \left(\frac{1}{\kappa_g} \mp \frac{w}{2}\right)^2.$$

Solving for  $V$ , we find

$$V_{\pm}(\eta, S) = \frac{1}{\kappa_g} \frac{1 - \sqrt{1 \mp (1 + \eta^2) w \kappa_g \left(1 \mp \frac{w \kappa_g}{4}\right)}}{1 + \eta^2} \quad (\text{annular ribbon}). \quad (9)$$

Letting the curvature of the annulus go to zero,  $\kappa_g \rightarrow 0$ , we recover  $V_{\pm} \rightarrow \pm w/2$  which is consistent with equation (8).

In the general case of a ribbon having a variable width, or when the center-line has non-constant curvature  $\kappa_g$ ,  $V_{\pm}$  may be available through an implicit equation and not necessarily in closed form.

### 2.3 Constraints expressing developability

The condition of inextensibility of the ribbon imposes some kinematical constraints on the unknowns listed in (1), and on the curvature and twisting strains  $\omega_i$  calculated by equations (6–7). These constraints are derived as follows.

The center-line is a curve drawn on the surface of the ribbon. Its geodesic curvature is defined by  $\kappa_g = \mathbf{x}'' \cdot \mathbf{d}_1 = \mathbf{d}'_3 \cdot \mathbf{d}_1 = \omega_2$ . It is a classical result of the differential geometry of surfaces [23] that the geodesic curvature is conserved upon isometric deformations of a surface. Therefore,  $\kappa_g$  is prescribed by the reference configuration:  $\kappa_g(S) = \mathbf{D}'_3(S) \cdot \mathbf{D}_1(S)$ . We write this geodesic constraint as

$$\mathcal{C}_g(\omega_2, S) = 0 \quad (10a^*)$$

where

$$\mathcal{C}_g(\omega_2, S) = \kappa_g(S) - \omega_2. \quad (10b^*)$$

Note that  $\omega_2 = 0$  when  $\kappa_g = 0$ , *i.e.* when the center-line is a geodesic. In that case, the derivative of the tangent is  $\mathbf{d}'_3 = \boldsymbol{\omega} \times \mathbf{d}_3 = \omega_1 (-\mathbf{d}_2)$  and  $(-\mathbf{d}_2)' \cdot \mathbf{d}_1 = \omega_3$ . Here, we recognize the definition of the Frenet–Serret frame  $(\mathbf{d}_3, -\mathbf{d}_2, \mathbf{d}_1)$  associated with the center-line: the Frenet–Serret curvature and torsion are  $\omega_1$  and  $\omega_3$ , respectively. Therefore, in the particular case when the center-line is a geodesic, our directors coincide with the Frenet–Serret frame. It is much more convenient to work with the directors in general.

The second constraint expresses the developability of the ruled surface spanned by the generatrices. It is found in classical textbooks of differential geometry [23], and is rederived in Appendix A:

$$\mathcal{C}_d(\omega_1, \omega_3, \eta) = 0 \quad (11a^*)$$

where

$$\mathcal{C}_d(\omega_1, \omega_3, \eta) = \eta \omega_1 - \omega_3. \quad (11b^*)$$

### 2.4 Area element

To integrate the elastic energy along the surface of the ribbon, we will need the expression for the area element  $da$ . It is calculated in Appendix A from the Jacobian of the transformation  $\phi$ :

$$da = |\partial_S \mathbf{y} \times \partial_V \mathbf{y}| dS dV = \left(1 - \frac{V}{V_c(\eta, S)}\right) dS dV, \quad (12)$$

where the auxiliary quantity  $V_c$  is defined as

$$V_c(\eta, \eta', S) = \frac{1}{(1 + \eta^2) \kappa_g(S) - \eta'}. \quad (13^*)$$

Since  $da = 0$  at  $V = V_c$ , the transformation  $\phi$  is singular there. The quantity  $V_c$  can be interpreted as the value of the transverse coordinate  $V$  where the

generatrix intersects neighboring generatrices, *i.e.* intersects its own caustic, called the *striction curve* [23]. We assume that the striction curve stays outside of the physical domain, so that the curvature tensor is nowhere singular on the ribbon: either  $V_c < V_- \leq V \leq V_+$  or  $V_c > V_+ \geq V \geq V_-$ . This implies that

$$\frac{1 - V_+/V_c}{1 - V_-/V_c} > 0 \quad (14a)$$

and

$$\left| \frac{V}{V_c} \right| < 1 \quad (14b)$$

## 2.5 Curvature tensor of the deformed ribbon

In order to write the elastic energy of the ribbon, we need the curvature tensor  $\mathbf{K}$  at the arbitrary point  $(S, V)$  of the ribbon surface. This geometrical calculation, at the heart of Wunderlich's energy, is given in Appendix A. The dependence of the curvature tensor,  $\mathbf{K}(\eta, \omega_1, S, V)$ , on the transverse coordinate is imposed by the developability condition to be

$$\mathbf{K}(\eta, \eta', \omega_1, S, V) = \frac{\mathbf{K}_0(\eta, \omega_1)}{1 - \frac{V}{V_c(\eta, \eta', S)}}, \quad (15)$$

where  $\mathbf{K}_0(\eta, \omega_1) = \mathbf{K}(\eta, \eta', \omega_1, S, V = 0)$  denotes the curvature tensor evaluated along the center-line, defined by

$$\mathbf{K}_0(\eta, \omega_1) = -\omega_1 (\mathbf{d}_3 \otimes \mathbf{d}_3 - \eta (\mathbf{d}_3 \otimes \mathbf{d}_1 + \mathbf{d}_1 \otimes \mathbf{d}_3) + \eta^2 \mathbf{d}_1 \otimes \mathbf{d}_1). \quad (16^*)$$

Here, the curvature tensor is expressed by its components in the orthonormal basis  $(\mathbf{d}_3(S), \mathbf{d}_1(S))$  spanning the plane tangent to the ribbon at  $(S, V)$ , as implied by the subscript notation. Inserting equation (16) into equation (15) yields  $\det \mathbf{K} = \det \mathbf{K}_0 = 0$  which is consistent with Gauss' *theorema egregium* (conservation of Gauss curvature by isometric deformations of a surface).

## 3 Elastic energy

Being modeled as an inextensible plate, the deformed ribbon is isometrically mapped to its planar underformed configuration. The constraint of isometric mapping is enforced by the equations (10–11). As a result, the stretching energy cancels. The only contribution to the elastic energy of the ribbon is the bending energy  $E$ . For a plate made up of a homogeneous isotropic Hookean solid,

$$E = \frac{D}{2} \iint \left[ (1 - \nu) \operatorname{tr}((\mathbf{K} - \mathbf{K}_r)^2) + \nu \operatorname{tr}^2(\mathbf{K} - \mathbf{K}_r) \dots \right. \\ \left. - (1 - \nu) \operatorname{tr}(\mathbf{K}_r^2) - \nu \operatorname{tr}^2 \mathbf{K}_r \right] da$$



where  $D = Y h^3 / (12(1 - \nu^2))$  is the bending modulus of the plate,  $\nu$  the Poisson's ratio,  $Y$  the Young's modulus,  $h$  the thickness of the ribbon. This expression is based on the classical formula for plates [18], and has been modified to take natural (or reference) curvature into account through the natural curvature tensor  $\mathbf{K}_r$ . The last two constant terms appearing in the second line are included to make the energy density zero when the ribbon is flat ( $\mathbf{K} = \mathbf{0}$ ), which is a convenient convention.

The energy can be simplified by expanding the squared matrices, using the identity  $\text{tr}^2 \mathbf{K} = \text{tr}(\mathbf{K}^2) + 2 \det \mathbf{K}$  valid for  $2 \times 2$  matrices, and the developability condition  $\det \mathbf{K} = 0$  that follows from equation (15):

$$E = \frac{D}{2} \iint (\text{tr}(\mathbf{K}^2) - 2 \mathbf{Q}_r : \mathbf{K}) da, \quad (17)$$

where the symmetric tensor  $\mathbf{Q}_r$  capturing the effect of the reference curvature is defined by

$$\mathbf{Q}_r = \mathbf{K}_r + \nu \text{cof } \mathbf{K}_r, \quad (18)$$

and  $\text{cof } \mathbf{K}_r$  denotes the cofactor matrix:

$$\text{cof } \mathbf{K}_r = (K_{11}^r \mathbf{d}_1 \otimes \mathbf{d}_1 - K_{13}^r (\mathbf{d}_1 \otimes \mathbf{d}_3 + \mathbf{d}_1 \otimes \mathbf{d}_3) + K_{33}^r \mathbf{d}_3 \otimes \mathbf{d}_3). \quad (19)$$

The symbols  $K_{ij}^r$  denote the components of the natural curvature  $\mathbf{K}_r$  in the tangent basis  $(\mathbf{d}_3, \mathbf{d}_1)$ , with  $i, j = 3, 1$ .

Inserting the expression of the area element  $da$  from equation (12) and the known dependence of  $\mathbf{K}$  on the transverse coordinate from equation (15) into equation (17) gives

$$E = \frac{D}{2} \int_0^L \left[ \text{tr}(\mathbf{K}_0^2) \int_{V_-}^{V_+} \frac{dV}{1 - V/V_c} - 2 \mathbf{Q}_r : \mathbf{K}_0 \int_{V_-}^{V_+} dV \right] dS.$$

Here, by factoring the natural curvature tensor  $\mathbf{Q}_r$  out of the integral along the generatrices, we have assumed that  $\mathbf{Q}_r$  varies on the typical length-scale  $L$  (length of the ribbon) and thus can be considered constant on the much smaller length-scale  $w$  (width of the ribbon). Our model thus handles non-uniform geometries  $\mathbf{Q}_r(S)$ , even though we omit the argument  $S$  in  $\mathbf{Q}_r$  for the sake of legibility.

Integrating along  $V$ , we obtain the one-dimensional energy functional

$$E(\eta, \eta', \omega_1) = \frac{D}{2} \int_0^L \left[ \left( -V_c \ln \frac{1 - V_+/V_c}{1 - V_-/V_c} \right) \text{tr}(\mathbf{K}_0^2) \dots \right. \\ \left. - 2(V_+ - V_-) \mathbf{Q}_r : \mathbf{K}_0 \right] dS. \quad (20^*)$$

This functional extends Wunderlich's result to the case of a ribbon with natural out-of-plane curvature (through the term depending on  $\mathbf{Q}_r$ ), a variable width (through the functions  $V_{\pm}$ ), and geodesic curvature (through the dependence of  $V_{\pm}$  and  $V_c$  on  $\kappa_g$ ). The argument of the logarithm is always positive, as is noted in the inequality (14a).

## 4 Equations of equilibrium

In this section, we use the calculus of variations to derive the equations of equilibrium for a general ribbon model. Thanks to our parameterization based on directors, we do this simply by extending the classical derivation of the equations of equilibrium for thin elastic rods, using the principle of virtual work. We have used a similar approach in a recent paper [10, eqs. 9b and A.2b] to derive the equations of equilibrium of a thin annular strip folded along its central circle.

### 4.1 Principle of virtual work for a ribbon

If the external load is conservative, the equilibrium of the ribbon is found by minimizing the total potential energy, which is the sum of the potential energy associated with the external load and of the elastic energy  $E(\eta, \eta', \omega_1)$  in equation (20). This is a constrained minimization problem, and we need to extend the classical variational derivation [9, 7, 13, 26, 6, 2] of the equations of equilibrium for thin rods to take into account the presence of the *kinematical constraints*  $\mathcal{C}_d$  and  $\mathcal{C}_g$ , and of an *internal variable*  $\eta(S)$ .

To handle the case of non-conservative loads, we derive the equations of equilibrium using the more general framework of the principle of virtual work. A virtual motion of the ribbon is specified by a virtual displacement  $\hat{\mathbf{x}}(S)$  of the center-line, a virtual rotation  $\hat{\boldsymbol{\psi}}(S)$  of the orthonormal frame of directors, a virtual variation  $\hat{\mathbf{d}}_i(S)$  of the directors, a virtual variation  $\hat{\eta}(S)$  of the parameter defining the direction of the generatrices, and the virtual changes of material curvature and twisting strains  $\hat{\omega}_i(S)$ .

We start by noting that equations (2), (3) and (7) define what is known as an inextensible Euler–Bernoulli rod. When written in incremental form, they yield relations between the virtual quantities  $\hat{\mathbf{x}}$ ,  $\hat{\boldsymbol{\psi}}$ ,  $\hat{\mathbf{d}}_i$  and  $\hat{\omega}_i$ . In the principle of virtual work, these relations are viewed as constraints, and are handled by a constraint term  $\mathcal{W}_{\text{cEB}}$ , which we call the virtual work of the Euler–Bernoulli constraints. Our treatment of these constraints does not differ from the standard theory of inextensible Euler–Bernoulli rods, and we shall therefore omit the details.

The virtual work of a ribbon includes the following contributions. First, the virtual internal work is equal to minus the first variation of the elastic energy  $\mathcal{W}_i = -\hat{E}$ , as usual for elasticity problems. Second, the external virtual work reads  $\mathcal{W}_e = \int_0^L (\mathbf{p} \cdot \hat{\mathbf{x}} + \mathbf{c} \cdot \hat{\boldsymbol{\psi}}) dS$  where  $\mathbf{p}$  and  $\mathbf{c}$  are the density of applied force and moment onto the center-line, per unit arc-length. For static problems, the virtual work of acceleration is zero. The constraints are treated by constraint terms involving Lagrange multipliers: the constraints applicable to an inextensible Euler–Bernoulli model are included in a constraint term  $\mathcal{W}_{\text{cEB}}$  as explained above; the constraints  $\mathcal{C}_g$  and  $\mathcal{C}_d$ , which are specific to a ribbon, are treated by two contributions  $\mathcal{W}_{\text{cg}} = \int_0^L \lambda_g \hat{\mathcal{C}}_g dS$  and  $\mathcal{W}_{\text{cd}} = \int_0^L \lambda_d \hat{\mathcal{C}}_d dS$ , which we call, respectively, the virtual work of the geodesic and developability

constraints. Here,  $\lambda_g(S)$  and  $\lambda_d(S)$  denote two Lagrange multipliers, and  $\hat{C}_i$  is the first variation of any of the constraints,  $i \in \{d, g\}$ . Combining all the contributions, we write the principle of virtual work as

$$-\int_0^L \left( \frac{\partial \mathcal{E}}{\partial \eta'} \hat{\eta}' + \frac{\partial \mathcal{E}}{\partial \eta} \hat{\eta} + \frac{\partial \mathcal{E}}{\partial \omega_1} \hat{\omega}_1 \right) dS + \mathcal{W}_e + \mathcal{W}_{\text{cEB}} \cdots \\ + \int_0^L \lambda_g \frac{\partial \mathcal{C}_g}{\partial \omega_2} \hat{\omega}_2 dS + \int_0^L \lambda_d \left( \sum_{i=1}^3 \frac{\partial \mathcal{C}_d}{\partial \omega_i} \hat{\omega}_i + \frac{\partial \mathcal{C}_d}{\partial \eta} \hat{\eta} \right) dS = 0,$$

where  $\mathcal{E}$  is the elastic energy density defined by  $E = \int_0^L \mathcal{E}(\eta, \eta', \omega_1) dS$ .

Rearranging the terms, we have

$$\mathcal{W}_\eta + \mathcal{W}_i^{\text{rod}} + \mathcal{W}_e + \mathcal{W}_{\text{cEB}} = 0, \quad (21)$$

where we have grouped the terms depending on the internal variable  $\eta$ ,

$$\mathcal{W}_\eta = - \int_0^L \left[ \frac{\partial \mathcal{E}}{\partial \eta'} \hat{\eta}' + \left( \frac{\partial \mathcal{E}}{\partial \eta} - \lambda_d \frac{\partial \mathcal{C}_d}{\partial \eta} \right) \hat{\eta} \right] dS, \quad (22)$$

and the terms depending on the virtual changes of strain,

$$\mathcal{W}_i^{\text{rod}} = - \int_0^L \sum_{i=1}^3 \left( \frac{\partial \mathcal{E}}{\partial \omega_i} - \sum_{j \in \{g, d\}} \lambda_j \frac{\partial \mathcal{C}_j}{\partial \omega_i} \right) \hat{\omega}_i dS. \quad (23)$$

Equation (21) expresses the principle of virtual work for a ribbon: equilibrium configurations are such that this equation is satisfied for any kinematically admissible virtual displacement. Making use of the strong similarity with the principle of virtual work for a thin rod, we can now derive the equations of equilibrium for ribbons easily.

## 4.2 Equations of equilibrium

The equation of equilibrium with respect to  $\eta$  comes from the term  $\mathcal{W}_\eta$ . Integrating by parts and canceling the coefficient of  $\hat{\eta}$ , we find, as in [25],

$$-\frac{d}{dS} \left( \frac{\partial \mathcal{E}}{\partial \eta'} \right) + \frac{\partial \mathcal{E}}{\partial \eta} - \lambda_d \frac{\partial \mathcal{C}_d}{\partial \eta} = 0. \quad (24)$$

Through its first term, this equation depends on the second arc-length derivative of  $\eta$ . Boundary terms have been omitted: the derivation of the boundary condition for  $\eta$  is irrelevant to the case of a closed ribbon, and is left to the reader.

We proceed to the second term  $\mathcal{W}_i^{\text{rod}}$  in equation (21). We observe that it can be put into the usual form of the internal virtual work of a thin rod [7, 26, 6, 2], namely

$$\mathcal{W}_i^{\text{rod}} = - \int_0^L \mathbf{M} \cdot \sum_{i=1}^3 (\hat{\omega}_i \mathbf{d}_i) dS, \quad (25)$$

when we identify the internal moment  $\mathbf{M}(S)$  based on equation (23):

$$\begin{aligned} \mathbf{M} &= \sum_{i=1}^3 \left( \frac{\partial \mathcal{E}}{\partial \omega_i} - \lambda_d \frac{\partial \mathcal{C}_d}{\partial \omega_i} - \lambda_g \frac{\partial \mathcal{C}_g}{\partial \omega_i} \right) \mathbf{d}_i \\ &= \left( \frac{\partial \mathcal{E}}{\partial \omega_1} - \eta \lambda_d \right) \mathbf{d}_1 + \lambda_g \mathbf{d}_2 + \lambda_d \mathbf{d}_3. \end{aligned} \quad (26)$$

This equation is one of the main results of our paper. It expresses the constitutive law of the ribbon in the language of thin rods. The case of a folded annular strip has been considered recently by the same authors, and similar constitutive laws have been obtained [10, eqs. 9b and A.2b]. In equation (26), the first term in the parenthesis yields the usual constitutive law  $\mathbf{M} = \sum_i \frac{\partial \mathcal{E}}{\partial \omega_i} \mathbf{d}_i$  for an unconstrained, non-linearly elastic thin rod. The two other terms are constraint terms (the isotropic pressure term entering in the constitutive law of an incompressible elastic solid is a constraint term of the same kind).

Let us now consider the contributions to the principle of virtual work that involve the virtual motion of the center-line and directors, *i.e.* those depending on  $(\hat{\mathbf{x}}, \hat{\boldsymbol{\psi}}, \hat{\mathbf{d}}_i, \hat{\omega}_i)$ . These contributions are  $\mathcal{W}_i^{\text{rod}} + \mathcal{W}_e + \mathcal{W}_{\text{CEB}}$ , as the other contribution  $\mathcal{W}_\eta$  concerns the internal degree of freedom only. These are exactly the same three terms as those entering in the principle of virtual work for an inextensible Euler–Bernoulli rod. It is well known [7, 26, 6, 2] that the corresponding equations of equilibrium are the Kirchhoff equations expressing the balance of forces and moments on a small chunk of the center-line,

$$\mathbf{R}'(S) + \mathbf{p}(S) = \mathbf{0}, \quad (27a^*)$$

$$\mathbf{M}'(S) + \mathbf{x}'(S) \times \mathbf{R}(S) + \mathbf{c}(S) = \mathbf{0}. \quad (27b^*)$$

This remark saves us the effort of rederiving these equations of equilibrium. Note that there is no constitutive law associated with the internal force  $\mathbf{R}(S)$ , as  $\mathbf{R}(S)$  is the Lagrange multiplier associated with the Euler–Bernoulli constraint in equation (2).

To sum up, an elastic ribbon is governed by the same equations as an inextensible Euler–Bernoulli rod, up to two small changes: (*i*) the presence of the internal degree of freedom  $\eta$  yields the new equation of equilibrium (24), and (*ii*) the presence of constraint terms in the constitutive law; see equation (26).

### 4.3 Complete set of equations for an elastic ribbon

The complete set of equations for the equilibrium of a ribbon are summarized as follows. The geometry and the natural shape of the ribbon are prescribed by the geodesic curvature  $\kappa_g(S)$ , by the edge functions  $V_\pm(\eta, S)$ , and by the tensor  $\mathbf{Q}_r$  defined in equation (18). The external loading is prescribed by the functions  $\mathbf{p}(S)$  and  $\mathbf{c}(S)$ , subject to the global balance of forces and moments. The equilibrium configuration of the ribbon is sought in terms of the following *unknowns*: the center-line  $\mathbf{x}(S)$ , the directors  $\mathbf{d}_i(S)$ , the parameter

$\eta(S)$  capturing the direction of the generatrices, and the Lagrange multipliers  $M_2(S) = \lambda_g(S)$  and  $M_3(S) = \lambda_d(S)$ .

The *geometrical* equations are the compatibility of the tangent and the third director in equation (2), the orthonormality of the directors in equation (3), and the developability constraints in equations (10–11). The rotation gradient  $\omega_i(S)$  is found using equation (7).

In terms of the unknowns, one expresses the coordinate  $V_c$  of the striction curve using equation (13), and the curvature tensor  $\mathbf{K}_0$  along the center-line using equation (16). Based on the definition of the energy functional in equation (20), one can then calculate the internal moment by the *constitutive law*, which we rewrite from equation (26) as

$$\mathbf{M}(S) = \left( \frac{\partial \mathcal{E}}{\partial \omega_1} - \eta M_3 \right) \mathbf{d}_1(S) + M_2(S) \mathbf{d}_2(S) + M_3(S) \mathbf{d}_3(S). \quad (28^*)$$

The *equations of equilibrium* are the standard Kirchhoff equations (27) together with equation (24) for the equilibrium of  $\eta$ , which we rewrite as

$$-\frac{d}{dS} \left( \frac{\partial \mathcal{E}}{\partial \eta'} \right) + \frac{\partial \mathcal{E}}{\partial \eta} - M_3 \omega_1 = 0. \quad (29^*)$$

If the ribbon is closed, these equations must be complemented by periodic boundary conditions<sup>3</sup>. If it is open, boundary conditions corresponding to the type of support must be enforced — their derivation has been left to the reader.

The equations tagged by a star, such as equations (28) and (29) above, form the complete set of equations governing the equilibrium of a ribbon.

## 5 Special cases

We review ribbon models known in the literature, that are specific to the case  $\kappa_g = 0$ , in the light of our general model.

### 5.1 Naturally straight, rectangular ribbons

This model was first introduced by Wunderlich in his study of the shape of a Möbius band [28]. The Möbius band is a naturally flat and straight ribbon having a constant width  $w$ : we set  $\mathbf{Q}_r = \mathbf{0}$ ,  $\kappa_g = 0$ . Then, by equation (13), the coordinate of the striction curve is  $V_c = -1/\eta'$ . The relevant edge functions

<sup>3</sup> Finding the conditions for a continuous curve in space to be closed is not a trivial problem and it may not have a close solution. This problem was posed by N. V. Efimov [12] and W. Frenkel [14]. Frenkel posed the problem asking what are the necessary and sufficient conditions of closure given the curvature and torsion of a space curve. By integrating the Frenet–Serret equations, the result yields an infinite series of integrals with no closed form [4].

are given by equations (8) as  $V_{\pm} = \pm w/2$ . Inserting into the energy functional in equation (20), we have

$$E_W = \frac{D w}{2} \int_0^L \omega_1^2 (1 + \eta^2)^2 \frac{1}{\eta' w} \ln \left( \frac{1 + \eta' w/2}{1 - \eta' w/2} \right) dS, \quad (30)$$

where we have used  $\text{tr}(\mathbf{K}_0^2) = \omega_1^2 (1 + \eta^2)^2$ . Wunderlich's energy functional is recovered [28]. It has been analyzed recently and is relevant to the elastic Möbius strip [19, 25], and to open developable ribbons [17].

Starostin and van der Heijden [25] have worked out the corresponding equations of equilibrium (for the naturally flat, rectangular ribbon). Here, we have recovered their results as a special case: their equations [4] are identical<sup>4</sup> to the Kirchhoff equations (27) derived above; their equations [5] are the constitutive law (28) and the equilibrium for the internal variable  $\eta$ .

## 5.2 Sadowsky's limit: narrow rectangular ribbons

The limiting case of Wunderlich's energy in equation (30) corresponding to  $w \rightarrow 0$  was derived by Sadowsky [21] much before Wunderlich. It reads

$$E_S = \frac{D w}{2} \int_0^L \omega_1^2 (1 + \eta^2)^2 dS. \quad (31)$$

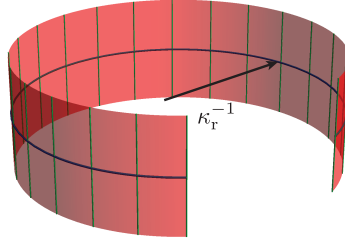
This *narrow strip* model applies to a rectangular, naturally flat ribbon, when the deformation is sufficiently small for the striction curve to remain far from the physical edge of the ribbon,  $|V_c| \gg w$ . This model captures the geometrically non-linear coupling between bending and twisting modes. It has been applied to the statistical mechanics of developable ribbons [15] and to the analysis of elastic strips comprising a central fold [10].

The equations governing the equilibrium of Wunderlich's strip model, derived in reference [25], apply as a special case to the equilibrium of Sadowsky's narrow strip model. An alternative derivation appears in reference [10, section 2].

## 5.3 Helical ribbons

The case of a naturally helical ribbon has been studied recently [24, 5]. Its geodesic curvature is zero,  $\kappa_g = 0$ . Assuming that the helical shape is stress-

<sup>4</sup> The quantities  $(2w, \mathbf{t}, \mathbf{n}, \mathbf{b}, \tau, \kappa, \eta, g, \mathbf{M}, \mathbf{F}, M_t, M_b)$  in their notation must be identified with the quantities  $(w, \mathbf{d}_3, -\mathbf{d}_2, \mathbf{d}_1, \omega_3, \omega_1, \eta, E, -\mathbf{M}, -\mathbf{R}, -M_3, -M_1)$  in our notation, respectively. The last four minus signs introduced here arise because they use a non-standard convention for the sign of the internal force  $\mathbf{R}$  and moment  $\mathbf{M}$  — by contrast, we use the usual convention that  $\mathbf{R}$  and  $\mathbf{M}$  measure the force and moment applied across an imaginary cut by the downstream part of the ribbon onto the upstream part, where 'downstream' and 'upstream' refer to the direction of increasing arc-length coordinate  $S$ .



**Fig. 2** A ribbon cut out in a cylindrical shell having natural curvature  $\kappa_r$  (§6.1).

free, the natural curvature tensor reads, from equation (16),

$$\mathbf{K}_r = -\omega_1^r \mathbf{d}_3 \otimes \mathbf{d}_3 + \omega_3^r (\mathbf{d}_3 \otimes \mathbf{d}_1 + \mathbf{d}_1 \otimes \mathbf{d}_3) - \frac{(\omega_3^r)^2}{\omega_1^r} \mathbf{d}_1 \otimes \mathbf{d}_1$$

where  $\omega_1^r$  and  $\omega_3^r$  are the two parameters defining the radius and the step of the helix. The equations of equilibrium, derived in [24] by the same method as in [25], can be recovered from the general model derived above.

When subjected to moderate forces and moments compatible with the helical symmetries, these ribbons remain helical — such helical configurations are relevant to the opening of chiral seed pods [1], a phenomenon driven by residual internal stresses [27]. In the presence of larger or less symmetric loads, non-helical solutions are possible [24, 5].

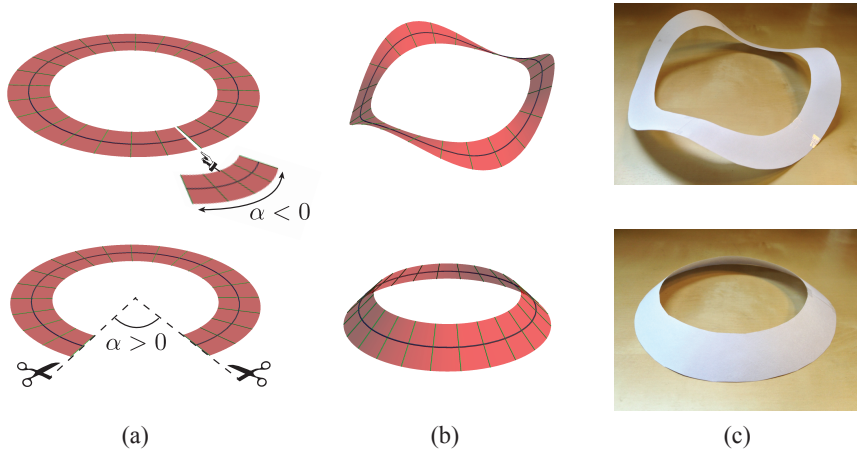
## 6 Illustrations

With the aim to illustrate our results and motivate further studies, we present a few problems that could be solved using the equations derived in this work.

### 6.1 Buckling of a cylindrical ribbon

We consider the cylindrical ribbon shown in figure 2. Its natural curvature is denoted by  $\kappa_r$ . When laid flat, the ribbon is a rectangle of size  $L \times w$ . In its natural configuration, the direction supported by its width  $w$  is aligned with the axis of the cylinder: its natural curvature is an out-of-plane curvature  $\kappa_r = \omega_1^r$ , its geodesic curvature being zero,  $\kappa_g = 0$ . The tensor of natural curvature reads  $\mathbf{K}_r = -\kappa_r \mathbf{d}_3 \otimes \mathbf{d}_3$ , as shown by inserting  $\omega_1 = \kappa_r$  and  $\eta = \omega_3/\omega_1 = 0$  into equation (16). Using the definition of the internal moment  $\mathbf{Q}_r$  in equation (18), we have

$$\mathbf{Q}_r = -\kappa_r (\mathbf{d}_3 \otimes \mathbf{d}_3 + \nu \mathbf{d}_1 \otimes \mathbf{d}_1). \quad (32)$$



**Fig. 3** Undercurved (top row) and overcurved (bottom row) annuli (§6.2) : (a) preparation of the two states, (b) sketch of typical equilibrium shapes, (c) experiments using paper models.

By equation (20), the energy functional governing this strip model reads

$$E = \frac{Dw}{2} \int_0^L \left[ \omega_1^2 (1 + \eta^2)^2 \frac{1}{w\eta'} \ln \left( \frac{1 + \eta'w/2}{1 - \eta'w/2} \right) - 2\kappa_r \omega_1 (1 + \nu\eta^2) \right] dS, \quad (33)$$

which is simply Wunderlich's energy, complemented by a term depending on natural curvature. For a narrow strip (Sadowsky's limit,  $\eta'w \rightarrow 0$ ), this becomes

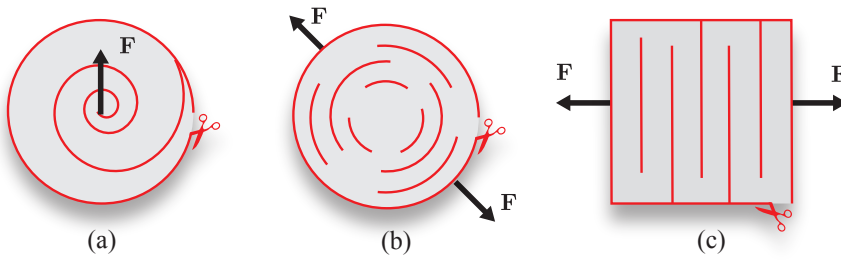
$$\begin{aligned} E &= \frac{Dw}{2} \int_0^L \left[ \omega_1^2 (1 + \eta^2)^2 - 2\kappa_r \omega_1 (1 + \nu\eta^2) \right] dS \\ &= \frac{Dw}{2} \int_0^L \left[ (\omega_1 (1 + \eta^2) - \kappa_r)^2 - \kappa_r^2 \right] dS. \end{aligned} \quad (34)$$

The buckling of a closed ribbon of this type, caused by a mismatch of the natural curvature and the curvilinear length of the ribbon ( $\kappa_r \neq 2\pi/L$ ), is studied in another article [22] based on the equations derived here.

## 6.2 Overcurved and undercurved annular ribbons

The undercurved and overcurved annuli shown in figure 3 are simple examples of ribbons having non-zero natural curvature. They can be obtained as a paper model by cutting out an annulus from a sheet of paper. Assume the reference line to be middle line of the annular strip: its curvature defines the geodesic curvature  $\kappa_g \neq 0$ . If a sector with angle  $\alpha$  is removed from the annulus and the two newly formed ends are pasted together, the annulus becomes overcurved: its curvilinear length  $L = (2\pi - \alpha)/\kappa_g$  is less than that of a circle





**Fig. 4** Patterns cut out in a piece of paper can produce various 3D shapes when pulled (§6.3): (a) spiral cut pattern, (b) concentric, circular cut patterns, (c) straight, alternating cut patterns.

with curvature  $\kappa_g$ , namely  $2\pi/\kappa_g$ . The undercurved annuli can be achieved by pasting together two identical annuli, each one being such that  $\alpha < \pi$  — the effective value of  $\alpha$  after pasting is then negative,  $\alpha < 0$ . In the presence of undercurvature or overcurvature, the annulus will buckle out-of-plane. This problem is a variant of the problem of the *folded* annular strip, which we have studied recently [11, 10]. The buckling of the annular ribbon without a fold has not yet been analyzed to the best of our knowledge. This can be done using the equations derived in this article.

### 6.3 3D kirigami from 2D cut-out patterns

Complex 3D shapes can be obtained by pulling on a thin sheet of paper that has been cut along arbitrary lines; see figure 4. This special form of kirigami (the art of *cutting paper*) involves tuning the geometry of the cuts to produce various 3D shapes. For instance, the alternated concentric cuts shown in figure 4b are the basis of a commercial paper model that produces a bowl-like shape. In all these examples, the center-line is not a geodesic. The example shown in figure 4c, for instance, is made by patching rectangular strips together, and it can be viewed as a single strip having a variable width and a zigzagging center-line. The unfolding of these 3D kirigami could be analyzed using the general ribbon model derived in this article.

## 7 Conclusion

We have presented a general theory of ribbons that fits into the well-established framework of thin elastic rods. By working with a frame of directors (material frame) instead of the theory of Frenet–Serret frame, we could extend the energy functional and the equilibrium equations to a general ribbon geometry. In particular, geodesic curvature, out-of-plane curvature, and a variable width have been taken into account. Instead of using the inextensibility conditions to eliminate degrees of freedom, as in previous work, we have treated them

as constraints in the sense of the calculus of variations. This allowed us to view an elastic ribbon as a special kind of a thin elastic rod — namely, a kinematically constrained, hyperelastic rod possessing an internal degree of freedom. We could also explain the deep similarities between the theories of ribbons and thin rods.

This unifying view bridges the gap between classical rod theory and the theory of elastic ribbons — the latter has been developed largely as an independent subject so far. This makes it possible to reuse the large body of numerical and analytical methods available for thin rods. Numerical methods for simulating elastic ribbons, for instance, are currently limited to the calculation of non-linear equilibria, using numerical continuation, see e.g. [25]. Numerical continuation is a powerful tool but its use can be quite impractical. Even though this is standard for thin rods [8], we are not aware of any simulation method that can predict the dynamics of ribbons, or account for self-contact. When extended to the dynamical case by including the virtual power of acceleration, the constrained variational formulation presented here offers a natural way of porting existing numerical methods for the dynamics of thin rods, such as the finite-element method [29] or methods based on discrete differential geometry [3], to ribbons.

Here we have focused on developable configurations. Under large loads involving a combination of tension and twisting, ribbons can adopt non-developable configurations [16, 20, 5]. In future work, it would be interesting to extend the parameterization of ribbons presented here to account for deviations from the constraint of local area preservation.

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## A Curvature tensor of a developable surface

Here, we use the notation introduced in section 2.1 to prove the geometrical identities announced in sections 2.3 to 2.5, relevant to developable surfaces. We consider a general ruled surface, enforce the condition of developability, and derive the expression of the curvature tensor at an arbitrary point on the surface. By doing so, we extend the expressions obtained by Wunderlich [28] and by Starostin and van der Heijden [25] to account for the geodesic curvature  $\kappa_g$  of the center-line.

Let us first calculate the tangent vectors at an arbitrary point  $\mathbf{y}(S, V)$  of the surface:

$$\mathbf{y}_{,S}(S, V) = \mathbf{d}_3(S) + V \mathbf{q}'(S) \quad (35a)$$

$$\mathbf{y}_{,V}(S, V) = \mathbf{q}(S), \quad (35b)$$

where  $\mathbf{q}'(S)$  denotes the total derivative of  $\mathbf{q}$  defined in equation (4) as  $\mathbf{q} = \eta \mathbf{d}_3 + \mathbf{d}_1$ .

Using equation (6), we write the following equation

$$\mathbf{q}' = \eta \omega_2 \mathbf{d}_1 + (\omega_3 - \eta \omega_1) \mathbf{d}_2 + (\eta' - \omega_2) \mathbf{d}_3, \quad (36)$$

which implies

$$\mathbf{q} \times \mathbf{q}' = (\eta \omega_1 - \omega_3) \mathbf{q}^\perp + \frac{1}{V_c} \mathbf{d}_2 \quad (37)$$

where  $V_c$  is the quantity defined by equation (13), and  $\mathbf{q}^\perp$  is the vector

$$\mathbf{q}^\perp = \mathbf{d}_2 \times \mathbf{q} = -\mathbf{d}_3 + \eta \mathbf{d}_1. \quad (38)$$

Later on, we shall show that  $\mathbf{q}^\perp$  is a vector perpendicular to  $\mathbf{q}$  lying in the plane tangent to the surface; hence the notation.

The classical condition for a ruled surface to be developable [23, section 3.II] is that the following three vectors are linearly dependent: the tangent  $\mathbf{d}_3$  to the center-line (called the directrix in the context of the geometry of surfaces), the vector  $\mathbf{q}$  spanning the generatrices, and its derivative with respect to the arc-length along the center-line. This is expressed by  $(\mathbf{q} \times \mathbf{q}') \cdot \mathbf{d}_3 = 0$ . In view of equation (37), this yields  $\eta \omega_1 = \omega_3$ , which is the constraint of developability announced in equation (11a).

As a result,  $\mathbf{q}' \cdot \mathbf{d}_2 = 0$ , and so  $\mathbf{y}_{,S} \cdot \mathbf{d}_2 = 0$ . On the other hand, equation (35b) shows that  $\mathbf{y}_{,V} \cdot \mathbf{d}_2 = 0$ . The director  $\mathbf{d}_2(S)$  is orthogonal to both tangents:  $\mathbf{d}_2$  is a unit normal at any point of the developable surface.

The element of area on the surface reads

$$da = |\mathbf{y}_{,S} \times \mathbf{y}_{,V}| dS dV = |\mathbf{d}_3 \times \mathbf{q} + V \mathbf{q}' \times \mathbf{q}| dS dV = \left| \left(1 - \frac{V}{V_c}\right) \mathbf{d}_2 \right| dS dV \quad (39)$$

Noting that  $1 - V/V_c > 0$  by the inequality (14b), we arrive at the result announced in equation (12).

To compute the curvature tensor  $\mathbf{K}(S, V)$ , we note that the direction of the generatrix is a principal direction of zero curvature, since the surface is developable. Therefore, there exists some scalar field  $k(S, V)$  such that

$$\mathbf{K} = k \mathbf{q}^\perp \otimes \mathbf{q}^\perp. \quad (40)$$

The quantity  $k$  in equation above can be found by contracting with  $\mathbf{y}_{,S}$  on both sides of the equation to give:

$$\mathbf{y}_{,S} \cdot \mathbf{K} \cdot \mathbf{y}_{,S} = k \left( \mathbf{q}^\perp \cdot (\mathbf{d}_3 + V \mathbf{q}') \right)^2 = k \left( -1 + \frac{V}{V_c} \right)^2. \quad (41)$$

By the definition of the curvature tensor (second fundamental form) [23], the left-hand side of the resulting identity is the normal projection of the second derivative  $\mathbf{y}_{,SS}$ :

$$\mathbf{y}_{,S} \cdot \mathbf{K} \cdot \mathbf{y}_{,S} = \mathbf{y}_{,SS} \cdot \mathbf{d}_2 = (\mathbf{d}'_3 + V \mathbf{q}'') \cdot \mathbf{d}_2 = -\omega_1 + V \left( \frac{d(\mathbf{q}' \cdot \mathbf{d}_2)}{dS} - \mathbf{q}' \cdot \mathbf{d}'_2 \right). \quad (42)$$

In this equation,  $\mathbf{q}' \cdot \mathbf{d}_2 = \omega_3 - \eta \omega_1 = 0$  by the developability condition, and  $\mathbf{q}' \cdot \mathbf{d}'_2 = \mathbf{q}' \cdot (\boldsymbol{\omega} \times \mathbf{d}_2) = \mathbf{q}' \cdot (\omega_1 \mathbf{q} \times \mathbf{d}_2) = -\omega_1 \mathbf{d}_2 \cdot (\mathbf{q} \times \mathbf{q}') = -\omega_1/V_c$ . Therefore,

$$\mathbf{y}_{,S} \cdot \mathbf{K} \cdot \mathbf{y}_{,S} = -\omega_1 \left( 1 - \frac{V}{V_c} \right) \quad (43)$$

From equations (41) and (43), we can solve for  $k$ , giving  $k = -\omega_1/(1 - V/V_c)$ . Inserting this result into equation (40) yields the expression of the curvature tensor announced in equations (15) and (16). The curvature tensor keeps the same form as in the case of zero geodesic curvature [28, 25] provided that the proper definition of  $V_c$  in terms of  $\kappa_g$  is used; see equation (13).

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